

## THE GROTHENDIECK-LEFSCHETZ THEOREM FOR NORMAL PROJECTIVE VARIETIES

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### Abstract

We prove that for a normal projective variety  $X$  in characteristic 0, and a base-point free ample line bundle  $L$  on it, the restriction map of divisor class groups  $\text{Cl}(X) \rightarrow \text{Cl}(Y)$  is an isomorphism for a general member  $Y \in |L|$  provided that  $\dim X \geq 4$ . This is a generalization of the Grothendieck-Lefschetz theorem, for divisor class groups of singular varieties.

We work over  $k$ , an algebraically closed field of characteristic 0.

Let  $X$  be a smooth projective variety over  $k$  and  $Y$  a smooth complete intersection subvariety of  $X$ . The Grothendieck-Lefschetz theorem states that if  $\dim Y \geq 3$ , the Picard groups of  $X$  and  $Y$  are isomorphic.

In this paper, we wish to prove an analogous statement for singular varieties, with the Picard group replaced by the divisor class group.

Let  $X$  be an irreducible projective variety which is regular in codimension 1 (for example,  $X$  may be irreducible and normal). Recall that for such  $X$ , the divisor class group  $\text{Cl}(X)$  is defined as the group of linear equivalence classes of Weil divisors on  $X$  (see [10], II, §6). If  $\dim X = d$ , then  $\text{Cl}(X)$  coincides with the Chow group  $CH_{d-1}(X)$  as defined in Fulton's book [7]. If  $Y \subset X$  is an irreducible Cartier divisor, which is also regular in codimension 1, there is a well-defined restriction homomorphism<sup>1</sup>

$$\text{Cl}(X) \rightarrow \text{Cl}(Y),$$

determined by the rule

$$[D] \mapsto [D \cap Y],$$

where  $D$  is any irreducible Weil divisor in  $X$  distinct from  $Y$ , and  $[D \cap Y]$  denotes the Weil divisor on  $Y$  associated to the intersection scheme  $D \cap Y$ .

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<sup>1</sup>The terminology is from the non-singular case, where one is considering restriction of line bundles.

This may be viewed as a particular case of the refined Gysin homomorphism  $CH_i(X) \rightarrow CH_{i-1}(Y)$  defined in [7], for  $i = \dim X - 1$ .

Now let  $X$  be an irreducible projective variety over  $k$ , regular in codimension 1, and let  $\mathcal{L}$  be an ample line bundle over  $X$ , together with a linear subspace  $V \subset H^0(X, \mathcal{L})$  which gives a base-point free ample linear system  $|V|$  on  $X$ . Let  $Y \in |V|$  be a general element of this linear system; by Bertini's theorem, we have  $Y_{sing} = Y \cap X_{sing}$ . In this context, our main result is the following, which is an analogue of the Grothendieck-Lefschetz theorem.

**Theorem 1.** *In the above situation, for a dense Zariski open set of  $Y \in |V|$ , the restriction map*

$$\mathrm{Cl}(X) \rightarrow \mathrm{Cl}(Y)$$

*is an isomorphism, if  $\dim X \geq 4$ , and is injective, with finitely generated cokernel, if  $\dim X = 3$ .*

Our proof is purely algebraic, in the style of the proof of the Grothendieck-Lefschetz theorem given in [11], Chapter IV. The above result has an application in the theory of Deligne's 1-motives (see [4]), which is discussed in §4 below; for this, it is of interest to have such an algebraic proof. In an appendix, we also sketch a different, transcendental proof of the theorem, when  $k = \mathbb{C}$ , due to N. Fakhruddin, using results from stratified Morse theory, and properties of the weight filtration on cohomology.

### 1. The Grothendieck-Lefschetz theorem for big linear systems

In the situation of Theorem 1, if  $\tilde{X} \xrightarrow{\pi} X$  is a desingularisation of  $X$ , we have the following (Cartesian) diagram:

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

Note that  $\tilde{Y}$  is a general member of the pull-back linear system  $\pi^*V$  on the smooth proper variety  $\tilde{X}$ , and therefore is smooth, by Bertini's theorem; hence  $\tilde{Y} \rightarrow Y$  is a desingularisation of  $Y$ . If  $X$  is singular, then  $\tilde{Y}$  is a general member of the linear system determined by  $\pi^*V \subset H^0(\tilde{X}, \pi^*\mathcal{L})$  where  $\pi^*\mathcal{L}$  is not ample, but is *big and base-point free*.

Let  $E_X = \pi^{-1}(X_{sing}) \subset \tilde{X}$  be the exceptional locus. Then  $E_Y = E_X \cap \tilde{Y}$  is the exceptional locus for  $\tilde{Y} \rightarrow Y$ . We have natural isomorphisms

$$\begin{aligned} \mathrm{Cl}(X) &\cong \mathrm{Cl}(X \setminus X_{sing}) \cong \mathrm{Pic}(X \setminus X_{sing}) \cong \mathrm{Pic}(\tilde{X} \setminus E_X), \\ \mathrm{Cl}(Y) &\cong \mathrm{Cl}(Y \setminus Y_{sing}) \cong \mathrm{Pic}(Y \setminus Y_{sing}) \cong \mathrm{Pic}(\tilde{Y} \setminus E_Y). \end{aligned}$$

This is because (i) the divisor class group is unchanged upon removal of a closed subset of codimension  $\geq 2$ , and (ii) the divisor class group coincides with the Picard group, for non-singular varieties. Thus, Theorem 1 may be viewed as an assertion about the natural restriction homomorphism

$$\text{Pic}(\tilde{X} \setminus E_X) \rightarrow \text{Pic}(\tilde{Y} \setminus E_Y).$$

Also, by a standard argument (repeated below),  $\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{X} \setminus E_X)$  is surjective, with kernel isomorphic to the free abelian group on the irreducible divisorial components of  $E_X$ , and a similar assertion holds for  $\text{Pic}(\tilde{Y}) \rightarrow \text{Pic}(\tilde{Y} \setminus E_Y)$ . Indeed, since  $\tilde{X}, \tilde{Y}$  are non-singular, any line bundle on any Zariski open subset extends to a line bundle on the variety, so the two restriction maps are surjective, with kernel given by the line bundles associated to divisors with support in  $E_X$  and  $E_Y$  respectively. However, if  $E$  is any non-zero divisor on  $\tilde{X}$  with support in  $E_X$ , then  $\mathcal{O}_{\tilde{X}}(E)$  is a non-trivial line bundle: if not, we would have a non-constant rational function  $f$  on  $\tilde{X}$  with divisor  $E$ ; then  $f$  determines a non-zero regular function on  $X \setminus X_{\text{sing}}$ , which must extend to a regular function on the normalization  $X_n$  of  $X$ . But  $X_n$  is an irreducible projective variety, so any global regular function on it is constant, which is a contradiction. This argument applies to  $\tilde{Y}$  as well.

Thus, Theorem 1 is a consequence of the following version of the Grothendieck-Lefschetz theorem for a big and base-point free linear system, describing the kernel and cokernel of the restriction map on Picard groups, for the inclusion  $\tilde{Y} \hookrightarrow \tilde{X}$  of a general member of the linear system. The statement is perhaps a bit technical, but there is an obvious geometric motivation for the conditions stated.

**Theorem 2.** *Let  $\tilde{X}$  be a non-singular projective  $k$ -variety,  $\mathcal{M}$  a big invertible sheaf,  $V \subset H^0(\tilde{X}, \mathcal{M})$  a  $k$ -subspace giving a base-point free linear system on  $\tilde{X}$ . Let  $\varphi : \tilde{X} \rightarrow \mathbb{P}_k^N$  be the morphism determined by  $|V|$ , and  $\tilde{X} \xrightarrow{\pi} X \rightarrow \mathbb{P}_k^N$  be the Stein factorization of  $\varphi$ . Suppose  $\dim \tilde{X} \geq 3$ . Then for a Zariski open subset of divisors  $\tilde{Y} \in |V|$ , the restriction map*

$$\rho : \text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})$$

*has the following properties:*

- (a)  $\rho$  has kernel (freely) generated by the classes of the irreducible divisors  $E \subset \tilde{X}$  with  $\dim \pi(E) = 0$ , and  $\rho$  has a finitely generated cokernel.
- (b) If  $F$  is a divisor on  $\tilde{Y}$  supported in  $E_Y$ , such that  $\mathcal{O}_{\tilde{Y}}(F) \in \text{image Pic}(\tilde{X})$ , then there is a divisor  $E$  on  $\tilde{X}$  supported in  $E_X$  with  $\mathcal{O}_{\tilde{X}}(E) \otimes \mathcal{O}_{\tilde{Y}} \cong \mathcal{O}_{\tilde{Y}}(F)$ .
- (c) If  $\dim \tilde{X} \geq 4$ , then the classes of  $\mathcal{O}_{\tilde{Y}}(E)$ , with  $E$  supported in  $E_Y$ , generate  $\text{Coker}(\rho)$ .

**2. Some lemmas on the vanishing of cohomology**

Next, we collect together some technical lemmas used in the proof of Theorem 2.

We first state a lemma due to Grothendieck (see [9, 13]).

**Lemma 2.1** (Artin-Rees formula). *Let  $f : V \rightarrow W$  be a proper morphism of Noetherian schemes,  $\mathcal{F}$  a coherent sheaf on  $V$ ,  $\mathcal{I} \subset \mathcal{O}_W$  a coherent ideal sheaf, and  $\mathcal{J} = f^{-1}\mathcal{I}\mathcal{O}_V$  the inverse image ideal sheaf on  $V$ . There exists  $n_0 \geq 0$  such that for all  $n \geq n_0$ , the natural map*

$$\mathcal{I}^{n-n_0} \otimes \mathbb{R}^i f_*(\mathcal{J}^{n_0} \mathcal{F}) \rightarrow \mathbb{R}^i f_*(\mathcal{J}^n \mathcal{F})$$

is a surjection.

Here,  $\mathcal{J}^n \mathcal{F}$  denotes the image of the multiplication map  $\mathcal{J}^n \otimes \mathcal{F} \rightarrow \mathcal{F}$ .

**Lemma 2.2.** *Let  $\widetilde{W} \xrightarrow{\pi} W$  be a proper surjective morphism, where  $\widetilde{W}$  is an irreducible non-singular variety of dimension  $d$ , and  $\dim W \geq 2$ . Let  $\mathcal{F}$  be a coherent sheaf on  $\widetilde{W}$ , with  $\mathbb{R}^{d-1} \pi_* \mathcal{F} \neq 0$ . Then there exists an effective divisor  $E \subset \widetilde{W}$  whose support has 0-dimensional image under  $\pi$ , such that  $\mathbb{R}^{d-1} \pi_* \mathcal{F}(-E) \rightarrow \mathbb{R}^{d-1} \pi_* \mathcal{F}$  is the zero map.*

*Proof.* Let  $S \subset W$  be the support of  $\mathbb{R}^{d-1} \pi_* \mathcal{F}$ . Then  $S$  consists of points  $w \in W$  with  $\dim \pi^{-1}(w) \geq d - 1$ , and under our hypotheses, this forces  $\dim S = 0$ .

Let  $\mathcal{I} \subset \mathcal{O}_W$  be the ideal sheaf of  $S$ , and let

$$\mathcal{J} = \text{image} (\pi^* \mathcal{I} \rightarrow \mathcal{O}_{\widetilde{W}})$$

be the inverse image ideal sheaf in  $\mathcal{O}_{\widetilde{W}}$ . Lemma 2.1 above implies that there exists an  $m_0 \geq 0$  such that the map

$$\mathcal{I}^{m-m_0} \otimes \mathbb{R}^{d-1} \pi_*(\mathcal{J}^{m_0} \mathcal{F}) \rightarrow \mathbb{R}^{d-1} \pi_*(\mathcal{J}^m \mathcal{F})$$

is a surjection.

We claim that, for large enough  $m$ , the map  $\mathbb{R}^{d-1} \pi_*(\mathcal{J}^m \mathcal{F}) \rightarrow \mathbb{R}^{d-1} \pi_*(\mathcal{F})$  is the zero map. This is because by Lemma 2.1, we have a diagram

$$\begin{array}{ccc} \mathcal{I}^{m-m_0} \otimes \mathbb{R}^{d-1} \pi_*(\mathcal{J}^{m_0} \mathcal{F}) & \twoheadrightarrow & \mathbb{R}^{d-1} \pi_*(\mathcal{J}^m \mathcal{F}) \\ \downarrow & & \downarrow \\ \mathcal{I}^{m-m_0} \otimes \mathbb{R}^{d-1} \pi_*(\mathcal{F}) & \longrightarrow & \mathbb{R}^{d-1} \pi_*(\mathcal{F}) \end{array}$$

where the top horizontal arrow is surjective. The bottom horizontal map is 0, if  $m$  is large enough, since  $\mathcal{I}$  is the ideal defining the support of  $\mathbb{R}^{d-1} \pi_*(\mathcal{F})$ ; hence the right vertical arrow is 0.

Since  $\widetilde{W}$  is non-singular, there exists an effective (Cartier) divisor  $E_0$  in  $\widetilde{W}$  such that  $\mathcal{J} = \mathcal{O}_{\widetilde{W}}(-E_0) \otimes J$ , where  $J \subset \mathcal{O}_{\widetilde{W}}$  defines a subscheme of

codimension  $\geq 2$ . In particular, we have inclusions of ideal sheaves  $\mathcal{J}^m \subset \mathcal{O}_{\widetilde{W}}(-mE_0) \subset \mathcal{O}_{\widetilde{W}}$ .

Now consider the exact sequence

$$0 \rightarrow \mathcal{J}^m \mathcal{F} \rightarrow \mathcal{F}(-mE_0) \rightarrow \mathcal{F}(-mE_0) \otimes \mathcal{O}_{mZ} \rightarrow 0.$$

Here  $mZ \subset \widetilde{W}$  is the subscheme defined by the ideal sheaf  $\mathcal{J}^m \subset \mathcal{O}_{\widetilde{W}}$ . This gives a long exact sequence

$$\mathbf{R}^{d-1} \pi_*(\mathcal{J}^m \mathcal{F}) \rightarrow \mathbf{R}^{d-1} \pi_*(\mathcal{F}(-mE_0)) \rightarrow \mathbf{R}^{d-1} \pi_*(\mathcal{F}(-mE)|_{mZ}).$$

We note that the last term is zero since the codimension of  $Z$  in  $\widetilde{W}$  is  $\geq 2$ . Hence the first arrow is a surjection.

To conclude the proof of the lemma, we note that the (zero) map  $\mathbf{R}^{d-1} \pi_*(\mathcal{J}^m \mathcal{F}) \rightarrow \mathbf{R}^{d-1} \pi_*(\mathcal{F})$  factors as

$$\mathbf{R}^{d-1} \pi_*(\mathcal{J}^m \mathcal{F}) \rightarrow \mathbf{R}^{d-1} \pi_*(\mathcal{F}(-mE_0)) \rightarrow \mathbf{R}^{d-1} \pi_*(\mathcal{F}).$$

Since the first map is a surjection, the second is necessarily the zero map. The lemma thus holds with  $E = mE_0$ , where  $m$  is sufficiently large.  $\square$

Another version of the above lemma is available, in a situation analogous to Lemma 3.1.

**Lemma 2.3.** *Let  $\widetilde{W} \xrightarrow{\pi} W$  be a desingularisation of a normal projective variety of dimension  $d$  and  $\mathcal{F}$  be a coherent sheaf on  $\widetilde{W}$ . Assume that there exists an effective divisor  $E$  on  $\widetilde{W}$  with  $\pi$ -exceptional support, such that  $-E$  is  $\pi$ -ample.*

- (a) *There exists a positive integer  $r_0$  such that  $\mathbf{R}^i \pi_* \mathcal{F}(-rE) = 0$  for all  $r \geq r_0$  and all  $i > 0$ .*
- (b) *Suppose  $\mathcal{L}$  is ample on  $W$ , and  $\mathcal{F}$  is locally free on  $\widetilde{W}$ . Then there exists a positive integer  $r_1$  such that for each  $r \geq r_1$ , and for all  $n \geq n_1$  (depending on  $r$ ), we have*

$$H^i(\widetilde{W}, \mathcal{F}(rE) \otimes \pi^* \mathcal{L}^{-n}) = 0 \quad \text{for all } i < d,$$

$$H^i(\widetilde{W}, \mathcal{F}(-rE) \otimes \pi^* \mathcal{L}^n) = 0 \quad \text{for all } i > 0.$$

*Proof.* The assertion in (a) is just Serre’s vanishing theorem, since  $\mathcal{O}_{\widetilde{W}}(-E)$  is  $\pi$ -ample. The two assertions in (b) are equivalent (with perhaps different values of  $r_1, n_1$ ), using Serre duality on  $\widetilde{W}$ . The second assertion in (b) follows from (a), using the Leray spectral sequence for  $\pi$ , and Serre’s vanishing on  $W$  for the ample line bundle  $\mathcal{L}$ .  $\square$

We recall a form of the Grauert-Riemenschneider theorem which we need below.

**Theorem 3.** *Let  $W$  be a non-singular projective variety over  $k$ , and  $\mathcal{M}$  a big and base-point free line bundle on  $W$ . Then  $H^i(W, \mathcal{M}^{-1}) = 0$  for  $i < \dim W$ .*

*Proof.* A proof of this statement can be found in [6], Cor. 5.6(b).  $\square$

### 3. Proof of Theorem 2

**3.1. Some preliminary reductions.** From now on, we fix the following notation (used in Theorem 2):  $\tilde{X}$  is a smooth projective  $k$ -variety,  $\mathcal{M}$  a big line bundle on  $\tilde{X}$ ,  $V \subset H^0(\tilde{X}, \mathcal{M})$  a subspace giving a linear system  $|V|$  without base points,  $\tilde{Y}$  a general member of this linear system,  $\pi : \tilde{X} \rightarrow X$  obtained by Stein factorization of the morphism determined by  $|V|$ ,  $\tilde{Y} \rightarrow Y$  the induced morphism (which is also the Stein factorization of the restriction to  $\tilde{Y}$  of the original morphism on  $\tilde{X}$ ), and  $\mathcal{L}$  the invertible sheaf on  $X$  such that  $\pi^* \mathcal{L} = \mathcal{M}$ .

We first make a further reduction.

**Lemma 3.1.** *To prove Theorem 2, it suffices to do it in the case when the morphism  $\pi : \tilde{X} \rightarrow X$  (obtained by Stein factorization) has a purely divisorial exceptional locus, with non-singular irreducible components, and there exists a  $\pi$ -ample divisor of the form  $-E$  where  $E$  is an effective divisor with  $\pi$ -exceptional support.*

*Proof.* Since  $\tilde{X} \rightarrow X$  and  $\tilde{Y} \rightarrow Y$  are obtained from Stein factorizations of the morphisms determined by the base-point free linear system  $|V|$ , we have that  $X, Y$  are normal projective varieties, such that  $Y$  is a Cartier divisor in  $X$ . There is an induced restriction homomorphism  $\text{Cl}(X) \rightarrow \text{Cl}(Y)$ .

It is easy to see that the conclusions of Theorem 2 hold for  $\text{Pic } \tilde{X} \rightarrow \text{Pic } \tilde{Y}$  if and only if the conclusions of Theorem 1 hold for  $\text{Cl}(X) \rightarrow \text{Cl}(Y)$  (i.e., the restriction map on class groups is an isomorphism if  $\dim X \geq 4$ , and an inclusion with finitely generated cokernel if  $\dim X = 3$ ).

In particular, if we replace  $\tilde{X} \rightarrow X$  by another resolution of singularities  $\pi' : \tilde{X}' \rightarrow X$ , and  $\tilde{Y}$  by the inverse image  $\tilde{Y}'$  of  $Y$  in that resolution, it suffices to prove Theorem 2 for this new pair  $(\tilde{X}', \tilde{Y}')$ , and the pull-back linear system from  $X$  (note that there is an open subset of the linear system consisting of divisors which are “general” for both  $\tilde{X}$  and  $\tilde{X}'$ ).

Now by Hironaka’s theorem, we can find a resolution of singularities  $\pi' : \tilde{X}' \rightarrow X$  such that the exceptional locus is divisorial, with non-singular irreducible components. Further, there is an effective exceptional divisor  $E$  such

that  $-E$  is  $\pi'$ -ample; this is because the resolution may be obtained by successive blow-ups at centres lying over the singular locus, so that the resolution may be viewed as a blow-up of an ideal sheaf whose radical defines the singular locus. The pull-back ideal sheaf is an invertible sheaf which is  $\pi'$ -ample, and is the ideal sheaf on  $\tilde{X}'$  of some subscheme with exceptional support.  $\square$

**Remark 3.2.** This reduction is needed only in the proof that, if  $\dim X \geq 4$ , then  $\text{Cl}(X) \rightarrow \text{Cl}(Y)$  is surjective.

We follow the line of proof of the Grothendieck-Lefschetz theorem as given in [11]. The idea is to pass from  $\tilde{Y}$  to the formal completion  $\mathfrak{X}$  of  $\tilde{X}$  along  $\tilde{Y}$ . Then from  $\mathfrak{X}$  we pass to a neighbourhood  $U$  of  $\tilde{Y}$  in  $\tilde{X}$ , using a version of the Lefschetz conditions, and then to  $\tilde{X}$  itself.

**Lemma 3.3.** *With notation as above, if  $\dim \tilde{X} \geq 4$ , then  $\text{Pic}(\mathfrak{X}) \cong \text{Pic}(\tilde{Y})$ . If  $\dim \tilde{X} = 3$ , then  $\text{Pic}(\mathfrak{X}) \rightarrow \text{Pic}(\tilde{Y})$  is injective, with finitely generated cokernel.*

*Proof.* Consider the short exact sequence

$$0 \rightarrow \mathcal{M}^{-m} \otimes \mathcal{O}_{\tilde{Y}} \rightarrow \mathcal{O}_{\mathfrak{X}_{m+1}}^\times \rightarrow \mathcal{O}_{\mathfrak{X}_m}^\times \rightarrow 0$$

where  $\mathfrak{X}_m$  is the  $m$ th infinitesimal neighbourhood of  $\tilde{Y} \subset \tilde{X}$  and in particular  $\mathfrak{X}_1 = \tilde{Y}$ . As usual,  $\mathcal{O}_T^\times$  denotes the (multiplicative) sheaf of invertible regular functions on  $T$ . The first horizontal sheaf map is the “exponential map”, defined on sections by  $s \mapsto 1 + s$ . Taking the cohomology long exact sequence, one has:

$$\begin{aligned} \rightarrow H^1(\tilde{Y}, \mathcal{M}^{-m} \otimes \mathcal{O}_{\tilde{Y}}) &\rightarrow H^1(\mathfrak{X}_{m+1}, \mathcal{O}_{\mathfrak{X}_{m+1}}^\times) \\ &\rightarrow H^1(\mathfrak{X}_m, \mathcal{O}_{\mathfrak{X}_m}^\times) \rightarrow H^2(\tilde{Y}, \mathcal{M}^{-m} \otimes \mathcal{O}_{\tilde{Y}}) \rightarrow \end{aligned}$$

By the Grauert-Riemenschneider vanishing theorem (Theorem 3 above), the extreme terms vanish if  $\dim \tilde{Y} \geq 3$ , and thus we have  $\text{Pic}(\mathfrak{X}_{m+1}) \cong \text{Pic}(\mathfrak{X}_m)$  for each  $m \geq 1$ . From the Grothendieck formula (see for example [10], II, Ex. 9.6.)

$$\text{Pic}(\mathfrak{X}) \cong \varinjlim_m \text{Pic}(\mathfrak{X}_m),$$

we then get  $\text{Pic}(\mathfrak{X}) \cong \text{Pic}(\tilde{Y})$ .

If  $\dim \tilde{Y} = 2$ , then the same argument shows that  $\text{Pic}(\mathfrak{X}_m) \rightarrow \text{Pic}(\tilde{Y})$  is injective for each  $m$ . On the other hand,  $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$  is an isomorphism, since  $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\tilde{Y}))$  vanishes for  $i < 3$  (Grauert-Riemenschneider vanishing, Theorem 3 above). Hence  $\text{Pic}^0(\tilde{X}) \rightarrow \text{Pic}^0(\tilde{Y})$  is an isogeny, and in particular is surjective. Hence  $\text{Coker Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})$  is a quotient of the Neron-Severi group of  $\tilde{Y}$ , and is finitely generated. A similar conclusion then clearly holds for  $\text{Coker Pic}(\mathfrak{X}) \rightarrow \text{Pic}(\tilde{Y})$ .  $\square$

**3.2. The condition  $\text{Lef}(\tilde{X}, \tilde{Y})$ .** In the proof of the Grothendieck-Lefschetz theorem (see [11], Ch. IV), one considers the *Lefschetz condition* which implies the injectivity of the morphism between the Picard groups. We will show that it holds in our situation as well.

If  $\mathcal{E}$  is a coherent sheaf on some open neighbourhood of  $\tilde{Y}$  in  $\tilde{X}$ , then  $\widehat{\mathcal{E}}$  denotes the corresponding (formal) coherent sheaf on the formal completion  $\mathfrak{X}$  of  $\tilde{X}$  along  $\tilde{Y}$ . With this notation, recall the following definition (see [11], page 164; this is a slight modification of Grothendieck’s definition in [9], page 112, as remarked by the referee).

**Definition 1.** The pair  $(\tilde{X}, \tilde{Y})$  satisfies the *Lefschetz condition*  $\text{Lef}(\tilde{X}, \tilde{Y})$  if for every open set  $U \subset \tilde{X}$  containing  $\tilde{Y}$ , and every locally free sheaf  $\mathcal{E}$  on  $U$ , there exists an open set  $U'$  with  $\tilde{Y} \subset U' \subset U$  such that the natural map

$$H^0(U', \mathcal{E}|_{U'}) \rightarrow H^0(\mathfrak{X}, \widehat{\mathcal{E}})$$

is an isomorphism.

Note that there is a finite (perhaps empty) set  $S_0 \subset X$  of (closed) points  $x \in X$  with  $\dim \pi^{-1}(x) = \dim X - 1$ . Since  $\tilde{Y}$  is general, we may assume that  $Y \cap S_0 = \emptyset$ . Further note that any divisor in  $\tilde{X}$  whose support is disjoint from  $\tilde{Y}$  must be supported in  $\pi^{-1}(S_0)$ . If  $E$  is any divisor on  $\tilde{X}$  with support in  $\pi^{-1}(S_0)$ , then  $\widehat{\mathcal{O}_{\tilde{X}}(E)} \cong \mathcal{O}_{\mathfrak{X}}$ .

Recall that the dual of a coherent sheaf  $\mathcal{N}$  on  $\tilde{X}$  is  $\mathcal{N}^\vee = \text{Hom}_{\mathcal{O}_{\tilde{X}}}(\mathcal{N}, \mathcal{O}_{\tilde{X}})$ ; recall also that  $\mathcal{N}$  is *reflexive* if  $\mathcal{N} \rightarrow (\mathcal{N}^\vee)^\vee$  is an isomorphism.

**Lemma 3.4.** *Let  $\mathcal{N}$  be a reflexive, coherent sheaf on  $\tilde{X}$  which is locally free in a neighbourhood of  $\tilde{Y}$ . Then there exists an effective divisor  $E$  on  $\tilde{X}$ , where either  $E = 0$  or  $\dim \pi(\text{supp } E) = 0$ , such that the natural map*

$$H^0(\tilde{X}, \mathcal{N}(E)) \rightarrow H^0(\mathfrak{X}, \widehat{\mathcal{N}})$$

is an isomorphism.

*Proof.* Let  $n = \dim \tilde{X}$ . Using Serre duality on  $\tilde{X}$  and formal duality on  $\mathfrak{X}$  (see [11], III, Theorem 3.3), we reduce to proving that

$$H^n_{\tilde{Y}}(\tilde{X}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}}) \rightarrow H^n(\tilde{X}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}})$$

is an isomorphism, for appropriate  $E$ . Here, we note that though  $\mathcal{N}$  may not be locally free, Serre duality implies that the dual of the finite-dimensional vector space  $H^n(\tilde{X}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}})$  is

$$\text{Hom}(\mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}}, \omega_{\tilde{X}}) = H^0(\tilde{X}, \mathcal{N}(E)),$$

since  $\mathcal{N}$  is reflexive.



For any effective divisor  $E$  supported in  $\pi^{-1}(S_0)$ , consider the commutative diagram with exact rows:

$$\begin{array}{ccc}
\mathrm{H}^{n-1}(\tilde{X} \setminus \tilde{Y}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}}) & \xrightarrow{\delta_1} & \mathrm{H}_Y^n(\tilde{X}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}}) \\
\downarrow \phi_1 & & \downarrow \phi_2 \\
\mathrm{H}^{n-1}(\tilde{X} \setminus \tilde{Y}, \mathcal{N}^\vee \otimes \omega_{\tilde{X}}) & \xrightarrow{\delta_2} & \mathrm{H}_Y^n(\tilde{X}, \mathcal{N}^\vee \otimes \omega_{\tilde{X}}) \\
& & \rightarrow \mathrm{H}^n(\tilde{X}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}}) \\
& & \downarrow \phi_3 \\
& & \rightarrow \mathrm{H}^n(\tilde{X}, \mathcal{N}^\vee \otimes \omega_{\tilde{X}})
\end{array}$$

The last maps in the sequences are onto since  $\tilde{X} \setminus \tilde{Y}$  has cohomological dimension at most  $n-1$ . Moreover, the map  $\phi_2$  is an isomorphism by excision since  $\tilde{Y} \cap E = \emptyset$ . The Leray spectral sequence for the map  $\pi : \tilde{X} \setminus \tilde{Y} \rightarrow X \setminus Y$  applied to the cohomology group  $\mathrm{H}^{n-1}(\tilde{X} \setminus \tilde{Y}, \mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}})$  has  $E_2^{p,q} = 0$  for  $p > 0$  ( $X \setminus Y$  is affine!). By Lemma 2.2, there exists an  $E$  as in the statement of the lemma such that the map  $\mathrm{R}^{n-1} \pi_*(\mathcal{N}^\vee(-E) \otimes \omega_{\tilde{X}}) \rightarrow \mathrm{R}^{n-1} \pi_*(\mathcal{N}^\vee \otimes \omega_{\tilde{X}})$  is the zero map, and thus the map  $\phi_1$  is the zero map, for this choice of  $E$ . This in turn implies that the corresponding map  $\delta_1$  is zero.

We thus have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_Y^n(\tilde{X}, \mathcal{N}^\vee(-mE) \otimes \omega_{\tilde{X}}) & \cong & \mathrm{H}^n(\tilde{X}, \mathcal{N}^\vee(-mE) \otimes \omega_{\tilde{X}}) \\
\downarrow \cong & & \downarrow \\
\mathrm{H}_Y^n(\tilde{X}, \mathcal{N}^\vee \otimes \omega_{\tilde{X}}) & \twoheadrightarrow & \mathrm{H}^n(\tilde{X}, \mathcal{N}^\vee \otimes \omega_{\tilde{X}})
\end{array}$$

Dualising, we have

$$\mathrm{H}^0(\tilde{X}, \mathcal{N}(mE)) \cong \mathrm{H}^0(\mathfrak{X}, \widehat{\mathcal{N}})$$

□

**Corollary 3.5.** *The condition  $\mathrm{Lef}(\tilde{X}, \tilde{Y})$  holds.*

*Proof.* For any open set  $U \supset \tilde{Y}$  in  $\tilde{X}$ , and any locally free sheaf  $\mathcal{N}_U$  on  $U$ , we can find a reflexive sheaf  $\mathcal{N}$  on  $\tilde{X}$  extending  $\mathcal{N}_U$ , i.e., with  $\mathcal{N}|_U \cong \mathcal{N}_U$  (first choose a coherent extension, then replace it by its double dual). For a suitable divisor  $E$  with  $\dim(\mathrm{supp} E) = 0$ , we have a commutative diagram induced by restriction maps:

$$\begin{array}{ccc}
\mathrm{H}^0(\tilde{X}, \mathcal{N}(E)) & \cong & \mathrm{H}^0(\mathfrak{X}, \widehat{\mathcal{N}}) \\
\searrow & & \nearrow \\
& \mathrm{H}^0(U, \mathcal{N}(E)) &
\end{array}$$

In particular, for any open  $V$  such that  $\tilde{Y} \subset V \subset U$  and  $V \cap \mathrm{supp} E = \emptyset$  the above factorisation gives a surjection

$$\mathrm{H}^0(V, \mathcal{N}(E)) \cong \mathrm{H}^0(V, \mathcal{N}) \twoheadrightarrow \mathrm{H}^0(\mathfrak{X}, \widehat{\mathcal{N}}).$$

But since  $\mathcal{N}$  is locally free on  $V$  and  $V$  is irreducible, the map is also an injection. Thus  $\text{Lef}(\tilde{X}, \tilde{Y})$  holds.  $\square$

**Corollary 3.6.** *For normal  $X, Y$  as above, the condition  $\text{Lef}(X, Y)$  holds.*

*Proof.* Since  $X$  and  $Y$  are normal, one has  $\pi_*\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$  and  $\pi_*\mathcal{O}_{\tilde{Y}} \cong \mathcal{O}_Y$ .  $\text{Lef}(X, Y)$  then follows from  $\text{Lef}(\tilde{X}, \tilde{Y})$  applied to sheaves which are pull-backs of locally free sheaves on neighbourhoods of  $Y$  in  $X$ .  $\square$

**Corollary 3.7.** *The kernel of the restriction map  $\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})$  is freely generated by the classes of irreducible effective divisors which map to points in  $X$ .*

*Proof.* It is obvious that the classes of such divisors are contained in the kernel since  $Y \cap S_0 = \emptyset$  (as  $Y$  is general). On the other hand, if  $\mathcal{N}$  is a line bundle on  $\tilde{X}$  with  $\mathcal{N} \otimes \mathcal{O}_{\tilde{Y}} \cong \mathcal{O}_{\tilde{Y}}$ , then we first note that  $\hat{\mathcal{N}} \cong \mathcal{O}_{\mathfrak{X}}$  by Lemma 3.3, and there is thus an invertible element of  $H^0(\mathfrak{X}, \hat{\mathcal{N}})$ ; by Lemma 3.4, this formal global section is obtained from a global section on  $\tilde{X}$  of  $\mathcal{N}(E)$  for some divisor  $E$  on  $\tilde{X}$  supported over  $S_0 \subset X$ . This section of  $\mathcal{N}(E)$  has no zeroes when restricted to  $\tilde{Y}$ , so its divisor of zeroes  $E'$  is also supported over  $S_0$ , and hence  $\mathcal{N} \cong \mathcal{O}_{\tilde{X}}(E' - E)$ .  $\square$

**Corollary 3.8.** *For each  $n \geq 0$ , and any effective divisor  $F$  on  $\tilde{X}$  with  $\pi$ -exceptional support, the natural maps*

$$H^0(\tilde{X}, \mathcal{M}^{\otimes n}) \rightarrow H^0(\tilde{X}, \mathcal{M}^{\otimes n}(F)) \rightarrow H^0(\mathfrak{X}, \widehat{\mathcal{M}^{\otimes n}(F)})$$

are isomorphisms.

*Proof.* Since  $\mathcal{M} = \pi^*\mathcal{L}$ , where  $\mathcal{L}$  is invertible on the normal variety  $X$ , and  $\pi_*\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ , it follows that  $\mathcal{M}^{\otimes n} \hookrightarrow \mathcal{M}^{\otimes n}(E + F)$  is an isomorphism on global sections for any effective divisor  $E$  supported in  $\pi^{-1}(S_0)$ , and any  $n \geq 0$ .  $\square$

**3.3. The condition  $\text{ALeff}(\tilde{X}, \tilde{Y})$ .** We now introduce a second condition  $\text{ALeff}(\tilde{X}, \tilde{Y})$  (Almost Effective Lefschetz), which is a variation of Grothendieck’s Effective Lefschetz Condition (denoted by “Leff” in [11]). In the proof in [11] of the Grothendieck-Lefschetz theorem, the condition Leff is used to show the surjectivity of the restriction map between the Picard groups;  $\text{ALeff}$  has a similar role here.

**Definition 2.** We say the pair  $(\tilde{X}, \tilde{Y})$  satisfies the *ALeff condition* if

- (1)  $\text{Lef}(\tilde{X}, \tilde{Y})$  holds and
- (2) for any (formal) invertible sheaf  $\mathcal{E}$  on  $\mathfrak{X}$  there exists an open set  $U$  containing  $\tilde{Y}$  and an invertible sheaf  $\mathcal{E}$  on  $U$ , together with a map  $\hat{\mathcal{E}} \rightarrow \mathcal{E}$ , which is an isomorphism outside the exceptional locus of  $\pi : \tilde{Y} \rightarrow Y$ .

Note that the formal scheme  $\mathfrak{X}$  is a ringed space with underlying topological space  $\tilde{Y}$ , so the second condition above is meaningful.

For any formal coherent sheaf  $\mathcal{F}$  on  $\mathfrak{X}$ , we will make the following abuses of notation: For any divisor  $D$  on  $\tilde{X}$ , let  $\mathcal{F}(D)$  denote the formal coherent sheaf  $\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \widehat{\mathcal{O}_{\tilde{X}}(D)}$ , and for any coherent  $\mathcal{G}$  on  $\tilde{X}$ , let  $\mathcal{F} \otimes \mathcal{G}$  denote the formal coherent sheaf  $\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \hat{\mathcal{G}}$ .

**Proposition 3.9.** *Let  $\tilde{X}, \tilde{Y}$  be as in Lemma 3.1, with  $\dim \tilde{X} \geq 3$ , and let  $E$  be an effective divisor on  $\tilde{X}$  with exceptional support such that  $-E$  is  $\pi$ -ample. Then for any formal locally free sheaf  $\mathcal{F}$  on  $\mathfrak{X}$ , there exists  $r > 0$  such that for any  $m \geq 0$ , if we set*

$$\mathcal{G}_m = \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m},$$

then for all  $m \gg 0$ ,

$$\text{Coker}(H^0(\mathfrak{X}, \mathcal{G}_m) \otimes_k \mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{G}_m)$$

is supported on  $\tilde{Y} \cap E$ .

*Proof.* The proof is in several steps. Let  $\mathcal{F}_n = \mathcal{F} \otimes \mathcal{O}_{\mathfrak{X}_n}$ , for  $n \geq 1$ , be the sequence of locally free sheaves (on the sequence of schemes  $\mathfrak{X}_n$ ) associated to the formal locally free sheaf  $\mathcal{F}$ . We have exact sequences

$$0 \rightarrow \mathcal{F} \otimes \mathcal{M}^{\otimes m-n} \rightarrow \mathcal{F} \otimes \mathcal{M}^{\otimes m} \rightarrow \mathcal{F}_n \otimes \mathcal{M}^{\otimes m} \rightarrow 0$$

for each  $m \in \mathbb{Z}$  and  $n > 0$ , where the ideal sheaf of  $\tilde{Y}$  in  $\mathcal{O}_{\tilde{X}}$  is identified with  $\mathcal{M}^{-1}$ .

**Lemma 3.10.** *Let  $d = \dim \tilde{Y}$ . There exists  $r_0 > 0$  such that, for each  $r \geq r_0$ , all  $m \gg 0$  (depending on  $r$ ), and all  $i > 0$ , we have*

$$H^i(\tilde{Y}, \mathcal{F}_1 \otimes \mathcal{O}_{\tilde{X}}(-rE) \otimes \mathcal{M}^{\otimes m}) = 0,$$

$$H^{d-i}(\tilde{Y}, \mathcal{F}_1 \otimes \mathcal{O}_{\tilde{X}}(rE) \otimes \mathcal{M}^{\otimes -m}) = 0.$$

*Proof.* We have that  $\mathcal{M} = \pi^* \mathcal{L}$  where  $\mathcal{L}$  is ample on  $X$ . Now apply Lemma 2.3(b). □

**Lemma 3.11.** *There exists  $r_0 > 0$  so that, for any  $r \geq r_0$  and any  $m \in \mathbb{Z}$ , the vector space*

$$H^1(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m})$$

is finite dimensional.

*Proof.* We have exact sheaf sequences

$$0 \rightarrow \mathcal{F}_1(rE) \otimes \mathcal{M}^{\otimes m-n} \rightarrow \mathcal{F}_{n+1}(rE) \otimes \mathcal{M}^{\otimes m} \rightarrow \mathcal{F}_n(rE) \otimes \mathcal{M}^{\otimes m} \rightarrow 0.$$

Since  $\dim \tilde{Y} \geq 2$ , we have for each given  $m$  that

$$H^1(\tilde{Y}, \mathcal{F}_1(rE) \otimes \mathcal{M}^{\otimes m-n}) = 0$$

provided  $n \gg m$ , from Lemma 3.10; thus for  $n \gg m$ ,

$$H^1(\mathfrak{X}_{n+1}, \mathcal{F}_{n+1}(rE) \otimes \mathcal{M}^{\otimes m}) \rightarrow H^1(\mathfrak{X}_n, \mathcal{F}_n(rE) \otimes \mathcal{M}^{\otimes m})$$

is injective. Hence, in the Grothendieck formula

$$H^1(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m}) = \varprojlim_n H^1(\mathfrak{X}_n, \mathcal{F}_n(rE) \otimes \mathcal{M}^{\otimes m}),$$

the maps in the inverse system on the right are, for  $n \gg 0$ , injective maps of finite-dimensional  $k$ -vector spaces (the above inverse limit formula holds, because the corresponding inverse system for  $H^0$  is an inverse system of finite-dimensional  $k$ -vector spaces, hence satisfies the Mittag-Leffler condition (ML)). Thus the inverse limit  $H^1(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m})$  is a finite-dimensional vector space.  $\square$

Now consider the exact sequences

$$0 \rightarrow \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m} \rightarrow \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m+1} \rightarrow \mathcal{F}_1(-rE) \otimes \mathcal{M}^{\otimes m+1} \rightarrow 0.$$

**Lemma 3.12.** *There exists  $r_1 > 0$  so that, for each  $r \geq r_1$ , and all  $m \gg 0$  (depending on  $r$ ), the map*

$$H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m}) \rightarrow H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m+1})$$

is surjective.

*Proof.* It suffices to see that

$$H^1(\tilde{Y}, \mathcal{F}_1(-rE) \otimes \mathcal{M}^{\otimes m+1}) = 0.$$

This follows from Lemma 3.10.  $\square$

Fix an  $r$  large enough so that the conclusions of Lemma 3.11 and 3.12 hold. Define

$$V_m = \text{image } H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m}) \xrightarrow{\beta_m} H^1(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m}),$$

where the map is induced by the natural inclusion  $\mathcal{O}_{\tilde{X}}(-rE) \rightarrow \mathcal{O}_{\tilde{X}}(rE)$ , determined by the tautological section of  $\mathcal{O}_{\tilde{X}}(2rE)$ . From Lemma 3.11,  $V_m$  is a finite-dimensional vector space, and from Lemma 3.12, the natural maps

$$H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m}) \rightarrow H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m+1})$$

induce surjections  $V_m \rightarrow V_{m+1}$ , for all large enough  $m$ . Hence  $V_m \rightarrow V_{m+1}$  is in fact an isomorphism, for all large enough  $m$ . Consider the commutative diagram of formal sheaves with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m} & \rightarrow & \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m+1} & \rightarrow & \mathcal{F}_1(-rE) \otimes \mathcal{M}^{\otimes m+1} & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m} & \rightarrow & \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m+1} & \rightarrow & \mathcal{F}_1(rE) \otimes \mathcal{M}^{\otimes m+1} & \rightarrow & 0 \end{array}$$

The vertical arrows are induced by the natural inclusion  $\mathcal{O}_{\tilde{X}}(-rE) \rightarrow \mathcal{O}_{\tilde{X}}(rE)$  (as above, in defining  $V_m$ ). There is an induced cohomology diagram with exact rows:

$$\begin{array}{ccc}
 H^0(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m+1}) & \rightarrow & H^0(\tilde{Y}, \mathcal{F}_1(-rE) \otimes \mathcal{M}^{\otimes m+1}) \\
 \downarrow & & \downarrow \alpha \\
 H^0(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m+1}) & \rightarrow & H^0(\tilde{Y}, \mathcal{F}_1(rE) \otimes \mathcal{M}^{\otimes m+1}) \\
 & & \rightarrow H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m}) \rightarrow H^1(\mathfrak{X}, \mathcal{F}(-rE) \otimes \mathcal{M}^{\otimes m+1}) \\
 & & \downarrow \beta_m \qquad \qquad \qquad \downarrow \beta_{m+1} \\
 & & \rightarrow H^1(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m}) \rightarrow H^1(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m+1})
 \end{array}$$

Since  $V_m \rightarrow V_{m+1}$  is an isomorphism, we see that

$$\text{image} \left( H^0(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m+1}) \rightarrow H^0(\tilde{Y}, \mathcal{F}_1(rE) \otimes \mathcal{M}^{\otimes m+1}) \right)$$

contains the subspace

$$\text{image} H^0(\tilde{Y}, \mathcal{F}_1(-rE) \otimes \mathcal{M}^{\otimes m+1}).$$

Since  $m \gg 0$ , the global sections of the sheaf  $\mathcal{F}_1(-rE) \otimes \mathcal{M}^{\otimes m+1}$  generate it on  $\tilde{Y} \setminus E$  (since the direct image of this sheaf on  $Y$  is globally generated, and  $\tilde{Y} \rightarrow Y$  is an isomorphism outside  $\tilde{Y} \cap E$ ). Hence, the natural map between coherent formal sheaves

$$H^0(\mathfrak{X}, \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m+1}) \otimes_k \mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{F}(rE) \otimes \mathcal{M}^{\otimes m+1}$$

restricts to a surjection on  $\tilde{Y} \setminus E$ . This proves Proposition 3.9 □

**Remark 3.13.** The referee has pointed out that, in Proposition 3.9, we can in fact get the stated conclusion for any integer  $r$ , and all sufficiently large  $m$  (depending on  $r$ ). Consider the pairs  $(r, m)$  for which the conclusion of the proposition holds. Given  $r$ , we have seen already that there is a positive integer  $r_0 > r$  so that (i) the conclusion holds for  $(r, m)$  for all  $m \geq m_0$ , say, and (ii) so that  $\mathcal{O}_{\tilde{X}}(-(r_0 - r)E)$  is very ample for  $\pi$ . Then choose  $m_1$  so that  $\pi_* \mathcal{O}_{\tilde{X}}(-(r_0 - r)E) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$  is globally generated for all  $m \geq m_1$ . Then  $\mathcal{O}_{\tilde{X}}(-(r_0 - r)E) \otimes \mathcal{M}^{\otimes m}$  is globally generated, for all  $m \geq m_1$ . Hence the conclusion of the proposition holds for  $(r, m)$  with  $m \geq m_0 + m_1$ .

**Proposition 3.14.** *For  $\tilde{X}, \tilde{Y}$  as above, the condition  $\text{ALeff}(\tilde{X}, \tilde{Y})$  holds.*

*Proof.* By Proposition 3.9, for any invertible formal sheaf  $\mathcal{F}$  on  $\mathfrak{X}$ , one has a map of formal locally free sheaves

$$(1) \qquad (\widehat{\mathcal{M}}^{\otimes -M})^{\oplus s}(-F_1) \rightarrow \mathcal{F} \rightarrow 0$$

for some  $M \gg 0, s > 0$ , with cokernel supported in  $\tilde{Y} \setminus E$ , where  $F_1$  is an effective divisor on  $\tilde{X}$  with exceptional support.

Similarly, for the dual formal line bundle  $\mathcal{F}^\vee$ , we have a map, surjective outside the exceptional locus,

$$(\widehat{\mathcal{M}}^{\otimes -N})^{\oplus t}(-F_2) \rightarrow \mathcal{F}^\vee \rightarrow 0$$

for some  $N \gg 0, t > 0$ . Dualizing this we have an injection

$$(2) \quad \mathcal{F} \rightarrow (\widehat{\mathcal{M}}^{\otimes N})^{\oplus t}(F_2)$$

which is a split inclusion on stalks at any point of  $\widetilde{Y} \setminus E$ .

Composing the maps in (1) and (2), we have a map between formal locally free sheaves

$$(3) \quad (\widehat{\mathcal{M}}^{\otimes -M})^{\oplus s}(-F_1) \xrightarrow{\widehat{\tau}} (\widehat{\mathcal{M}}^{\otimes N})^{\oplus t}(F_2)$$

such that  $\text{Im } \widehat{\tau} \hookrightarrow \mathcal{F}$  with cokernel supported in  $\widetilde{Y} \setminus E$ .

The map  $\widehat{\tau}$  may be described by an  $s \times t$  matrix of elements of

$$H^0(\mathfrak{X}, \widehat{\mathcal{M}}^{\otimes N+M}(F_1 + F_2)).$$

By the condition  $\text{Lef}(\widetilde{X}, \widetilde{Y})$  (or rather Corollary 3.8),

$$H^0(\mathfrak{X}, \widehat{\mathcal{M}}^{\otimes N+M}(F_1 + F_2)) \cong H^0(\widetilde{X}, \mathcal{M}^{\otimes N+M}),$$

so that the map  $\widehat{\tau}$  is the formal completion of a map of locally free sheaves on  $\widetilde{X}$

$$(\mathcal{M}^{\otimes -M})^{\oplus s}(-F_1) \xrightarrow{\tau} (\mathcal{M}^{\otimes N})^{\oplus t}(F_2).$$

Thus we have  $\widehat{\text{Im}}(\widehat{\tau}) \hookrightarrow \mathcal{F}$ , with cokernel supported on  $\widetilde{Y} \setminus E$ .

Now  $\text{Im}(\tau)$  is a coherent sheaf on  $\widetilde{X}$ , such that for any point  $y \in \widetilde{Y} \setminus E$ , the stalk at  $y$  satisfies

$$\text{Im}(\tau)_y \otimes_{\mathcal{O}_{\widetilde{X},y}} \widehat{\mathcal{O}_{\widetilde{X},y}} \cong \mathcal{F}_y \cong \mathcal{O}_{\mathfrak{X},y} \cong \widehat{\mathcal{O}_{\widetilde{X},y}},$$

(where the completions are with respect to the ideal defining  $\widetilde{Y}$ ). Hence  $\text{Im}(\tau)$  is a coherent sheaf of rank 1, which is invertible on  $\widetilde{X}$  at all points in  $\widetilde{Y} \setminus E$ . Since  $\widetilde{X}$  is non-singular, the double dual of  $\text{Im}(\tau)$  is an invertible sheaf  $\widetilde{\mathcal{F}}$  on  $\widetilde{X}$ , such that

$$\widetilde{\mathcal{F}} \otimes_{\mathcal{O}_{\widetilde{Y}}} |_{\widetilde{Y} \setminus E} \cong \mathcal{F}_1 |_{\widetilde{Y} \setminus E}.$$

Thus we have  $\text{ALeff}(\widetilde{X}, \widetilde{Y})$ . □

**Remark 3.15.** The above argument, applied to an arbitrary formal locally free sheaf  $\mathcal{F}$ , implies the existence of a coherent, reflexive sheaf  $\mathcal{F}$  together with an injective map  $\widehat{\mathcal{F}} \rightarrow \mathcal{F}$  which restricts to an isomorphism on  $\widetilde{Y} \setminus E$ . We do not know if  $\mathcal{F}$  can be chosen to be locally free in a neighbourhood of  $\widetilde{Y}$ ; perhaps the “natural” extension to our situation of the Grothendieck “Lef” condition is for this property to hold.

**Corollary 3.16.** *If  $\dim \tilde{X} \geq 4$ , the cokernel of the restriction map  $\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})$  is generated by exceptional divisors of  $\tilde{Y}$  which map to points in  $Y$ .*

*Proof.* By Lemma 3.3 one has  $\text{Pic}(\mathfrak{X}) \cong \text{Pic}(\tilde{Y})$ . Let  $\mathcal{N}$  be a line bundle on  $\tilde{Y}$ , and  $\mathfrak{N}$  its unique lift to a formal line bundle on  $\mathfrak{X}$ . Proposition 3.14 implies that there exists an invertible sheaf  $\mathcal{G}$  on  $\tilde{X}$  such that  $\hat{\mathcal{G}} \cong \mathfrak{N}$  on  $\tilde{Y} \setminus E$ . Thus  $\mathcal{G}|_{\tilde{Y}} \otimes \mathcal{N}^\vee$  is a line bundle on  $\tilde{Y}$  which has trivial restriction to  $\tilde{Y} \setminus E$ , and is thus the line bundle associated to a divisor on  $\tilde{Y}$  with exceptional support. It remains to show that, up to tensoring with a line bundle restricted from  $\tilde{X}$ , it corresponds to a sum of exceptional divisors for  $\tilde{Y} \rightarrow Y$  with 0-dimensional image in  $Y$ .

Let  $E_1, \dots, E_r$  be the irreducible exceptional divisors of  $\tilde{X} \rightarrow X$ , indexed so that for some  $0 \leq s \leq t \leq r$ , we have:

- (i)  $E_1, \dots, E_s$  are the irreducible divisors in  $\tilde{X}$  with 0-dimensional image in  $X$ .
- (ii)  $E_{s+1}, \dots, E_t$  are the irreducible divisors with 1-dimensional image in  $X$ .
- (iii)  $E_{t+1}, \dots, E_r$  are the irreducible exceptional divisors for  $\tilde{X} \rightarrow X$  whose images in  $X$  have dimension  $\geq 2$ .

Since  $\tilde{Y} \in |\mathbf{V}|$  is a general member, we have:

- (a)  $\tilde{Y} \cap E_i = \emptyset$  for  $1 \leq i \leq s$ .
- (b) For each  $s + 1 \leq i \leq t$ , let  $\tilde{Y} \cap E_i = \bigcup_{j=1}^{s_i} F_{ij}$  be the irreducible (equivalently connected) components of the intersection; then  $\tilde{Y} \cap E_i$  is non-singular, reduced, and has no common irreducible component with  $\tilde{Y} \cap E_{i'}$  for any  $i' \neq i$ .
- (c) For  $t + 1 \leq i \leq r$ ,  $F_i = \tilde{Y} \cap E_i$  is reduced and irreducible.

Here, (a) is clear, and (c) follows from Bertini's theorem. For (b), note that the linear system  $|\mathbf{V}|$  restricts to a base-point free linear system  $|\mathbf{V}_i|$  on  $E_i$ , and the Stein factorization of the corresponding morphism has the form  $\pi_i : E_i \rightarrow C_i$  for some irreducible curve  $C_i$ , such that  $|\mathbf{V}_i|$  is the pull-back of a linear system from  $C_i$ . Hence the general member of  $|\mathbf{V}_i|$  is a disjoint union of a finite set of fibers of  $E_i \rightarrow C_i$ , over points of  $C_i$  for which  $E_i \rightarrow C_i$  is smooth.

Thus, the line bundle determined by an irreducible exceptional divisor  $F$  on  $\tilde{Y}$  lies in the image of the  $\text{Pic}(\tilde{X}) \rightarrow \text{Pic}(\tilde{Y})$ , except possibly when  $F$  is one of the  $F_{ij}$  in (b) above, and the image of each such  $F_{ij}$  in  $Y$  is a point.  $\square$

**Lemma 3.17.** *Assume  $\dim \tilde{X} \geq 3$ , and  $\tilde{Y} \in |\mathbf{V}|$  is general. Let  $F$  be a divisor on  $\tilde{Y}$  with exceptional support, such that  $\mathcal{O}_{\tilde{Y}}(F)$  is the restriction of a line bundle from  $\tilde{X}$ . Then there is a divisor  $\tilde{F}$  on  $\tilde{X}$  with exceptional support such that  $\mathcal{O}_{\tilde{X}}(\tilde{F})|_{\tilde{Y}} = \mathcal{O}_{\tilde{Y}}(F)$ .*

*Proof.* From the description of the irreducible exceptional divisors of  $\tilde{Y} \rightarrow Y$  given in (a), (b), and (c) of the proof of Corollary 3.16 above, we see that it suffices to assume (in the notation of (b)) that

$$(4) \quad F = \sum_{i=s+1}^t \sum_j n_{ij} F_{ij},$$

for some integers  $n_{ij}$ ,  $s + 1 \leq i \leq t$ ,  $1 \leq j \leq s_i$ . Here, the divisors

$$F_i := \sum_{j=1}^{s_i} F_{ij}$$

for each  $s + 1 \leq i \leq t$  satisfy  $\mathcal{O}_{\tilde{X}}(E_i)|_{\tilde{Y}} \cong \mathcal{O}_{\tilde{Y}}(F_i)$ .

So it suffices to show: If  $F$  in (4) is such that  $\mathcal{O}_{\tilde{Y}}(F)$  is the restriction of a line bundle from  $\tilde{X}$ , then  $n_{ij}$  is independent of  $j$ , for each  $s + 1 \leq i \leq t$ . This is done using a suitable computation with intersection numbers.

Let  $T \subset \tilde{X}$  be a general complete intersection of dimension 3, in some projective embedding of  $\tilde{X}$ . Then by Bertini’s theorem, we may assume that:

- (i)  $T$  is irreducible and non-singular, and the scheme theoretic intersection  $T \cap E_i$  is a reduced, irreducible surface, for each  $1 \leq i \leq r$ .
- (ii) The scheme theoretic intersection  $Z := T \cap \tilde{Y}$  is a (reduced, irreducible) non-singular surface in  $T$ .
- (iii)  $Z \cap E_i = \emptyset$  for  $1 \leq i \leq s$ , and  $Z \cap E_i$  is a reduced, irreducible curve for each  $t + 1 \leq i \leq r$ .
- (iv)  $Z \cap F_{ij}$  is a reduced, irreducible curve in  $Z$ , for each  $s + 1 \leq i \leq t$ , for all  $j$ .

If  $T_0$  is the image of  $T$  in  $X$ , let  $\bar{T} \rightarrow T_0$  be the normalization. Then  $\pi_T : T \rightarrow \bar{T}$  is a resolution of singularities, such that  $T \cap E_i$ ,  $1 \leq i \leq r$  are the irreducible exceptional divisors of  $\pi_T$ . The surface  $\pi_T(Z) = \bar{Z} \subset \bar{T}$  is a normal Cartier divisor in  $\bar{T}$  (it is a general member of the ample, base-point free linear system on  $\bar{T}$  determined by  $|\mathbf{V}|$ ). Let  $\pi_Z : Z \rightarrow \bar{Z}$  be the restriction of  $\pi_T$ ; then  $\pi_Z$  is a resolution of singularities of a normal, projective surface, with irreducible exceptional curves  $Z \cap E_i$ ,  $t + 1 \leq i \leq r$  and  $Z \cap F_{ij}$ ,  $s + 1 \leq i \leq t$ ,  $1 \leq j \leq s_i$ .

Since the linear equivalence class of  $F = \sum_{i,j} n_{ij} F_{ij}$  is assumed to lie in the image of the restriction  $\text{Pic } \tilde{X} \rightarrow \text{Pic } \tilde{Y}$ , we have that  $\mathcal{O}_Z(\sum_{i,j} n_{ij} Z \cap F_{ij})$  is in the image of  $\text{Pic } T \rightarrow \text{Pic } Z$ . For each  $i$ , the  $F_{ij}$  are irreducible components of general fibers of  $E_i \rightarrow C_i$ . Hence  $F_{ij} \cap Z$  are irreducible components of general fibers of  $T \cap E_i \rightarrow C_i$ , and are thus *algebraically equivalent* as 1-cycles on the smooth projective 3-fold  $T$ . Hence, for any divisor  $D$  on  $T$ , the intersection number  $(D \cdot (F_{ij} \cap T))_T$  is independent of  $j$ , for each  $i$ . Since  $F_{ij} \cap Z$  is a



Cartier divisor in  $Z$ , if  $D_Z$  is any divisor on  $Z$  representing  $\mathcal{O}_T(D)|_Z$ , the projection formula gives an equation between intersection numbers

$$(5) \quad (D_Z \cdot (F_{ij} \cap Z))_Z = (D \cdot (F_{ij} \cap T))_T$$

computed on the surface  $Z$  and the 3-fold  $T$ , respectively.

Apply this to our divisor  $F|_Z = \sum_{i,j} n_{ij} F_{ij} \cap Z$ , which is assumed to be of the form  $D_Z$  for some divisor  $D$  on  $T$ . We get that

$$(6) \quad \left( \sum_{ij} n_{ij} (F_{ij} \cap Z) \cdot (F_{i'j'} \cap Z) \right)_Z$$

is independent of  $j'$ , for each  $s + 1 \leq i' \leq t$ .

Since  $F_{ij} \cap Z$  are irreducible exceptional divisors for  $\pi_Z : Z \rightarrow \bar{Z}$ , which is a resolution of singularities of a normal surface, the intersection matrix

$$(F_{ij} \cap Z, F_{i'j'} \cap Z)_Z$$

is negative definite. Regard (6) as a system of linear equations satisfied by the  $n_{ij}$ , with coefficients given by intersection numbers. The solutions of the system (6) for the “unknowns”  $n_{ij}$  correspond to elements in the  $\mathbb{Z}$ -span of the  $F_{ij}$ , which are in the orthogonal complement of the span of all the differences  $(F_{ij_1} \cap Z) - (F_{ij_2} \cap Z)$ , for all  $1 \leq j_1 < j_2 \leq s_i$ , and all  $s + 1 \leq i \leq t$ . The span of these differences clearly has co-rank  $t - s$ , so the orthogonal complement has rank  $t - s$ . We have  $t - s$  elements  $\sum_{j=1}^{s_i} (F_{ij} \cap Z)$  which lie in the orthogonal complement, which are clearly independent, so must span the orthogonal complement after tensoring with  $\mathbb{Q}$ . This implies that  $F_Z = \sum_{i,j} n_{ij} (F_{ij} \cap Z)$  must be a rational linear combination of the divisors  $\sum_{j=1}^{s_i} (F_{ij} \cap Z)$ , and so  $n_{ij}$  must be independent of  $j$ , for each  $i$ , as desired.  $\square$

Assume now that  $\dim X \geq 4$ . The conditions Lef and ALef imply that we have the following diagram, with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & I'_X & \rightarrow & \mathbb{Z}[E_X] & \rightarrow & \mathbb{Z}[E_Y] & \rightarrow & I'_Y & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & I_X & \rightarrow & \text{Pic}(\tilde{X}) & \rightarrow & \text{Pic}(\tilde{Y}) & \rightarrow & I_Y & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & \text{Cl}(X) & \rightarrow & \text{Cl}(Y) & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & & 
 \end{array}$$

Here  $\mathbb{Z}[E_X]$  and  $\mathbb{Z}[E_Y]$  are the subgroups in the respective Picard groups freely generated by the irreducible exceptional divisors in  $\tilde{X}$  and  $\tilde{Y}$ , and  $I_X, I'_X, I_Y, I'_Y$  are defined by the exactness of the rows. Clearly  $I'_X, I'_Y$  are generated

by the irreducible exceptional divisors in  $\tilde{X}$  and  $\tilde{Y}$ , respectively, which have 0-dimensional image under  $\pi$ .

It is clear that  $I_X \cong I'_X$ : it is an injection since  $\mathbb{Z}[E_X] \hookrightarrow \text{Pic}(\tilde{X})$  is so. That it is a surjection follows from Corollary 3.7. Also Corollary 3.16 shows that that the map  $I'_Y \rightarrow I_Y$  is surjective, while it is also injective, by Lemma 3.17. Hence, from a diagram chase, we see that  $\text{Cl}(X) \rightarrow \text{Cl}(Y)$  is an isomorphism, completing the proof of Theorem 2 when  $\dim \tilde{X} \geq 4$ .

By a similar argument, Corollary 3.7 and Lemma 3.17 imply that, if  $\dim \tilde{X} = 3$ , then  $\text{Cl}(X) \rightarrow \text{Cl}(Y)$  is *injective*. The finite generation of the cokernel results from the fact that  $\text{Pic}^0 \tilde{X} \rightarrow \text{Pic}^0 \tilde{Y}$  is an isogeny (and hence an isomorphism), since the map on tangent spaces  $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$  is an isomorphism, from Theorem 3. This completes the proof of Theorem 2 in the case  $\dim \tilde{X} = 3$ .

For possible use elsewhere, we make explicit the following result, more or less implicit above. We thank the referee for some illuminating remarks about formal Cartier divisors.

**Theorem 4.** *Let  $\tilde{X}$  be as in Lemma 3.1, with  $\dim \tilde{X} = 3$ , and let  $E$  be an effective divisor on  $\tilde{X}$  with exceptional support such that  $-E$  is  $\pi$ -ample. Let  $\tilde{Y} \subset \tilde{X}$  be a general member of the linear system  $|V|$ , and let  $\mathfrak{X}$  denote the formal completion of  $\tilde{X}$  along  $\tilde{Y}$ . Then the map*

$$\rho_{\mathfrak{X}} : \text{Pic } \tilde{X} \rightarrow \text{Pic } \mathfrak{X}$$

*has the following properties.*

- (i) *The kernel of  $\rho_{\mathfrak{X}}$  is freely generated by the classes of irreducible  $\pi$ -exceptional divisors with 0-dimensional image under  $\pi$ .*
- (ii) *The cokernel of  $\rho_{\mathfrak{X}}$  is generated by the classes of exceptional divisors  $F$  on  $\tilde{Y}$  such that  $\dim \pi(\text{supp } F) = 0$  (in particular, the corresponding line bundles on  $\tilde{Y}$  do extend to formal line bundles).*
- (iii) *With the notation introduced above (proof of Corollary 3.16), let  $A$  denote the quotient of the free abelian group on  $F_{ij}$ ,  $s + 1 \leq i \leq t$ ,  $1 \leq j \leq s_i$ , by the subgroup generated by  $\sum_{j=1}^{s_i} F_{ij}$ , for  $s + 1 \leq i \leq t$ . Then there is a natural isomorphism  $\text{Coker}(\rho_{\mathfrak{X}}) \cong A$ .*
- (iv) *Let  $\pi : \tilde{X} \rightarrow X$ ,  $\pi : \tilde{Y} \rightarrow Y$  be the Stein factorizations. There is a natural isomorphism*

$$\text{Coker}(\text{Cl}(X) \rightarrow \text{Cl}(Y)) \cong \text{Coker}(\text{Pic}(\mathfrak{X}) \rightarrow \text{Pic}(\tilde{Y})).$$

Most of these conclusions have already been obtained in the course of the above proof. The only remaining assertion to prove is that all the line bundles  $\mathcal{O}_{\tilde{Y}}(F_{ij})$  do extend to formal line bundles on  $\mathfrak{X}$ . We thank the referee for suggesting a proof.

In fact, for each  $i$  as in (iii), the divisor  $E_i$  defines a formal Cartier divisor  $\widehat{E}_i$  on  $\mathfrak{X}$  with support  $E_i \cap \widetilde{Y}$ , which is a disjoint union of the closed sets  $F_{ij}$ ,  $1 \leq j \leq s_i$ . Hence each connected component of the support defines a formal Cartier divisor  $\widehat{F}_{ij}$  on  $\mathfrak{X}$  (one may first check a similar assertion for Cartier divisors on each of the schemes  $\mathfrak{X}_m$ , all of which have reduced scheme  $\widetilde{Y}$ , for example). The associated formal line bundles are the desired extensions.

#### 4. Application to 1-motives

In [4], §10, Deligne defined 1-motives over  $k$  as complexes  $[L \rightarrow G]$ , where  $L$  is a lattice (free abelian group of finite rank with a continuous action of the absolute Galois group of  $k$ ), and  $G$  a semi-abelian  $k$ -variety. This gives an algebraic way of “defining” certain (co)homology groups of a variety, in a manner analogous to the way in which the Jacobian of a non-singular projective curve “defines” its first (co)homology group algebraically. Over  $\mathbb{C}$ , 1-motives have a transcendental description using certain special types of mixed Hodge structures, and there is an equivalence of categories between 1-motives over  $\mathbb{C}$  and the full subcategory of these special types of mixed Hodge structures. In particular, there is an underlying philosophy (“Deligne’s Conjecture”, some aspects of which have been proved in [1], [14]) that, if one can construct a 1-motive transcendently using some “part” of the mixed Hodge structure of an algebraic variety, then there must be an algebraic construction of that 1-motive as well, valid over more general ground fields. Further, if some operation between 1-motives can be constructed transcendently, there must be an algebraic construction of it as well, and properties of such an operation (e.g. injectivity, isomorphism) should have algebraic proofs.

In [2], a 1-motive  $\text{Alb}^+(X)$ , the *cohomological Albanese 1-motive*, has been associated to any variety  $X$  over a field  $k$  of characteristic 0. If  $X$  is proper, this is a semi-abelian variety over  $k$ , and if  $X$  is also non-singular, it coincides with the “classical” Albanese variety. If  $k = \mathbb{C}$ ,  $\text{Alb}^+(X)$  can be constructed analytically, using the mixed Hodge structure on  $H^{2n-1}(X, \mathbb{Z}(n))$ , where  $n = \dim X$ , in a manner generalizing the analytic construction of the Albanese variety of a non-singular proper complex variety. For a proper, possibly singular complex variety  $X$ , one has a formula

$$\text{Alb}^+(X)(\mathbb{C}) = \text{Ext}_{\text{MHS}}^1(\mathbb{Z}, H^{2n-1}(X, \mathbb{Z}(n)))$$

where the right side is the group of extensions in the (abelian) category of mixed Hodge structures.

If  $X$  is projective,  $Y \subset X$  is a (reduced, effective) Cartier divisor, then there is a *Gysin map*  $\text{Alb}^+(Y) \rightarrow \text{Alb}^+(X)$ , constructed algebraically in [2], and which in case  $k = \mathbb{C}$  corresponds to the Gysin map (modulo torsion)  $H^{2n-3}(Y, \mathbb{Z}(n-1)) \rightarrow H^{2n-1}(X, \mathbb{Z}(n))$  in topology (which is a morphism of mixed Hodge structures).

In case  $X$  is projective over  $\mathbb{C}$ , and  $Y$  is a general hyperplane section (here, “general” means “in a Zariski open set of the parameter variety”), then it is shown in [3] that  $\text{Alb}^+(Y) \rightarrow \text{Alb}^+(X)$  is an isomorphism when  $\dim X = n \geq 3$ ; this is an important step in the proof of the Roitman theorem for singular projective complex varieties (the main result of [3]). In case  $X$  (and hence  $Y$ ) is non-singular, this is a particular case of the Lefschetz hyperplane theorem. The proof of this isomorphism in [3] is transcendental, ultimately relying on the local structure (in the Euclidean topology) of a morphism of complex varieties, which is given by the theory of Whitney stratifications (see [15], or [8], for example).

This suggests that, if  $X$  is projective over a field  $k$  of characteristic 0, of dimension  $\geq 3$ , and  $Y \subset X$  is a general hyperplane section, then the Gysin map  $\text{Alb}^+(Y) \rightarrow \text{Alb}^+(X)$  is an isomorphism; further, the “philosophy of 1-motives” suggests that there is a purely algebraic proof of this fact.

The validity of the isomorphism over an arbitrary  $k$  of characteristic 0 can be deduced from the case  $k = \mathbb{C}$ . An algebraic proof, on the other hand, can be obtained as follows.

It is easy to see that the general case follows from the case when  $k$  is algebraically closed, so we assume this holds. Next, the category of 1-motives admits a notion of Cartier duality, which is an auto-antiequivalence of the category. So it suffices to show that the Cartier dual to  $\text{Alb}^+(Y) \rightarrow \text{Alb}^+(X)$  is an isomorphism.

It is shown in [2] that the Cartier dual of  $\text{Alb}^+(X)$  is another 1-motive, explicitly described as follows (implicitly, this gives an algebraic description of  $\text{Alb}^+(X)$ ).

Let  $\pi : X' \rightarrow X$  be the normalization map,  $\text{Cl}(X')$  the divisor class group of the normal projective variety  $X'$ , and  $\text{Cl}^0(X')$  the largest divisible subgroup. Then  $\text{Cl}^0(X')$  is naturally identified with (the  $k$ -points of) an abelian variety (which can be identified with the Picard variety of any resolution of singularities). Let  $L_X$  denote the group of all Weil divisors  $D$  on  $X'$  such that:

- (i)  $\pi_*(D) = 0$  as a cycle on  $X$ .
- (ii)  $[D] \in \text{Cl}(X)$  lies in the subgroup  $\text{Cl}^0(X')$ .

If  $\text{Div}(X'/X)$  denotes the group of Weil divisors  $D$  on  $X'$  with  $\pi_*(D) = 0$  as

a cycle on  $X$ , then

$$(7) \quad L_X = \ker(\text{Div}(X'/X) \rightarrow \text{Cl}(X')/\text{Cl}^0(X')).$$

Note that  $\text{Div}(X'/X)$  is a subgroup of the group of Weil divisors on  $X'$  which are supported on  $\pi^{-1}(X_{\text{sing}})$ ; in particular,  $\text{Div}(X'/X)$  is a free abelian of finite rank. Thus  $L_X$  is a free abelian group of finite rank, and the obvious homomorphism  $L_X \rightarrow \text{Cl}^0(X')$  defines a 1-motive; this is the Cartier dual to  $\text{Alb}^+(X)$ .

We are given that  $Y$  is a general hyperplane section of  $X$  in a certain projective embedding; the pull-back of the corresponding very ample linear system to  $X'$  gives an ample, base-point free linear system on  $X'$ . Hence  $Y' = Y \times_X X'$  is a general member of this linear system on  $X'$ , and is thus normal, by Bertini's theorem, so that  $Y' \rightarrow Y$  is the normalization of  $Y$ . There is also an associated restriction map  $\text{Cl}(X') \rightarrow \text{Cl}(Y')$ , such that by Theorem 1 above, it is an isomorphism when  $n = \dim X \geq 4$ , and is an injection with finitely generated cokernel if  $n = 3$ . Hence it induces an isomorphism  $\text{Cl}^0(X') \rightarrow \text{Cl}^0(Y')$  between abelian varieties, and an inclusion on quotients  $\text{Cl}(X')/\text{Cl}^0(X') \rightarrow \text{Cl}(Y')/\text{Cl}^0(Y')$ , if  $\dim X \geq 3$ .

Now the Gysin map  $\text{Alb}^+(Y) \rightarrow \text{Alb}^+(X)$  is Cartier dual to a map of 1-motives (i.e., to a map between 2-term complexes)

$$[L_X \rightarrow \text{Cl}^0(X')] \longrightarrow [L_Y \rightarrow \text{Cl}^0(Y')]$$

where  $\text{Cl}^0(X') \rightarrow \text{Cl}^0(Y')$  is the above isomorphism, induced by the restriction homomorphism  $\text{Cl}(X') \rightarrow \text{Cl}(Y')$ . It remains to see that, for general  $Y$ , this map is an isomorphism of 1-motives, i.e., the map  $L_X \rightarrow L_Y$  is also an isomorphism.

Since  $Y' = Y \times_X X'$ , a functorial property of the refined Gysin homomorphism defined in [7] implies that if  $D$  is a Weil divisor on  $X'$  with  $\pi_*(D) = 0$  as a cycle on  $X$ , then the cycle  $[D \cap Y']$  has the property that  $\pi_*[D \cap Y'] = 0$  as a cycle on  $Y$ . This gives us a map  $L_X \rightarrow L_Y$ . It is shown in [2] that this is the map corresponding to the Cartier dual of the Gysin map.

Since  $Y$  is a general member of a very ample linear system on  $X$ , where  $\dim X \geq 3$ , it is clear (using Bertini's theorem) that if  $D$  is an irreducible Weil divisor in  $Y'$  lying over the singular locus of  $Y$ , then there is a unique irreducible Weil divisor  $D_1$  in  $X'$ , lying over the singular locus of  $X$ , such that  $D = D_1 \cap Y'$  as divisors. This is because  $Y_{\text{sing}} = X_{\text{sing}} \cap Y$ , giving a bijection between the codimension 1 irreducible components of  $X_{\text{sing}}$  and  $Y_{\text{sing}}$ , which also gives a bijection between the codimension 1 irreducible components of  $\pi^{-1}(X_{\text{sing}}) \subset X'$  and  $\pi^{-1}(Y_{\text{sing}}) \subset Y'$ . Hence, for general  $Y$  as above, we have that  $\text{Div}(X'/X) \rightarrow \text{Div}(Y'/Y)$  is an isomorphism. The formula

(7) applied to  $X$  and to  $Y$ , together with the fact (from Theorem 1) that  $\mathrm{Cl}(X')/\mathrm{Cl}^0(X') \rightarrow \mathrm{Cl}(Y')/\mathrm{Cl}^0(Y')$  is injective, implies that  $L_X \rightarrow L_Y$  is an isomorphism, as was to be shown.

### 5. Some refinements, and statements in any characteristic

In this brief section, we make connections with the classical theory of Weil divisors, in the style of Weil [16] and Lang, as exposed in Lang's book [12], following comments of the referee. This gives another perspective to the above results, and yields also some statements in arbitrary characteristic.

First, in Theorem 1, one can improve the statement in the following ways.

- (i) There is a dense Zariski open set  $\Omega \subset |V|$  so that, if  $K$  is any algebraically closed extension field of  $k$ ,  $X_K = X \times_k K$ , and  $Y_K \subset X_K$  is a member of the base-changed linear system  $|V \otimes_k K|$  on  $X_K$ , corresponding to a  $K$ -point of  $\Omega$ , then the theorem holds for the pair  $(X_K, Y_K)$ . As stated, Theorem 1 does yield such an open subset of  $|V \otimes_k K|$ , but in fact it may be taken to be  $\Omega_K$ . It is not difficult to modify the proof given above to yield this conclusion as well.
- (ii) When  $\dim X = 3$ , the cokernel of the (injective) map on class groups is in fact torsion-free. This follows from the proof given, since from Theorem 4, it boils down to the assertions that the cokernel of  $\mathrm{Pic} \mathfrak{X} \rightarrow \mathrm{Pic} \tilde{Y}$  is torsion-free. If  $\mathcal{K} = \ker(\mathcal{O}_{\mathfrak{X}}^* \rightarrow \mathcal{O}_{\tilde{Y}}^*)$ , then  $\mathcal{K}$  is a sheaf of  $\mathbb{Q}$ -vector spaces on the topological space  $\tilde{Y}$  (as may be immediately verified on suitable affine open subsets), and we have an exact sequence

$$\mathrm{Pic} \mathfrak{X} \rightarrow \mathrm{Pic} \tilde{Y} \rightarrow H^2(\tilde{Y}, \mathcal{K})$$

where the last term is a  $\mathbb{Q}$ -vector space.

A different way of seeing (ii) is by a transcendental argument, using a suitable Lefschetz theorem, as in the Appendix: in Theorem 7, for  $i = \hat{n}$ , the cokernel is torsion-free and the conclusions of that theorem also hold for cohomology with  $\mathbb{Z}/n\mathbb{Z}$  coefficients, for any  $n > 0$ .

The referee has also pointed out that for a geometrically integral projective variety  $X$  over a field  $k$ , which is smooth in codimension 1, one can associate to it an abelian  $k$ -variety  $\mathrm{Pic}_W(X)$  (the ‘‘Picard variety in the sense of Weil’’), such that when  $k = \bar{k}$ , the group of  $k$ -rational points  $\mathrm{Pic}_W(X)(k)$  coincides with the group  $\mathrm{Cl}^0(X)$ , the maximal divisible subgroup of the group  $\mathrm{Cl}(X)$  of Weil divisors modulo linear equivalence. Further, the Weil-Neron-Severi

group  $NS_W(X) = Cl(X)/Cl^0(X)$  is finitely generated. With this notation, we sketch the referee’s argument to prove the following result.

**Theorem 5.** *Let  $X$  be an irreducible projective  $k$ -variety of dimension  $d \geq 3$ , which is regular in codimension 1, where  $k$  is an algebraically closed field of any characteristic. Let  $f : X \rightarrow \mathbb{P}_k^N$  be an embedding. Let  $\overline{K}$  be the algebraic closure of the function field  $K = k(\widehat{\mathbb{P}}_k^N)$  of the dual projective space, and let  $Y_K \subset X_K$  be the generic hyperplane section. Then,*

- (i)  $Pic_W(X)_K \rightarrow Pic_W(Y_K)$  is an isomorphism;
- (ii) the composition

$$NS_W(X) \rightarrow NS_W(X_{\overline{K}}) \rightarrow NS_W(Y_{\overline{K}})$$

is injective.

The isomorphism between Weil-Picard varieties is a consequence of [12] VIII, Theorem 4. The injectivity on Weil-Neron-Severi groups is reduced to a result of Weil [16]. We must show that, if  $\mathcal{L}$  is a line bundle on  $X$ , whose pull-back to  $(Y_K)_{reg}$ , the smooth locus of  $Y_K$ , is algebraically equivalent to 0, then  $\mathcal{L}$  is algebraically equivalent to 0 on  $X$ . From [12] VI, Theorem 1, the pull-back of  $\mathcal{L}$  to  $(Y_K)_{reg}$  determines a  $K$ -rational point of  $Pic_W(Y_K) = Pic_W(X)_K$ . Since  $K$  is a pure transcendental extension of  $k$ , this must determine a  $k$ -rational point of  $Pic_W(X)$ . Hence, changing  $\mathcal{L}$  by the class of some point of  $Pic_W(X)$ , we may assume  $\mathcal{L}$  has trivial pull-back to  $(Y_K)_{reg}$ . Now Theorem 2 of Weil [16] implies that  $\mathcal{L}$  is itself trivial on  $X$ .

### 6. Appendix: Grothendieck-Lefschetz theorem for complex projective varieties

In this appendix, we shall sketch the proof of the following theorem using results from stratified Morse theory, as explained to us by Najmuddin Fakhruddin.

**Theorem 6.** *Let  $X$  be a smooth projective variety of dimension at least 4 defined over the field of complex numbers,  $\mathbb{C}$ . Let  $\mathcal{L}$  be a big line bundle over  $X$  generated by global sections. If  $Y$  denotes a general member of the linear system  $|\mathcal{L}|$ , then one has an exact sequence*

$$0 \rightarrow K \rightarrow Pic(X) \rightarrow Pic(Y) \rightarrow Q \rightarrow 0$$

where  $K$  is the (free) subgroup generated by divisors in  $X$  which map to points under the generically finite map  $X \xrightarrow{\pi} \mathbb{P}(H^0(X, \mathcal{L}))$  and  $Q$  is the group generated by the irreducible components of the restriction of divisors in  $X$  which map to curves under  $\pi$ .

Theorem 6 is an immediate consequence of Corollaries 6.4 and 6.6 below.

In what follows, all cohomologies that we consider are singular cohomology of the underlying analytic space(s), with  $\mathbb{Z}$ -coefficients. Recall that, for any  $\mathbb{C}$ -variety, these cohomology groups support mixed Hodge structures, which are functorial for morphisms between varieties (see [4]). The proofs of Corollaries 6.4 and 6.6 are reduced to assertions about the homomorphisms  $H^i(X) \rightarrow H^i(Y)$  for  $i = 1, 2$ , using the following standard lemma.

**Lemma 6.1.** *Let  $W$  be a smooth proper  $\mathbb{C}$ -variety. Then there are isomorphisms, functorial for morphisms of  $\mathbb{C}$ -varieties,*

$$\text{Pic}^0(W) \cong \frac{H^1(W) \otimes \mathbb{C}}{F^1 H^1(W) \otimes \mathbb{C} + H^1(W)},$$

$$\text{NS}(W) = \ker \left( H^2(W) \rightarrow \frac{H^2(W) \otimes \mathbb{C}}{F^1 H^2(W) \otimes \mathbb{C}} \right).$$

*Proof.* From Serre’s GAGA, it follows that  $\text{Pic}(W) \cong \text{Pic}(W_{an})$ , where the latter denotes the group of isomorphism classes of analytic line bundles. Using the exponential sheaf sequence, and the Hodge decomposition, we obtain the above isomorphisms in a standard way, where  $\text{Pic}^0(W) = \ker(\text{Pic}(W) \rightarrow H^2(W))$  is the maximal divisible subgroup, and the Neron-Severi group  $\text{NS}(W) = \text{image}(\text{Pic}(W) \rightarrow H^2(W))$  is finitely generated.  $\square$

We now state the following consequence of the relative Lefschetz theorem with large fibres (see [8], page 195).

**Theorem 7.** *Let  $W$  be an  $n$ -dimensional non-singular connected algebraic variety. Let  $\pi : W \rightarrow \mathbb{P}^N$  be a morphism and let  $H \subset \mathbb{P}^N$  be a general linear subspace of codimension  $c$ . Define  $\phi(k)$  to be the dimension of the set of points  $z \in \mathbb{P}^N$  such that the fibre  $\pi^{-1}(z)$  has dimension  $k$ . (If this set is empty, define  $\phi(k) = -\infty$ .) Then the homomorphism induced by restriction,*

$$H^i(W, \mathbb{Z}) \rightarrow H^i(\pi^{-1}(H), \mathbb{Z})$$

*is an isomorphism for  $i < \hat{n}$  and is an injection for  $i = \hat{n}$ , where*

$$\hat{n} = n - \sup_k (2k - (n - \phi(k)) + \inf(\phi(k), c - 1)) - 1.$$

In the situation of Theorem 6, we first take  $W = X$ ,  $\pi$  a generically finite map and  $H$  a general hyperplane. Then one can easily check that  $\hat{n} \geq 1$  in this case. Let  $X'$  be the (open) subvariety of  $X$  defined by removing all divisors which map to points under  $\pi$ , and  $X''$  be the subvariety obtained by further removing divisors which map to curves under  $\pi$ . In these two cases, for the restriction of  $\pi$  to  $X'$  and  $X''$ , one can check that  $\hat{n} \geq 2$  and  $\geq 3$  respectively.

Let  $Y = \pi^{-1}(H)$ , and let  $Y'$  and  $Y''$  be defined similarly in  $X'$  and  $X''$  respectively. Since a general hyperplane section in  $\mathbb{P}^N$  misses points, one notes immediately that  $Y' = Y$ .



**Lemma 6.2.** *If  $V \subset W$  is a dense Zariski open subset of a non-singular proper variety  $W$ , then*

- (i)  $H^1(W, V) = 0$ , and  $H^2(W, V)$  is a free abelian group, with a basis given by the irreducible divisors supported on  $W \setminus V$  (in particular, it is pure of weight 2);
- (ii)  $H^3(W, V)$  is a free abelian group, supporting a mixed Hodge structure with weights  $\geq 3$ .

*Proof.* Let  $W \setminus V = D$ , and let  $S \subset D$  be the union of the singular locus of  $D$ , together with all irreducible components of  $D$  of codimension  $\geq 2$  in  $X$ . Then  $D \setminus S = \coprod_j D_j$  where  $D_j \subset X \setminus S$  are irreducible, non-singular divisors.

We first observe that  $H^i(W, W \setminus S) = 0$  for  $i \leq 3$ , since  $S \subset W$  has (complex) codimension  $\geq 2$ . This implies that  $H^i(W, V) \rightarrow H^i(W \setminus S, V)$  are isomorphisms for  $i \leq 3$ . Since  $(W \setminus S) \setminus V = \coprod_j D_j$ , we have Thom-Gysin isomorphisms  $H^i(W \setminus S, V) \cong \bigoplus_j H^{i-2}(D_j)(-1)$  for all  $i \geq 0$  (where the Tate twist  $(-1)$  increases the weights by 2). In particular, we have  $H^i(W \setminus S, V) = 0$  for  $i < 2$ ,  $H^0(D_j) = \mathbb{Z}$  (the trivial MHS), and  $H^1(D_j) = \text{Hom}(H_1(D_j, \mathbb{Z}), \mathbb{Z})$  is a torsion-free abelian group, which supports a MHS of weights  $\geq 1$ .  $\square$

**Corollary 6.3.**  $H^1(X) \cong H^1(Y)$ .

*Proof.* We have a factorization  $H^1(X) \rightarrow H^1(X') \rightarrow H^1(Y') = H^1(Y)$ , since  $Y = Y' \subset X' \subset X$ . By Theorem 7, we have that  $H^1(X') \rightarrow H^1(Y') = H^1(Y)$  is an isomorphism. In particular,  $H^1(X')$  supports a pure Hodge structure of weight 1. Consider the exact cohomology sequence

$$H^1(X, X') \rightarrow H^1(X) \rightarrow H^1(X') \rightarrow H^2(X, X') \rightarrow \dots$$

By Lemma 6.2 with  $W = X$ ,  $V = X'$ , we have  $H^1(X, X') = 0$ , while  $H^2(X, X')$  is torsion free, and it is pure of weight 2, generated by the cohomology classes of the irreducible divisors in  $X \setminus X'$ . Hence the boundary map  $H^1(X') \rightarrow H^2(X, X')$  is the zero map. Thus  $H^1(X) \rightarrow H^1(X')$  is an isomorphism.  $\square$

**Corollary 6.4.**  $\text{Pic}^0(X) \rightarrow \text{Pic}^0(Y)$  is an isomorphism.

*Proof.* The isomorphism in Corollary 6.3 is compatible with the respective Hodge structures, and so by Lemma 6.1, induces an isomorphism on  $\text{Pic}^0$  groups.  $\square$

**Proposition 6.5.** *Let  $X$  and  $Y$  be as in Theorem 6. One then has an exact sequence*

$$0 \rightarrow K \rightarrow H^2(X) \rightarrow H^2(Y) \rightarrow Q \rightarrow 0$$

where  $K$  (as in Theorem 6) is generated by divisors which map to points under  $\pi$  and  $Q$  is generated by the divisors in  $Y$  which map to points under  $\pi$ .

*Proof.* Consider the diagram:

$$\begin{array}{ccccccc}
 H^1(X) & \xrightarrow{a_1} & H^1(X'') & \xrightarrow{a_2} & H^2(X, X'') & \xrightarrow{a_3} & H^2(X) \\
 \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_3 & & \downarrow \psi_4 \\
 H^1(Y) & \xrightarrow{b_1} & H^1(Y'') & \xrightarrow{b_2} & H^2(Y, Y'') & \xrightarrow{b_3} & H^2(Y) \\
 & & & & & & \\
 & & & & & \xrightarrow{a_4} & H^2(X'') & \xrightarrow{a_5} & H^3(X, X'') \\
 & & & & & \downarrow \psi_5 & & & \downarrow \\
 & & & & & \xrightarrow{b_4} & H^2(Y'') & \xrightarrow{b_5} & H^3(Y, Y'')
 \end{array}$$

Here the horizontal sequences are the cohomology long exact sequences corresponding to suitable pairs. The proposition amounts to the assertions that there are isomorphisms

$$\ker \psi_3 \xrightarrow{\cong} \ker \psi_4,$$

$$\text{Coker } \psi_3 \xrightarrow{\cong} \text{Coker } \psi_4.$$

From Theorem 7,  $\psi_2$  and  $\psi_5$  are isomorphisms, while  $\psi_1$  is an isomorphism from Corollary 6.3. We claim that  $\psi_5$  induces an isomorphism

$$\text{image } a_4 \xrightarrow{\cong} \text{image } b_4.$$

This follows by an argument using weights. Let  $W_2 H^2(X'') \subset H^2(X'')$ ,  $W_2 H^2(Y'') \subset H^2(Y'')$  be the subgroups obtained as inverse images of the corresponding weight subspaces of cohomology with rational coefficients. Since  $\psi_5$  is an isomorphism of mixed Hodge structures, it induces an isomorphism  $W_2 H^2(X'') \cong W_2 H^2(Y'')$ . By Lemma 6.2,  $H^3(X, X'')$  and  $H^3(Y, Y'')$  are torsion free, and have weights  $\geq 3$ , while  $H^2(X)$ ,  $H^2(Y)$  are pure of weight 2. Hence we have

$$W_2 H^2(X'') = \text{image } a_4,$$

$$W_2 H^2(Y'') = \text{image } b_4,$$

and so  $\psi_5$  induces an isomorphism between these image subgroups.

Thus we have a commutative diagram with exact rows, and vertical isomorphisms as shown:

$$\begin{array}{ccccccccccc}
 H^1(X) & \xrightarrow{a_1} & H^1(X'') & \xrightarrow{a_2} & H^2(X, X'') & \xrightarrow{a_3} & H^2(X) & \xrightarrow{a_4} & \text{image } a_4 & \rightarrow & 0 \\
 \cong \downarrow \psi_1 & & \cong \downarrow \psi_2 & & \downarrow \psi_3 & & \downarrow \psi_4 & & \cong \downarrow \psi_5 & & \\
 H^1(Y) & \xrightarrow{b_1} & H^1(Y'') & \xrightarrow{b_2} & H^2(Y, Y'') & \xrightarrow{b_3} & H^2(Y) & \xrightarrow{b_4} & \text{image } b_4 & \rightarrow & 0
 \end{array}$$

A version of the 5-lemma now implies that  $\ker \psi_3 \rightarrow \ker \psi_4$  and  $\text{Coker } \psi_3 \rightarrow \text{Coker } \psi_4$  are isomorphisms, as desired.  $\square$

**Corollary 6.6.** *There is an exact sequence*

$$0 \rightarrow K \rightarrow \text{NS}(X) \rightarrow \text{NS}(Y) \rightarrow Q \rightarrow 0.$$

*Proof.* This follows from the proposition, and Lemma 6.1, since the explicit descriptions of  $K$  and  $Q$  imply that  $K \subset \text{NS}(X)$ , and  $\text{NS}(Y) \twoheadrightarrow Q$ .  $\square$

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