ON THE PICARD BUNDLE

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ABSTRACT. Fix a holomorphic line bundle ξ over a compact connected Riemann surface X of genus g, with $g \geq 2$, and also fix an integer r such that degree$(\xi) > r(2g-1)$. Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles over X of rank r and determinant ξ. The Fourier–Mukai transform, with respect to a Poincaré line bundle on $X \times J(X)$, of any $F \in \mathcal{M}_\xi(r)$ is a stable vector bundle on $J(X)$. This gives an embedding of $\mathcal{M}_\xi(r)$ in a moduli space associated to $J(X)$. If $g = 2$, then $\mathcal{M}_\xi(r)$ becomes a Lagrangian subvariety.

Résumé

Sur le fibré de Picard. Soient ξ un fibré en droites holomorphe sur une surface de Riemann compacte connexe X de genre $g \geq 2$, et r un entier tel que degré$(\xi) > r(2g-1)$. Notons $\mathcal{M}_\xi(r)$ l'espace de modules des fibrés vectoriels stables sur X, de rang r et de déterminant ξ. Ayant choisi un fibré de Poincaré sur $X \times J(X)$, la transformée de Fourier–Mukai associée fait correspondre à un fibré $F \in \mathcal{M}_\xi(r)$ un fibré vectoriel stable sur $J(X)$. Ceci fournit un plongement de $\mathcal{M}_\xi(r)$ dans un espace de modules associé à $J(X)$. Lorsque $g = 2$, $\mathcal{M}_\xi(r)$ s'identifie ainsi à une sous-variété lagrangienne de cet espace de modules.

1. INTRODUCTION

Let $(X\,, x_0)$ be a one–pointed compact connected Riemann surface of genus g, with $g \geq 2$. Let $\mathcal{L}$ be the Poincaré line bundle on $X \times J(X)$ constructed using x_0, where $J(X)$ is the Jacobian of X. Fix an integer $r \geq 2$ and a holomorphic line bundle ξ over X with degree$(\xi) > r(2g-1)$. Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles over X of rank r and determinant ξ.

In Lemma 2.1 we show that for any $F \in \mathcal{M}_\xi(r)$,

$$\mathcal{V}_F := \phi_{J*}(\mathcal{L} \otimes \phi_X^* F)$$

is a stable vector bundle with respect to the canonical polarization on $J(X)$, where ϕ_J (respectively, ϕ_X) is the projection of $X \times J(X)$ to $J(X)$ (respectively, X).

Rational characteristic classes of $\mathcal{V}_F$, as well as the line bundle $\bigwedge^{\text{top}} \mathcal{V}_F$, are independent of F. Let $\mathcal{M}(J(X))$ be the moduli space of stable vector bundles W over $J(X)$ with $\text{rank}(W) = \text{rank}(\mathcal{V}_F)$, $c_i(W) = c_i(\mathcal{V}_F)$ and $\bigwedge^{\text{top}} W = \bigwedge^{\text{top}} \mathcal{V}_F$. The map $\mathcal{M}_\xi(r) \longrightarrow \mathcal{M}(J(X))$ defined by $F \longmapsto \mathcal{V}_F$ is an embedding (see Corollary 2.3).

We next assume that $g = 2$, and if degree(ξ) is even, then also assume that $r \geq 3$. Let $\mathcal{M}^0(J(X)) \subset \mathcal{M}(J(X))$ be the locus of all W for which the image of $C_1(W)^2 - 2\cdot C_2(W) \in \text{CH}^2(J(X))$ in the Deligne–Beilinson cohomology vanishes.
Notation: The i–th Chern class with values in the Chow group will be denoted by C_i.

We show that the image of $\mathcal{M}_\xi(r)$ lies in $\mathcal{M}^0(J(X))$, and $\mathcal{M}_\xi(r)$ is a Lagrangian subvariety of the symplectic variety $\mathcal{M}^0(J(X))$.

2. Fourier–Mukai transform of a stable vector bundle

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. Fix once and for all a point $x_0 \in X$.

Let $J(X) := \mathrm{Pic}^0(X)$ be the Jacobian of X. There is a canonical principal polarization on $J(X)$ given by the cup product of $H^1(X, \mathbb{Z})$. All stable vector bundles over $J(X)$ considered here will be with respect to this polarization.

Let $\mathcal{L}$ be a holomorphic line bundle over $X \times J(X)$ such that

- for each point $\xi \in J(X)$, the restriction of $\mathcal{L}$ to $X \times \{\xi\}$ is in the isomorphism class of holomorphic line bundles represented by ξ, and
- the restriction of $\mathcal{L}$ to $\{x_0\} \times J(X)$ is a holomorphically trivial line bundle over $J(X)$.

Such a line bundle $\mathcal{L}$ exists [1, p. 166–167]. Moreover, from the see–saw theorem (see [7, p. 54, Corollary 6]) it follows that $\mathcal{L}$ is unique up to a holomorphic isomorphism. We will call $\mathcal{L}$ the *Poincaré line bundle* for the pointed curve $(X\,, x_0)$.

Fix an integer $r \geq 2$. Fix a holomorphic line bundle ξ over X with

$$\mathrm{degree}(\xi) > r(2g-1)\,. \tag{1}$$

Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles E over X with $\mathrm{rank}(E) = r$ and $\bigwedge^r E = \xi$.

Let ϕ_J (respectively, ϕ_X) denote the projection of $X \times J(X)$ to $J(X)$ (respectively, X).

Lemma 2.1. *For each vector bundle $F \in \mathcal{M}_\xi(r)$,*

$$R^1\phi_{J*}(\mathcal{L} \otimes \phi_X^* F) = 0\,,$$

where $\mathcal{L}$ is the Poincaré line bundle. The direct image

$$\mathcal{V}_F := \phi_{J*}(\mathcal{L} \otimes \phi_X^* F)$$

is a stable vector bundle over $J(X)$ of rank $\delta :=$ degree$(\xi) - r(g-1)$.

Proof. For a stable vector bundle W over X of rank r and degree $d > 2r(g-1)$, we have $H^0(X, W^* \otimes K_X) = 0$ because a stable vector bundle of negative degree does not admit any nonzero sections. Hence by Serre duality we have $H^1(X, W) = 0$. Therefore, using (1) it follows that $R^1\phi_{J*}(\mathcal{L} \otimes \phi_X^* F) = 0$.

Since $R^1\phi_{J*}(\mathcal{L} \otimes \phi_X^* F) = 0$, we know that the direct image $\mathcal{V}_F$ in the statement of the lemma is a vector bundle of rank degree$(\xi) - r(g-1)$.

The stability of $\mathcal{V}_F$ is derived from [2, p. 5, Theorem 1.2] as follows. Consider the embedding

$$f : X \longrightarrow J(X)$$

defined by $x \longmapsto \mathcal{O}_X(x_0 - x)$. Therefore,

$$(\mathrm{Id}_X \times f)^*\mathcal{L} = \mathcal{O}_{X\times X}((\{x_0\} \times X) - \Delta)\,, \tag{2}$$

where $\Delta \subset X \times X$ is the diagonal divisor.

Set E in [2, Theorem 1.2] to be $F \otimes \mathcal{O}_X(x_0)$. Using (2) it follows that the vector bundle M_E in [2, Theorem 1.2] is identified with $f^*\mathcal{V}_F$. From [2, Theorem 1.2] we know that $f^*\mathcal{V}_F$ is stable. Now using the openness of the stability condition (see [4, p. 635, Theorem 2.8(B)]) it follows that there is a Zariski open dense subset

$$U \subset J(X) \tag{3}$$

such that for each $z \in U$, the pullback $f^*\tau_z^*\mathcal{V}_F$ is a stable vector bundle, where $\tau_z \in \mathrm{Aut}(J(X))$ is the translation defined by $y \longmapsto y+z$.

If $\mathcal{W} \subset \mathcal{V}_F$ violates the stability condition of $\mathcal{V}_F$ for the canonical polarization, then take a point $z_0 \in U$ (see (3)) such that $\tau_{z_0} \circ f$ intersects the Zariski open dense subset of $J(X)$ over which $\mathcal{W}$ is locally free. Now it is straight–forward to check that

$$f^*\tau_{z_0}^*\mathcal{W} \subset f^*\tau_{z_0}^*\mathcal{V}_F$$

contradicts the stability condition of $f^*\tau_{z_0}^*\mathcal{V}_F$. Therefore, we conclude that $\mathcal{V}_F$ is stable. This completes the proof of the lemma. □

Fix a holomorphic line bundle L over $J(X)$ such that $c_1(L)$ coincides with the canonical polarization on $J(X)$. As in [7, p. 123], set

$$M := m^*L \otimes p_1^*L^* \otimes p_2^*L^* \tag{4}$$

on $J(X) \times J(X)$, where

$$p_i : J(X) \times J(X) \longrightarrow J(X) \tag{5}$$

is the projection to the i–th factor, and m is the addition map on $J(X)$; the dual abelian variety $J(X)^\vee$ is identified with $J(X)$ using the Poincaré line bundle $\mathcal{L}$. Let

$$\varphi : X \longrightarrow J(X) \tag{6}$$

be the morphism defined by $x \longmapsto \mathcal{O}_X(x-x_0)$. Then

$$(\varphi \times \mathrm{Id}_{J(X)})^*M = \mathcal{L}\,.$$

Proposition 2.2. *Consider the vector bundle $\mathcal{V}_F$ is Lemma 2.1. For all $i \neq g$,*

$$R^ip_{1*}(M^* \otimes p_2^*\mathcal{V}_F) = 0\,,$$

and

$$R^gp_{1*}(M^* \otimes p_2^*\mathcal{V}_F) = \varphi_*F\,,$$

where M and φ are is defined in (4) *and* (6) *respectively, and p_1 and p_2 are the projections in* (5)*.*

Proof. The proof of the proposition is identical to the proof of Theorem 2.2 in [5, p. 156]. We note that the key input is the result in [7, p. 127] which says that $R^ip_{1*}M = 0$ for $i \neq g$, and $R^gp_{1*}M = \mathbb{C}$ is supported at the point $e_0 = \mathcal{O}_X$ with stalk $H^g(J(X) \times J(X),\, M) \cong \mathbb{C}$. (see also [7, p. 129, Corollary 1]). □

Let $\underline{c} := c_1(\mathcal{V}_F) \in H^2(J(X),\,\mathbb{Z})$. Note that since $\mathcal{M}_\xi(r)$ is connected, for all $i \geq 0$, the Chern class $c_i(\mathcal{V}_F) \in H^{2i}(J(X),\,\mathbb{Z})$ is independent of the choice of $F \in \mathcal{M}_\xi(r)$. We have a morphism

$$\alpha : \mathcal{M}_\xi(r) \longrightarrow \mathrm{Pic}^{\underline{c}}(J(X))$$

defined by $E \longmapsto \bigwedge^\delta \mathcal{V}_E$ (see Lemma 2.1). Since $\mathcal{M}_\xi(r)$ is a Zariski open subset of a unirational variety (the moduli space of semistable vector bundles over X of rank r and determinant ξ is unirational), the morphism α constructed above must be a constant one.

Let $\mathcal{M}(J(X))$ denote the moduli space of stable vector bundles $\mathcal{W}$ over $J(X)$ with $\text{rank}(\mathcal{W}) = \delta := \text{degree}(\xi) - r(g-1)$, $\bigwedge^{\text{top}} \mathcal{W} = \text{image}(\alpha)$, and $c_i(\mathcal{W}) = c_i(\mathcal{V}_F)$ for all $i \geq 2$.

Corollary 2.3. *We have a morphism*

$$\beta : \mathcal{M}_\xi(r) \longrightarrow \mathcal{M}(J(X)) \tag{7}$$

defined by $F \longmapsto \mathcal{V}_F$. This morphism β is an embedding.

Proof. The map β is well defined by Lemma 2.1. That β is an embedding follows immediately from Proposition 2.2, because we have a morphism

$$\gamma : \beta(\mathcal{M}_\xi(r)) \longrightarrow \mathcal{M}_\xi(r)$$

defined by $W \longmapsto \varphi^* R^g p_{1*}(M^* \otimes p_2^* W)$ such that $\gamma \circ \beta$ is the identity map of $\mathcal{M}_\xi(r)$. ☐

3. The case of $g = 2$

Henceforth, we will assume that $g = 2$. If degree(ξ) is even, then we will also assume that $r > 2$.

Lemma 3.1. *Take any $F \in \mathcal{M}_\xi(r)$. Then the image of $C_1(\mathcal{V}_F)^2 - 2 \cdot C_2(\mathcal{V}_F) \in \text{CH}^2(J(X))$ in the Deligne–Beilinson cohomology $\text{H}^4_{\mathcal{D}}(J(X),\, \mathbb{Z}(2))$ (see* [3, p. 85, Corollary 7.7]*) is independent of F. More precisely, it vanishes.*

Proof. Since $\text{H}^4_{\mathcal{D}}(J(X),\, \mathbb{Z}(2))$ is an extension of a discrete group by a complex torus [3, p. 86, (7.9)], and $\mathcal{M}_\xi(r)$ is connected and unirational, there is no nonconstant morphism from $\mathcal{M}_\xi(r)$ to $\text{H}^4_{\mathcal{D}}(J(X),\, \mathbb{Z}(2))$. In particular, the image of $C_1(\mathcal{V}_F)^2 - 2 \cdot C_2(\mathcal{V}_F)$ in $\text{H}^4_{\mathcal{D}}(J(X),\, \mathbb{Z}(2))$ is independent of the choice of $F \in \mathcal{M}_\xi(r)$.

From [8, § 4] (reproduced in [5, p. 164, Theorem 4.3(2)]) we know that $C_1(\mathcal{V}_F) = r \cdot \lambda^*_{x_0}\Theta$, where $\Theta \in \text{Pic}^1(X)$ is the theta divisor, and $\lambda_{x_0} : \text{Pic}^0(X) \longrightarrow \text{Pic}^0(X)$ is defined by $\zeta \longmapsto \zeta \otimes \mathcal{O}_X(x_0)$. Similarly, $C_2(\mathcal{V}_F) = r^2 \cdot e_0$, where $e_0 = \mathcal{O}_X$ is the identity element. On the other hand, the image of $\Theta^2 - 2e_0$ in $\text{H}^4_{\mathcal{D}}(J(X),\, \mathbb{Z}(2))$ vanishes (see the proof of Theorem 1.3 in [1, p. 212]). ☐

Consider the moduli space $\mathcal{M}(J(X))$ in (7). Let

$$\mathcal{M}^0(J(X)) \subset \mathcal{M}(J(X)) \tag{8}$$

be the subvariety defined by the locus of all E such that image of

$$C_1(E)^2 - 2 \cdot C_2(E) \in \text{CH}^2(J(X))$$

in $\text{H}^4_{\mathcal{D}}(J(X),\, \mathbb{Z}(2))$ vanishes. From Lemma 3.1 we know that the image of the map β in Corollary 2.3 lies in $\mathcal{M}^0(J(X))$.

Since $J(X)$ is an abelian surface, the moduli space $\mathcal{M}^0(J(X))$ in (8) is smooth, and it has a canonical symplectic structure [6, p. 102, Corollary 0.2].

Theorem 3.2. *The image of the embedding β in Corollary 2.3 is a Lagrangian subvariety of the symplectic variety $\mathcal{M}^0(J(X))$.*

Proof. We note that $\mathcal{M}_\xi(r)$ is the smooth locus of the moduli space of semistable vector bundles over X of rank r and determinant ξ. In particular, $\mathcal{M}_\xi(r)$ is the smooth locus of a normal unirational variety. Therefore, $\mathcal{M}_\xi(r)$ does not admit any nonzero algebraic two–forms. Consequently, the pull back to $\mathcal{M}_\xi(r)$ of the symplectic form on $\mathcal{M}^0(J(X))$ vanishes identically. Therefore, to prove the theorem it suffices to show that

$$\dim \mathcal{M}^0(J(X)) \,=\, 2\cdot \dim \mathcal{M}_\xi(r) \,=\, 2(r^2-1)\,. \tag{9}$$

Let $\theta \in H^2(J(X),\mathbb{Z})$ denote the canonical polarization. In the proof of Lemma 3.1 we noted that $c_1(\mathcal{V}_F) = r\cdot\theta$, and $ch_2(\mathcal{V}_F) = c_1(\mathcal{V}_F)^2/2 - c_2(\mathcal{V}_F) = 0$. Hence $ch_2(\mathcal{E}nd(\mathcal{V}_F))([J(X)]) = -r^2$. Therefore, using Hirzebruch–Riemann–Roch,

$$\dim H^1(J(X),\, \mathcal{E}nd(\mathcal{V}_F)) \,=\, r^2+2\,.$$

Since $\dim \mathcal{M}^0(J(X)) = \dim \mathcal{M}(J(X)) - 2 = \dim H^1(J(X),\, \mathcal{E}nd(\mathcal{V}_F)) - 4$, we now conclude that (9) holds. This completes the proof of the theorem. □

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