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Zero cycles on certain surfaces in arbitrary characteristic

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Abstract. Let k be a field of arbitrary characteristic. Let S be a singular surface defined over k with multiple rational curve singularities and suppose that the Chow group of zero cycles of its normalisation $\tilde{S}$ is finite dimensional. We give numerical conditions under which the Chow group of zero cycles of S is finite dimensional.

Keywords. Fake projective planes; Bloch's conjecture; singular surfaces.

1. Introduction

This work arose out of an attempt to prove Bloch's conjecture for the fake projective planes constructed by Mumford [10] and later by Ishida and Kato [6]. An earlier (again unsuccessful) attempt to prove the same was by Barlow [1].

The strategy was to compute the Chow group of zero cycles of the special fibre of a model of the fake planes (it turns out to be isomorphic to $\mathbb{Z}$) and to use this to deduce that the generic fibre, which is the fake plane, too has Chow group of zero cycles isomorphic to $\mathbb{Z}$. Motivated by our example, we now pose (and partly answer) the following general question:

Question 1. Let S be a projective surface defined over an algebraically closed field such that $\tilde{S}$, the normalisation of S is smooth. Suppose now that $CH_0(\tilde{S})$ is finite dimensional (in a suitable sense). Then under what conditions is $CH_0(S)$ finite dimensional?

For instance, when S is defined over the complex numbers it is clear from Mumford's theorem [11] that the finite dimensionality of the Chow group of zero cycles forces that there are no non-trivial global two forms on $\tilde{S}$. So in this case, one is interested in knowing what are the conditions under which S itself has a finite dimensional Chow group.

Our main theorem in this context is the following:

Theorem 1. *Let S be a singular surface with multiple rational curve singularities such that the normalisation $\tilde{S}$ is non-singular. Further, assume that $H^i(S, \mathcal{O}_S) \cong H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})$ for $i = 1, 2$. Then if $CH_0(\tilde{S})$ is 'finite dimensional', then so is $CH_0(S)$.*

As an application we prove a consequence of Bloch's conjecture:

Theorem 2. *Let M_0 be the special fibre occurring in the construction of Mumford's fake projective plane. Then $CH_0(M_0) \cong \mathbb{Z}$.*

After this work was completed, we were informed that Gillet [4] has proved similar results but for surfaces defined over algebraically closed fields of characteristic zero. The interesting aspect of our work is that we can obtain results for fields of any characteristic. Further, we can show that the cohomological conditions stated above imply a numerical condition on the geometry of the surface which is very easy to check.

2. Preliminaries

2.1 *Definitions and generalities*

We work over a field k of arbitrary characteristic.

For any variety X, let $K_0(X)$ (resp. $G_0(X)$) be the Grothendieck group of vector bundles (resp. coherent sheaves). One knows that when X is smooth, $G_0(X)$ is isomorphic to $K_0(X)$.

DEFINITION 1.

Let S be an irreducible, reduced quasi-projective surface over any field. Then

$$F^2 K_0(S) := \{\alpha \in K_0(S) \mid \text{rank}(\alpha) = 0 = \det(\alpha)\}.$$

Theorem 3 (Bloch-Levine). *For S as above, there is a natural isomorphism*

$$CH_0(S) \cong F^2 K_0(S).$$

We now state Bloch's conjecture for surfaces, which is the main interest of this work. First some generalities: For any smooth projective variety X of dimension n, there is a cycle class map

$$CH_{n-p}(X) \xrightarrow{cl_X} H^{2p}(X),$$

where the group on the right-hand side is a suitable Weil cohomology group. Further, if we denote the kernel of this map by $CH_{n-p}(X)_{\text{hom}}$, then there is an Abel–Jacobi map for $p = n$,

$$CH_0(X)_{\text{hom}} \xrightarrow{AJ_X} \text{Alb}_X .$$

DEFINITION 2.

Let X be a projective variety. $CH_0(X)$ is said to be *finite dimensional* (see [11]) if for some $m > 0$, the map

$$\gamma_m \colon S^m(X_{\text{reg}}) \to CH_0(X)_{\text{hom}}$$

is surjective. Here X_{reg} refers to the smooth locus of the variety X and the map is the canonical one which after fixing a base point $[x_0]$ sends an element $[x]$ to the class of the difference $[x] - [x_0]$.

One can see that this is also equivalent to the statement that for some integer $m' > 0$, depending only on X, any element of $CH_0(X)_{\text{hom}}$ is represented by a 0-cycle $\sum_{i=1}^{r} \delta_i$,

where for each i, the cycle δ_i is a difference of two effective 0-cycles of degree m' supported in X_{reg}.

For a smooth projective surface S defined over an algebraically closed field k, Roitman [14] has shown that the Chow group (of zero cycles) is finite dimensional precisely when the natural map

$$CH_0(S)_{\text{hom}} \to \text{Alb}(S)$$

is an isomorphism.

Conjecture 1. (*Bloch*). Let S be a smooth projective surface defined over $\mathbb{C}$ with $p_g = 0$. Then $CH_0(S)$ is finite dimensional.

The conjecture above has been proved when S is not of general type. Further, in characteristic 0, the work of Gillet [4] and later on the work of Krishna and Srinivas [7] sheds some light on what controls the Chow group in the case when S is singular. Unfortunately, we do not know of any general conjecture for singular surfaces defined over positive characteristic.

2.2 *The surfaces of interest*

Let $\tilde{S}$ be a smooth projective surface defined over k and $Z' \subset \tilde{S}$ be a divisor. Suppose that there exists a push out diagram

$$
\begin{array}{ccc}
Z' & \to & \tilde{S} \\
\downarrow & & \\
Z & &
\end{array}
,
$$

where $Z' \to Z$ is a finite surjective morphism. Let $v \colon \tilde{S} \to S$ be the resulting morphism. Further assume that the curves Z' and Z have the following properties:

1. Z' is a union of smooth rational curves.
2. There are at most 3 components passing through any point in Z.
3. The components of Z intersect transversally.
4. Z' is a normal crossings divisor in $\tilde{S}$.

Further, let n_1 be the number of components, n_2 the number of points of two-fold intersections and n_3 the number of points of three-fold intersections in Z. Similarly, let m_1 (resp. m_2) be the number of components (resp. the number of points of two-fold intersections) in Z'.

Here, by making a base change to a larger field, we can assume without loss of generality that all points above are k-rational points.

Lemma 1. *In the situation above, Z is the subscheme defined by the conducting ideal.*

Proof. By definition we have to prove that Z is the subscheme of S defined by the $\mathscr{O}_S$-annihilator of the coherent sheaf $v_* \mathscr{O}_{\tilde{S}} / \mathscr{O}_S$. Since $\mathscr{I}_Z = v_* \mathscr{I}_{Z'}$, we have

$$v_* \mathscr{O}_{\tilde{S}} / \mathscr{O}_S = v_* \mathscr{O}_{Z'} / \mathscr{O}_Z.$$

The annihilator of the latter is clearly $\mathscr{I}_Z$ and thus we are done. (For any finitely generated module M, over a Noetherian ring A, the zero set of the annihilator $\text{Ann}(M)$ is equal to the support, $\text{supp}(M)$.) $\qquad\square$

PROPOSITION 1. (Localisation sequence) [7,13].

Let $(\tilde{S}, Z') \to (S, Z)$ be the normalisation with Z the subscheme defined by the conducting ideal and Z' its inverse image. Then there exists a commutative diagram

$$
\begin{array}{ccccccc}
\to & SK_1(Z) & \to & F^2 K_0(S,Z) & \to & F^2 K_0(S) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to & SK_1(Z') & \to & F^2 K_0(\tilde{S},Z') & \to & F^2 K_0(\tilde{S}) & \to 0
\end{array}
\tag{1}
$$

where for a curve C, $SK_1(C)$ denotes the cohomology group $H^1(C, \mathcal{K}_{2,C})$. Furthermore, in the diagram above the middle vertical arrow is an isomorphism.

COROLLARY 1.

There is an exact sequence

$$
0 \to \operatorname{coker}(SK_1(Z) \to SK_1(Z')) \to F^2 K_0(S) \to F^2 K_0(\tilde{S}) \to 0.
$$

Proof. Follows easily from diagram (1) in Proposition 1 above. □

Lemma 2. (Mayer–Vietoris sequences). For the subschemes Z and Z', there exist Mayer–Vietoris sequences such that the following diagram commutes.

$$
\begin{array}{ccccccc}
\mathcal{K}_{2,Z} & \to & \oplus_{i=1}^{n_1} \mathcal{K}_{2,Z_i} & \overset{\Phi}{\to} & \oplus_{1 \le i < j \le n_1} \mathcal{K}_{2,Z_{ij}} & \to \oplus_{1 \le i < j < k \le n_1} \mathcal{K}_{2,Z_{ijk}} \to 0 \\
\downarrow v_0^* & & \downarrow v_1^* & & \downarrow v_2^* & \\
\mathcal{K}_{2,Z'} & \to & \oplus_{r=1}^{m_1} \mathcal{K}_{2,Z'_r} & \overset{\Phi'}{\to} & \oplus_{1 \le r < s \le m_1} \mathcal{K}_{2,Z'_{rs}} & \to \qquad 0
\end{array}
\tag{2}
$$

Proof. The Mayer–Vietoris sequence for Z' (which is divisor of normal crossings in B) can be obtained as follows: The map $\mathcal{K}_{2,Z'} \to \oplus_i \mathcal{K}_{2,Z'_i}$ is the restriction map. The map $\oplus_{r=1}^{m_1} \mathcal{K}_{2,Z'_r} \to \oplus_{1 \le r < s \le m_1} \mathcal{K}_{2,Z'_{rs}}$ is given by $(\alpha_i) \mapsto (\phi_{i,j(i)}(\alpha_i) - \phi_{i,j(i)}(\alpha_{j(i)}))$ where for $i < j(i)$, $\phi_{i,j(i)}$ is the restriction map induced by the inclusion of $Z'_{ij(i)}$ in Z'_i (or $Z'_{j(i)}$). It is enough to prove surjectivity of this map at the stalks. This follows since $K_2(A)$ for a local ring A, is generated by Steinberg symbols [9] and hence the map on K_2 is surjective for surjective maps of local rings.

The extra term in the Mayer–Vietoris sequence for the scheme Z occurs because of the fact that the components do not necessarily meet transversally. There are points (corresponding to some of the two-fold intersections Z_{ij}) which occur more than once. Thus one has to apply (Theorem 6.4 of [9]) once more. Surjectivity now is obvious.

Furthermore, the maps v_1 and v_2 are injective.

For the diagram to commute, one needs to define the map Φ' carefully: Notice that since the components of Z are smooth, this means that no two rational curves which get identified in Z intersect in Z'. This implies that when we index the curves in Z', we need to index them according to the indexing in Z. For instance, for two curves above Z'_r and Z'_s, $r < s$ if and only if their images Z_i and Z_j are such that $i < j$. Once this is taken care of, it is easy to check that the diagram commutes. □

We rewrite the above diagram in a way which is suitable for cohomology computations. Let

$$
\overline{\mathcal{K}}_{2,Z} = \operatorname{Image}\left(\mathcal{K}_{2,Z} \to \oplus_{i=1}^{n_1} \mathcal{K}_{2,Z_i}\right),
$$

$$
\overline{\mathcal{K}}_{2,Z'} = \operatorname{Image}\left(\mathcal{K}_{2,Z'} \to \oplus_{r=1}^{m_1} \mathcal{K}_{2,Z'_r}\right).
$$

These give short exact sequences

$$
\begin{array}{ccccccc}
0 & \to & \overline{\mathscr{K}}_{2,Z} & \to & \oplus_{i=1}^{n_1} \mathscr{K}_{2,Z_i} & \to & Q & \to 0 \\
& & \downarrow v_0^* & & \downarrow v_1^* & & \downarrow v_2^* \\
0 & \to & \overline{\mathscr{K}}_{2,Z'} & \to & \oplus_{r=1}^{m_1} \mathscr{K}_{2,Z_r'} & \to & \oplus_{1\leq r<s\leq m_1} \mathscr{K}_{2,Z_{rs}'} & \to 0
\end{array}
\tag{3}
$$

$$
0 \to Q \to \oplus_{1\leq i<j\leq n_1} \mathscr{K}_{2,Z_{ij}} \to \oplus_{1\leq i<j<k\leq n_1} \mathscr{K}_{2,Z_{ijk}} \to 0.
\tag{4}
$$

We record a couple of useful lemmas now.

Lemma 3. For Z' above, the following hold:

1. $H^0(Z_r', \mathscr{K}_{2,Z_r'}) \cong K_2(k)$.
2. $H^0(Z_{rs}', \mathscr{K}_{2,Z_{rs}'}) \cong K_2(k)^{|Z_{rs}'|}$.
3. $H^1(Z', \overline{\mathscr{K}}_{2,Z'}) \cong H^1(Z', \mathscr{K}_{2,Z'})$.
4. $H^1(Z_r', \mathscr{K}_{2,Z_r'}) \cong k^*$.
5. $H^1(Z_{rs}', \mathscr{K}_{2,Z_{rs}'}) = 0$.

Proof. Recall that $Z_r' \cong \mathbb{P}^1$ (this is assumption (1) in § 2.2).

1. The result follows from the fact that $K_2(k[t]) = K_2(k)$.
2. Obvious.
3. $\ker(\mathscr{K}_{2,Z'} \quad \overline{\mathscr{K}}_{2,Z'})$ is supported on points.
4. Follows from the projective bundle formula and the BGQ spectral sequence.
5. Obvious. □

Associated to diagram (3), we have the following commutative diagram of cohomology long exact sequences:

$$
\begin{array}{ccccccccccc}
0 & \to & H^0(\overline{\mathscr{K}}_{2,Z}) & \to & \oplus H^0(\mathscr{K}_{2,Z_i}) & \overset{\Phi}{\to} & H^0(Q) & \to & H^1(\mathscr{K}_{2,Z}) & \to & \oplus H^1(\mathscr{K}_{2,Z_i}) & \to 0 \\
& & \downarrow \varepsilon_1 & & \downarrow \varepsilon_2 & & \downarrow \varepsilon_3 & & \downarrow \varepsilon_4 & & \downarrow \varepsilon_5 \\
0 & \to & H^1(\overline{\mathscr{K}}_{2,Z'}) & \to & \oplus H^0(\mathscr{K}_{2,Z_r'}) & \overset{\Phi'}{\to} & \oplus H^0(\mathscr{K}_{2,Z_{rs}'}) & \to & H^1(\mathscr{K}_{2,Z'}) & \to & \oplus H^1(\mathscr{K}_{2,Z_i'}) & \to 0
\end{array}
\tag{5}
$$

Lemma 4. The cohomology long exact sequence associated to the sequence (4) *yields* $H^0(Q) \cong K_2(k)^{\oplus n_2 - n_3}$.

Proof. Easily follows from the cohomology long exact sequence associated to the short exact sequence (4) and the fact that $H^1(Q) = 0$ as Q is supported on points. □

Lemma 5. Let C be a projective curve defined over k and $x \in C$ be a k-rational point. Then the restriction map $H^0(C, \mathscr{K}_{2,C}) \to H^0(\{x\}, \mathscr{K}_{2,x}) \cong K_2(k)$ is a split surjection.

Lemma 6. Let C be a smooth rational curve. Then $H^1(C, \mathscr{K}_{2,C})$ is finite dimensional.

Proof. This is immediate from the Gersten resolution for $\mathscr{K}_{2,C}$. In fact one gets a surjection $\text{Pic}(C) \otimes k^* \quad H^1(C, \mathscr{K}_{2,C})$. Since $\text{Pic}(C) \cong \mathbb{Z}$, we are done. □

Lemma 7. Suppose $CH_0(\tilde{S})$ is finite dimensional. Then $CH_0(S)$ is finite dimensional if and only if the groups $\text{Image}(\varepsilon_3)$ and $\text{Image}(\Phi')$ generate $\oplus_{1\leq r<s\leq m_1} H^0(\mathscr{K}_{2,Z_{rs}'})$ in diagram (5).

Proof. $\mathrm{coker}(\varepsilon_5)$ is a quotient of $\oplus_j H^1(\mathcal{K}_{2,Z'_j})$ and hence by Lemma 6 (and some diagram chasing) is finite dimensional. Thus the hypothesis in the statement is equivalent to the condition that $\mathrm{coker}(SK_1(Z) \xrightarrow{\varepsilon_4} SK_1(Z'))$ is finite dimensional. Corollary 1 implies that under this hypothesis $F^2K_0(S)$ is finite dimensional if and only if $F^2K_0(\tilde{S})$ is so and thus we are done. $\square$

Theorem 4. *The hypothesis in Lemma 7 holds if and only if $m_1 - m_2 \geq n_1 - n_2 + n_3$.*

Proof. By Lemma 5, diagram (5) reduces to the following:

$$
\begin{array}{ccccccccc}
0 & & 0 & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
K_2(k)^{n_1} & \xrightarrow{\Phi} & K_2(k)^{n_2-n_3} & \to & SK_1(Z) & \to & \oplus H^1(\mathcal{K}_{2,Z_i}) & \to & 0 \,. \\
\downarrow & & \downarrow{\scriptstyle \varepsilon_3} & & \downarrow{\scriptstyle \varepsilon_4} & & \downarrow{\scriptstyle \varepsilon_5} & & \\
K_2(k)^{m_1} & \xrightarrow{\Phi'} & K_2(k)^{m_2} & \to & SK_1(Z') & \to & \oplus H^1(\mathcal{K}_{2,Z'_i}) & \to & 0
\end{array}
\tag{6}
$$

Using the explicit description of the maps Φ and Φ' it is not hard to check (we explicitly check these in the example in the next section) that if $m_1 - m_2 \geq n_1 - n_2 + n_3$, then the first vertical map is an isomorphism and thus $\mathrm{coker}(SK_1(Z) \to SK_1(Z'))$ is isomorphic to $\mathrm{coker}(\varepsilon_5)$ which as mentioned above is finite dimensional. The converse is obvious.

$\square$

The proof of theorem 1 follows from the following.

PROPOSITION 2.

Suppose that the natural maps $H^1(S, \mathcal{O}_S) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}})$ and $H^2(S, \mathcal{O}_S) \to H^2(\tilde{S}, \mathcal{O}_{\tilde{S}})$ are isomorphisms. Then $m_1 - m_2 \geq n_1 - n_2 + n_3$.

Proof. Since S is the pushout of the diagram

$$
\begin{array}{ccc}
Z' & \to & \tilde{S} \\
\downarrow & & \\
Z & &
\end{array}
$$

we have seen earlier $v_* \mathcal{I}_{Z'} \cong \mathcal{I}_Z$ and therefore

$$
\frac{v_* \mathcal{O}_{\tilde{S}}}{\mathcal{O}_S} \cong \frac{v_* \mathcal{O}_{Z'}}{\mathcal{O}_Z} \,.
$$

Now consider the short exact sequence

$$
0 \to \mathcal{O}_S \to v_* \mathcal{O}_{\tilde{S}} \to \frac{v_* \mathcal{O}_{\tilde{S}}}{\mathcal{O}_S} \to 0.
$$

The associated cohomology long exact sequence yields

$$
\cdots H^1(S, \mathcal{O}_S) \to H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}) \to H^1\left(\frac{v_* \mathcal{O}_{Z'}}{\mathcal{O}_Z}\right) \to H^2(S, \mathcal{O}_S) \to H^2(S, v_* \mathcal{O}_{\tilde{S}}) \to 0.
$$

Here the right exactness follows from the fact that H^2 is an exact functor in our situation (S is a surface!!). The hypothesis implies that

$$
H^1\left(\frac{v_* \mathcal{O}_{\tilde{S}}}{\mathcal{O}_S}\right) \cong H^1\left(\frac{v_* \mathcal{O}_{Z'}}{\mathcal{O}_Z}\right) = 0.
$$

The sequence

$$0 \to \mathscr{O}_Z \to \nu_* \mathscr{O}_{Z'} \to \frac{\nu_* \mathscr{O}_{Z'}}{\mathscr{O}_Z} \to 0$$

yields a surjection

$$H^1(\mathscr{O}_Z) \to H^1(\nu_* \mathscr{O}_{Z'}).$$

To obtain the numerical condition, we consider the Mayer–Vietoris sequences for the sheaves $\nu_* \mathscr{O}_{Z'}$ and $\mathscr{O}_Z$. We imitate the computations done for comparing the sheaves $\mathscr{K}_2$. For this, put

$$Q' := \mathrm{coker}\{\mathscr{O}_Z \to \oplus_{i=1}^{n_1} \mathscr{O}_{Z_i}\}$$

to get

$$
\begin{array}{ccccccc}
0 \to & \mathscr{O}_Z & \to & \oplus_{i=1}^{n_1} \mathscr{O}_{Z_i} & \to & Q' & \to 0 \\
& \downarrow \nu_0^* & & \downarrow \nu_1^* & & \downarrow \nu_2^* & \\
0 \to & \mathscr{O}_{Z'} & \to & \oplus_{1 \le i \le m_1} \mathscr{O}_{Z_i'} & \to & \oplus_{1 \le i < j \le n_1} \mathscr{O}_{Z_{ij}'} & \to 0 \\
\\
0 \to & Q' & \to & \oplus_{1 \le i < j < n_1} \mathscr{O}_{Z_{ij}} & \to & \oplus_{1 \le i < j < k < n_1} \mathscr{O}_{Z_{ijk}} & \to 0
\end{array}
$$

Imitating the proof of theorem 4 the result now follows by noting that $H^0(Z, \mathscr{O}_Z) \cong H^0(Z', \mathscr{O}_{Z'}) \cong k$ and using the snake lemma to get a short exact sequence

$$0 \to \ker\{H^1(\mathscr{O}_Z) \to H^1(\nu_* \mathscr{O}_{Z'})\} \to k^{m_1 - n_1} \to k^{(m_2 - n_2 + n_3)} \to 0.$$

This implies that $(m_1 - m_2) \ge (n_1 - n_2 + n_3)$. $\square$

Remark 1. In the case when $k = \mathbb{C}$, we note that if the conditions of Proposition 2 are satisfied then $h^2(\mathscr{O}_S)$ is zero if and only if $h^2(\mathscr{O}_{\tilde{S}})$ is zero. Assuming Bloch's conjecture, we get $F^2 K_0(\tilde{S})$ is finite dimensional and thus by Lemma 7, $F^2 K_0(S)$, is also finite dimensional.

2.3 *Representability questions*

This section is inspired by Theorem B in [4]. By a theorem of Roitman [14], it is known that if the Chow group of 0-cycles of a smooth projective surface over an algebraically closed field k is finite dimensional then it is representable. In other words, there is an abelian variety, the Albanese of the surface whose geometric points 'represent' the degree 0, 0 cycles i.e., the (Albanese) map

$$CH_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$$

is an isomorphism.

Theorem B in [8] says that in the case of complex projective surfaces whose normalisation is smooth there is an isogeny

$$CH_0(S)_{\mathrm{hom}} \to J^2(S),$$

where now S is as above and $J^2(S)$ is the 'naive' Albanese i.e.,

$$J^2(S) := H^3(S, \mathbb{C})/F^2 H^3(X, \mathbb{C}) + H^3(S, \mathbb{Z}),$$

where $F^2 H^3$ is the Hodge filtration defined by Deligne.

Recently Esnault, Srinivas and Viehweg [3] have proved the following:

Theorem 5. *Let X be a projective variety of dimension n, defined over an algebraically closed field k.*

1. *There exists a smooth connected commutative algebraic group $A^n(X)$, together with a regular homomorphism ϕ: $CH^n(X)_{\mathrm{hom}} \to A^n(X)$, such that ϕ is universal among regular homomorphisms from the group $CH^n(X)_{\mathrm{hom}}$ to smooth commutative algebraic groups.*
2. *Over a universal domain k the Chow group is finite dimensional precisely when ϕ is an isomorphism.*
3. *$A^n(X \times_k K) = A^n(X) \times_k K$, for all algebraically closed fields K containing k.*

The following now follows almost tautologically from the theorems above.

Theorem 6. *Let S be a projective surface such that the normalisation $(\tilde{S}, Z') \to (S, Z)$ satisfies the conditions of Proposition 2 above. Then if $CH_0(\tilde{S})_{\mathrm{hom}}$ is finite dimensional (i.e., representable by $\mathrm{Alb}(\tilde{S})$), then $CH_0(S)_{\mathrm{hom}}$ is representable by $A^2(S)$, the generalised Albanese of Esnault, Srinivas and Viehweg.*

Proof. By Theorem 4, it follows that $CH_0(S)_{\mathrm{hom}}$ (which is $\cong F^2 K_0(S)$) is finite dimensional. By Theorem 5 this implies that this group is isomorphic to the generalised Albanese $A^2(S)$. □

3. An application: Bloch's conjecture for the special fibre

3.1 *A brief description of the fake plane*

Let $R = \mathbb{Z}_2$ denote the ring of 2-adic integers with quotient field $K = \mathbb{Q}_2$ and finite residue field $k = \mathbb{F}_2$ of characteristic 2. We describe here the fake plane constructed by Mumford (henceforth denoted MFP) and refer the reader to [6] for the other fake planes.

Mumford makes use of the method of *p*-adic uniformisation introduced by Mustafin and Kurihara (see [10] and the references therein) to construct a formal scheme which is of finite type over $\mathrm{Spec}(R)$. Take $\mathbb{P}^2_R$ and successively blow up all the *k*-rational lines and *k*-rational points in the special fibre. Let U be the union of the generic fibre $\mathbb{P}^2_K$ and a sufficiently small open neighbourhood of the proper transform of $\mathbb{P}^2_k$ in the blown-up scheme above. For each $\alpha \in PGL(3, K)$, we denote by U^α, the R-scheme such that the generic fibre is equal to $\mathbb{P}^2_K$ and that there exists an isomorphism $U \xrightarrow{\cong} U^\alpha$ which induces the natural action of α on the generic fibre. Then the union of all U^α above is patched together to get a regular scheme X. By construction, the action of $PGL(3, K)$ extends to X. Let $\mathscr{X}$ be the completion of the resulting scheme along the special fibre. For a certain discrete torsion-free co-compact group $\Gamma \subset PGL(3, K)$, he is then able to construct a quotient subscheme $\mathscr{X}/\Gamma$. Mumford (*op. cit.*) then shows that the canonical sheaf $\omega_{\mathscr{X}}$ is ample and descends to $\mathscr{X}/\Gamma$ and that the latter can be algebraized to a projective variety denoted by M. For this choice of the subgroup Γ, he shows that $\mathscr{X}/\Gamma$ has generic fibre, a smooth surface of general type with the same Betti numbers as that of the projective plane which we refer to as MFP. Further, the special fibre is an irreducible rational surface over k whose normalisation is $\mathbb{P}^2_k$ blown-up at the 7 rational points.

Kato and Ishida [6] use results on discrete groups by Cartwright *et al* [2] to construct two more fake projective planes.

3.2 *The configurations of lines*

Below we give a description of the special fibre M_0 of the fake planes. As noted above, the normalisation of the special fibre, which we denote by B is $\mathbb{P}^2_k$ blown up at its 7 rational points. The first and second tables give the configuration of all rational curves in B and the identifications that give the configuration of lines in the special fibre M_0 respectively and the last two tables give the configuration of lines in the fake planes constructed by Kato and Ishida. We will note here that while in the former one of the rational double lines contains a node as a singularity, the latter have no singular lines and the situation is as in the previous situation.

Let $E(lmn)$ be the exceptional divisor over the point $[l : m : n] \in \mathbb{P}^2_k$ and $C(lmn)$ be the proper transform of the line given by $lx + my + nz = 0$. We shall denote by Z'_i and Z_j respectively the rational curves in B and M_0. Further, the rational points in M_0 shall be denoted by a, b etc. and their preimages in B shall also be denoted by the same alphabet.

Tables 1 and 2 give the intersection data of the lines in the configurations Z' and Z.

Table 1. Intersection data in the normalisation Z'.

	$C(110)$ $= Z'_8$	$C(100)$ $= Z'_9$	$C(010)$ $= Z'_{10}$	$C(001)$ $= Z'_{11}$	$C(101)$ $= Z'_{12}$	$C(011)$ $= Z'_{13}$	$C(111)$ $= Z'_{14}$
$E(001)$ $= Z'_1$	a	b	a				
$E(100)$ $= Z'_2$			d	b		g	
$E(110)$ $= Z'_3$	a			c			d
$E(111)$ $= Z'_4$	b				e	c	
$E(011)$ $= Z'_5$		g				f	e
$E(101)$ $= Z'_6$			c		g		f
$E(010)$ $= Z'_7$		d		e	f		

Table 2. Intersection data in the special fibre Z.

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
Z_1			e	g	f, g	e, f	
Z_2			c	d	c	d	a, a
Z_3	e	c		b	c	e	b
Z_4	g	d	b		g	d	b
Z_5	f, g	c	c	g		f	
Z_6	f, e	d	e	d	f		
Z_7		a, a	b	b			a

Note that every exceptional curve meets a proper transform of a rational line in Z in each of its rational points (3 in number) and similarly a proper transform meets an exceptional line in each of its rational points (again 3). Further no two exceptional or proper transform intersect.

In the above one can easily check that Z_i' is identified with Z_{i+7}' to obtain the configuration Z.

The following are the intersection tables (tables 3, 4) of the lines in the special fibres of fake planes constructed by Kato–Ishida.

3.3 *Precise statement of the conjecture*

Let F be a universal domain containing k, W the Witt vectors of F and let M denote the base change of the original M (over $\mathbb{Z}_2$) to W. Let M_0 and M_η be the special fibre and the generic fibre of M defined over F (which is now an infinite extension of k) and Ω (which denotes the quotient field of W) respectively. Let $B \to M_0$ be the normalisation. One knows from above that B is isomorphic to the base change to F of $\mathbb{P}_k^2$ blown up at all k-rational points.

By Riemann–Roch one has $G_0(M_0) \otimes \mathbb{Q} \cong \oplus CH_i(M_0) \otimes \mathbb{Q}$. Since $B \to M_0$ is a finite map, one has a surjection $G_0(B) \otimes \mathbb{Q} \to G_0(M_0) \otimes \mathbb{Q}$. Since the first group is finite rank $\mathbb{Q}$-vector space (for instance, by the Grothendieck–Riemann–Roch theorem) so is the second.

Table 3. .

	Z_8'	Z_9'	Z_{10}'	Z_{11}'	Z_{12}'	Z_{13}'	Z_{14}'
Z_1'			f		e		g
Z_2'				d		c	b
Z_3'	g			a	f		
Z_4'		c			a		d
Z_5'	f		a			e	
Z_6'	e	b		c			
Z_7'		d	g			b	

Table 4. .

	Z_8'	Z_9'	Z_{10}'	Z_{11}'	Z_{12}'	Z_{13}'	Z_{14}'
Z_1'			e		g		f
Z_2'		b	b			c	
Z_3'		b		a	e		
Z_4'	g				a	d	
Z_5'	e		a	g			
Z_6'				d		d	c
Z_7'	f	c					f

We note the following important consequence of Bloch's conjecture for fake projective planes.

Lemma 8. *If Bloch's conjecture is true for the fake planes, then the group $F^2K_0(M_0) \otimes \mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-vector space.*

Proof. Consider the following diagram:

$$
\begin{array}{ccccccc}
G_0(M_0) \otimes \mathbb{Q} & \to & G_0(M) \otimes \mathbb{Q} & \to & G_0(M_\eta) \otimes \mathbb{Q} & \to & 0 \\
& & \downarrow & & & & \\
& & K_0(M_0) \otimes \mathbb{Q} & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}
\qquad .
$$

Here the horizontal sequence is the localisation theorem for the K-theory of (coherent sheaves on) M and the vertical map is the restriction of K-groups. (Since M is regular, in this case we have $G_0(M) \cong K_0(M)$.) The surjectivity of the vertical map follows from the fact that any line bundle on M_0 lifts since $p_g = 0 = q$ for M_0 and since any element of $K_0(M_0)$ is equivalent to a sum of line bundles. If Bloch's conjecture holds for M_η then this implies that $G_0(M_\eta) \otimes \mathbb{Q} (\cong \oplus CH_i(M_\eta) \otimes \mathbb{Q}) \cong \mathbb{Q}^3$. Further from the previous paragraph, we know that $G_0(M_0) \otimes \mathbb{Q}$ is of finite rank. □

3.4 *The Chow group of the special fibre*

In this section, we compute the Chow group of the special fibre M_0 of the fake projective planes. Ishida [5] and Kato–Ishida [6] have explicitly described the identifications of the seven pairs of lines $C(a,b,c)$ and $E(a,b,c)$ of the normalisation B. We use these identifications in an essential way in our computation of the Chow group $F^2K_0(M_0)$. We compute the above group in the case when M is Mumford's fake projective plane and leave the fake projective planes of Kato and Ishida as an exercise to the reader.

Let $Z = \cup_{i=1}^7 Z_i$ be the union of the F-rational lines Z_i in the special fibre M_0 and $Z_i' = \cup_{i=1}^{14} Z_i'$ be the union of the 7 pairs of F-rational lines in the normalisation B of M_0 (namely the 7 exceptional divisors and the 7 proper transforms of the lines in $\mathbb{P}_F^2$).

From the previous section we have for the map $\nu_0 \colon Z' \to Z$ (the restriction of the normalisation $\nu \colon B \to M_0$), a diagram of $\mathscr{K}$-sheaves:

$$
\begin{array}{ccccccc}
\mathscr{K}_{2,Z} & \to & \oplus_{i=1}^7 \mathscr{K}_{2,Z_i} & \to & \oplus_{1 \le i < j \le 7} \mathscr{K}_{2,Z_{ij}} & \to & \oplus_{1 \le i < j < k \le 7} \mathscr{K}_{2,Z_{ijk}} \to 0 \\
\downarrow \nu_0^* & & \downarrow \nu_1^* & & \downarrow \nu_2^* & & \\
\mathscr{K}_{2,Z'} & \to & \oplus_{r=1}^{14} \mathscr{K}_{2,Z_r'} & \to & \oplus_{1 \le r < s \le 14} \mathscr{K}_{2,Z_{rs}'} & \to & 0
\end{array}
\tag{7}
$$

Using the intersection table, it is easy to check that the above diagram is indeed commutative. From the earlier section, we have short exact sequences

$$
\begin{array}{ccccccc}
0 & \to & \overline{\mathscr{K}}_{2,Z} & \to & \oplus_{i=1}^7 \mathscr{K}_{2,Z_i} & \to & Q & \to 0 \\
& & \downarrow \nu_0^* & & \downarrow \nu_1^* & & \downarrow \nu_2^* \\
0 & \to & \overline{\mathscr{K}}_{2,Z'} & \to & \oplus_{r=1}^{14} \mathscr{K}_{2,Z_r'} & \to & \oplus_{1 \le r < s \le 14} \mathscr{K}_{2,Z_{rs}'} \to 0
\end{array}
\tag{8}
$$

$$
0 \to Q \to \oplus_{1 \le i < j \le 7} \mathscr{K}_{2,Z_{ij}} \to \oplus_{1 \le i < j < k \le 7} \mathscr{K}_{2,Z_{ijk}} \to 0.
\tag{9}
$$

Similar statements as in Lemma 3 hold for the variety Z but in this case one needs the following lemma.

Lemma 9. *Let C be a nodal rational curve. Then there are surjections* $H^i(\mathcal{K}_{2,C}) \rightarrow H^i(\mathcal{K}_{2,\mathbb{P}^1})$ *for* $i = 0, 1$.

Proof. The desingularisation $\mathbb{P}^1 \rightarrow C$ induces a sequence

$$0 \rightarrow K \rightarrow \mathcal{K}_{2,C} \rightarrow \mathcal{K}_{2,\mathbb{P}^1} \rightarrow L \rightarrow 0.$$

Here the extreme two terms are supported on the singular point. The cohomology sequence associated to this yields a sequence

$$0 \rightarrow H^0(K) \rightarrow H^0(\mathcal{K}_{2,C}) \xrightarrow{f} H^0(\mathcal{K}_{2,\mathbb{P}^1}) \rightarrow H^0(L) \rightarrow H^1(\mathcal{K}_{2,C}) \rightarrow H^1(\mathcal{K}_{2,\mathbb{P}^1}) \rightarrow 0.$$

Since C has a point, this implies that the map f splits. Thus we see that the maps $H^i(\mathcal{K}_{2,C}) \rightarrow H^i(\mathcal{K}_{2,\mathbb{P}^1})$ are surjective for $i = 0, 1$.   □

Consider the exact sequence (9). Taking the long exact sequence of cohomology one gets

$$0 \rightarrow H^0(Q) \rightarrow \oplus_{1 \leq i < j \leq 7} H^0(\mathcal{K}_{2,Z_{ij}}) \rightarrow \oplus_{1 \leq i < j < k \leq 7} H^0(\mathcal{K}_{2,Z_{ijk}})$$

$$\rightarrow H^1(Q) \rightarrow \oplus_{1 \leq i < j \leq 7} H^1(\mathcal{K}_{2,Z_{ij}}) \rightarrow 0.$$

Since Q is supported on points, $H^1(Q)$ is zero. Using the previous lemma, one gets

$$0 \rightarrow H^0(Q) \rightarrow K_2(F)^{\oplus 20} \rightarrow K_2(F)^{\oplus 6} \rightarrow 0.$$

This implies that $H^0(Q) \cong K_2(F)^{\oplus 14}$.

Lemma 10. $H^0(Z, \overline{\mathcal{K}}_{2,Z}) \cong H^0(Z', \overline{\mathcal{K}}_{2,Z'}) \cong K_2(F)$.

Proof. That $H^0(Z', \overline{\mathcal{K}}_{2,Z'}) \cong K_2(F)$ follows from an inspection of the cohomology long exact sequence of the Mayer–Vietoris sequence of Z'.

Note that the map

$$\oplus_{1 \leq r \leq 14} H^0(\mathcal{K}_{2,Z'_r}) \rightarrow \oplus_{1 \leq r < s \leq 14} H^0(\mathcal{K}_{2,Z'_{rs}})$$

is abstractly the map $K_2(F)^{\oplus 14} \rightarrow K_2(F)^{\oplus 21}$ which takes the rth component of an element $(\underline{\alpha}) \in K_2(F)^{\oplus 14}$, $\alpha_r \mapsto (\beta_{ij})$ where $\beta_{rs} = \alpha_r - \alpha_s$ for s such that Z'_s intersects Z'_r and $\beta_{ij} = 0$ otherwise. Following table 1, it is easy to see that the kernel of this map is isomorphic to $K_2(F)$.

To see that a similar statement holds true for Z, one observes that the map $H^0(Q) \rightarrow \oplus_{1 \leq r < s \leq 14} H^0(\mathcal{K}_{2,Z_{rs}})$ is injective. Using the identifications that follow from tables 1 and 2 and the explicit description of maps in the Mayer–Vietoris sequences, one can then check that the kernel of the composite map

$$K_2(F)^{\oplus 7} \rightarrow K_2(F)^{\oplus 14} \rightarrow K_2(F)^{\oplus 21}$$

is isomorphic to $K_2(F)$.

This implies that $H^0(Z, \overline{\mathcal{K}}_{2,Z}) \cong K_2(F)$.   □

Taking the long exact sequence of cohomology for the diagram (8) one gets

$$\begin{array}{ccccccccc}
0 \rightarrow & K_2(F) & \rightarrow & K_2(F)^{\oplus 7} & \rightarrow & K_2(F)^{\oplus 14} & \rightarrow & SK_1(Z) & \rightarrow & (F^*)^{\oplus 7} & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow & K_2(F) & \rightarrow & K_2(F)^{\oplus 14} & \rightarrow & K_2(F)^{\oplus 21} & \rightarrow & SK_1(Z') & \rightarrow & (F^*)^{\oplus 14} & \rightarrow 0
\end{array} \quad (10)$$

Remark 2. Here one of the components say Z_0 is a nodal curve. We replace the group $H^0(\mathscr{K}_{2,Z_0})$ by its image in $K_2(F)$. By Lemma 9, this is $K_2(F)$.

Using the earlier computations we get a commutative diagram

$$
\begin{array}{ccccccc}
0 \to & K_2(F)^{\oplus 8} & \to & SK_1(Z) & \to & (F^*)^{\oplus 7} & \to 0 \\
& \downarrow \cong & & \downarrow & & \downarrow & \\
0 \to & K_2(F)^{\oplus 8} & \to & SK_1(Z') & \to & (F^*)^{\oplus 14} & \to 0
\end{array}
. \tag{11}
$$

Lemma 11. *$SK_1(Z')$ is generated by $SK_1(Z)$ and $SK_1(B)$. In particular,* coker$(SK_1(Z) \to SK_1(Z'))$ *is finite dimensional.*

Proof. To understand the image of the map $SK_1(B) \to SK_1(Z')$ we consider the following diagram:

$$
\begin{array}{ccc}
SK_1(B) & H^1(B, \mathscr{K}_{2,B}) & \mathrm{Pic}(B) \otimes K_1(F) \\
& \downarrow & \\
& SK_1(Z') &
\end{array}
.
$$

Thus the Image$(SK_1(B) \to SK_1(Z')) =$ Image$(\mathrm{Pic}(B) \otimes K_1(F) \to SK_1(Z'))$. From diagram (5), it is clear that all we need to check is that $\oplus_r H^1(\mathscr{K}_{2,Z'_r})$ is generated by the images of $\oplus_i H^1(\mathscr{K}_{2,Z_i})$ and $SK_1(B)$. Abstractly this is just saying that a free F^*-module with generators $\{z_r\}_{r=1}^{14}$ is generated by the elements $\{z_r + z_{r+7}\}_{r=1}^{7}$ and $\{z_r\}_{r=1}^{7}$ which is indeed obvious. $\qquad\square$

Lemma 12. *In the localisation sequence* (*diagram* (1)) *for the map $B \to M_0$, the vertical map $F^2K_0(M_0, Z) \to F^2K_0(B, Z')$ is an isomorphism.*

Proof. Since $v: B \to M_0$ is the normalisation, the above statement follows from Proposition 1, if one checks that $Z \subset M_0$ is the subscheme defined by the conducting subscheme for the normalisation morphism v. Since M_0 is obtained as a pushout of the diagram

$$
\begin{array}{ccc}
Z' & \hookrightarrow & B \\
\downarrow & & \\
Z & &
\end{array}
,
$$

this follows from the same argument as in Lemma 1. $\qquad\square$

Now we come to the main result of this section.

Theorem 7. *The Chow group $F^2K_0(M_0)$ is isomorphic to $\mathbb{Z}$.*

Proof. The result follows by a trivial diagram chase in the localisation sequence below:

$$
\begin{array}{ccccccc}
SK_1(Z) & \to & F^2K_0(M_0, Z) & \to & F^2K_0(M_0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
SK_1(Z') & \to & F^2K_0(B, Z') & \to & F^2K_0(B) & \to & 0
\end{array}
. \tag{12}
$$

By Lemma 11, coker$(SK_1(Z) \to SK_1(Z'))$ is finite dimensional. Moreover the middle vertical arrow is an isomorphism by Lemma 12. A simple diagram chase will then show that $F^2K_0(M_0)$ is isomorphic to $F^2K_0(B)$ up to a finitely generated free F^*-module $(= \mathrm{coker}(SK_1(Z) \to SK_1(Z')))$. Since B is a smooth rational surface, $F^2K_0(B) \cong \mathbb{Z}$. Thus $F^2K_0(M_0)$ being a finitely generated abelian group, it is finite dimensional. Now by Theorem 5, the Albanese map is an isomorphism. Since this generalised Albanese is connected this implies that $F^2K_0(M_0) \cong \mathbb{Z}$. $\qquad\square$

Remark 3. We direct the attention of the reader interested in further examples to the paper of Gillet [4] mentioned in the Introduction.

Acknowledgements

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