EXTENSIONS OF VECTOR BUNDLES WITH APPLICATION TO NOETHER-LEFSCHETZ THEOREMS

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Abstract. Given a smooth, projective variety Y over an algebraically closed field of characteristic zero, and a smooth, ample hyperplane section $X \subset Y$, we study the question of when a bundle E on X, extends to a bundle $\mathcal{E}$ on a Zariski open set $U \subset Y$ containing X. The main ingredients used are explicit descriptions of various obstruction classes in the deformation theory of bundles, together with Grothendieck-Lefschetz theory. As a consequence, we prove a Noether-Lefschetz theorem for higher rank bundles, which recovers and unifies the Noether-Lefschetz theorems of Joshi and Ravindra-Srinivas.

1. Introduction

We work over an algebraically closed field of chacteristic zero, which we denote by $\mathbf{k}$.

One of the most fundamental results in algebraic geometry are the *Lefschetz theorems* which state that if Y is a smooth, projective variety and $X \subset Y$ is a smooth member of an ample linear system, then the Picard groups of Y and X are isomorphic provided $\dim X \geq 3$; when $\dim X = 2$, the same is true, if in addition, we assume that X is a *very general* member of a *sufficiently ample* linear system. These theorems imply in particular, that any line bundle on X *extends* to a line bundle on Y. From this point of view, one may ask if there are analogous results for higher rank bundles.

Let L be an ample line bundle on a smooth projective variety Y, and X be a smooth member of the associated linear system $|\mathrm{H}^0(Y, L)|$. For $k \geq 0$, let X_k denote the k-th order thickening of X in Y, so that $X_0 = X$. The obstruction for a bundle E on X_{k-1} to lift to a bundle on X_k is a class $\eta_E \in \mathrm{H}^2(X, \mathcal{E}nd\, E \otimes \mathcal{O}_Y(-kX)_{|X})$ (see §2). Clearly, the vanishing of these classes is a necessary condition for E to extend to the ambient variety Y. The fact that these classes depend on the bundle E is one of the main points of departure when we study extension questions for higher rank bundles; when E is a line bundle, $\mathcal{E}nd\, E \cong \mathcal{O}_X$, and so the obstruction classes do not depend on the bundle *per se*. Consequently, one cannot hope to get a uniform result for *all* bundles of any given rank.

Another noteworthy point when studying higher rank bundles is that even if these obstruction classes vanish, in most cases the bundle extends only as a reflexive sheaf on Y, and would need to satisfy additional conditions in order for the extension to be a bundle. Consider for example, the inclusion $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$, and let $\pi_x : \mathbb{P}^{n+1} \setminus \{x\} \to \mathbb{P}^n$ denote the projection map for $x \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$. Since the composition $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \setminus \{x\} \to \mathbb{P}^n$ is the identity map, the pull-back bundle $\pi_x^* E$ for any bundle E on $\mathbb{P}^n$ is an extension of E on the variety $\mathbb{P}^{n+1} \setminus \{x\}$. Even if $\pi_x^* E$ extends to a bundle on $\mathbb{P}^{n+1}$, note that there exists an $N > n$ such that E does not extend to a bundle on $\mathbb{P}^N$. For if this were to be so, then by the *Babylonian theorems* (see [1] for the rank 2 case, and [18] for arbitrary rank), E would have to be a sum of line bundles.

The following result (see also [6]) summarises the discussion above.

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Theorem 1. *Let Y be a smooth, projective variety of dimension at least 4, and $X \subset Y$ be a smooth, ample hyperplane section. Let E be a bundle on X satisfying the property that* $\mathrm{H}^2(X, \mathcal{E}nd\, E \otimes \mathcal{O}_Y(-kX)_{|X})$ *for all* $k \in \mathbb{Z}_{>0}$. *Then there exists a Zariski open set* $U \subset Y$ *containing X and a bundle $\mathcal{E}$ on U such that* $\mathcal{E} \otimes \mathcal{O}_U \cong E$.

When Y is a threefold, we have the following result.

Theorem 2. *Let Y be a smooth 3-fold and $X \subset Y$ be a general, ample hyperplane section of Y. Let E be a bundle on X such that the "multiplication" map*

$$\mathrm{H}^0(X, \mathcal{E}nd\, E \otimes K_X(a)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(b)) \to \mathrm{H}^0(X, \mathcal{E}nd\, E \otimes K_X(a+b)) \tag{1}$$

is surjective $\forall\, a,\ b \geq 0$. *Then there exists a Zariski open set $U \subset Y$ containing X and a bundle* $\widetilde{E}$ *on U such that* $\widetilde{E} \otimes \mathcal{O}_U \cong E$.

When E is a line bundle, the above result yields the *Formal Noether-Lefschetz theorem* proved in [17]. Using our formalism, and a remark by M.V.Nori, we are also able to prove the following version of the Noether-Lefschetz theorem for divisor class groups (compare with the result of [17]) which generalises the main result of a paper of Joshi [12].

Theorem 3. *Let Y be a normal projective 3-fold over k, $\mathcal{O}_Y(d)$ be an ample, invertible sheaf such that the linear system* $\mathrm{V} := \mathrm{H}^0(\mathrm{Y}, \mathcal{O}_\mathrm{Y}(\mathrm{d}))$ *is base point free. Let X denote a very general member of $|\mathrm{V}|$ and, $\pi : \widetilde{Y} \to Y$ and $\widetilde{X} := \widetilde{Y} \times_Y X$ be the desingularisations of Y and X respectively. Assume further that,*

(i) $\mathrm{H}^1(\widetilde{Y}, \Omega^2_{\widetilde{Y}} \otimes \pi^*\mathcal{O}_Y(d)) = 0$.
(ii) *The multiplication map*

$$\mathrm{H}^0(Y, \mathcal{O}_Y(d)) \otimes \mathrm{H}^0(Y, K_Y \otimes \mathcal{O}_Y(d)) \to \mathrm{H}^0(Y, K_Y \otimes \mathcal{O}_Y(2d))$$

is surjective.

Then the restriction map of divisor class groups $\mathrm{Cl(Y)} \to \mathrm{Cl(X)}$ *is an isomorphism.*

1.1. **Outline of the proof and some remarks.** Given a bundle E on a smooth hypersurface $X \subset Y$, we suppose that we have been able to extend it to a bundle E_{k-1} on X_{k-1}, the $(k-1)$-st order thickening of X in Y. Then the obstruction for E_{k-1} to lift to a bundle E_k on X_k, is an element of the cohomology group $\mathrm{H}^2(X, \mathcal{E}nd\, E(-kX)_{|X})$. Thus we see that if the hypothesis of Theorem 1 is satisfied, then the bundle lifts to X_k, $\forall\, k > 0$ and so we have a projective system of bundles $\{E_k\}$. Let $\widehat{E}$ denote the inverse limit of this system; then $\widehat{E}$ is a formal vector bundle on $\widehat{Y}$. The conclusion then follows from Grothendieck's Lefschetz theory (see definition 2, section 4).

When Y is a threefold (so X is a surface), by Serre's theorems, $\mathrm{H}^2(X, \mathcal{E}nd\, E(kX)_{|X}) \neq 0$, for $k \ll 0$. The assumption that E lives on a "general" hypersurface X implies that the obstruction classes for E to deform to a "nearby" fibre X' in the universal family $\mathcal{X} \to S$ are all zero. This implies, by the hypothesis of Theorem 2, that the obstructions for E to extend across infinitesimal thickenings also vanish. The conclusion again follows by Grothendieck-Lefschetz theory.

As mentioned above, Theorem 3 recovers the theorem of Joshi [12] when X is smooth, and L is sufficiently ample. The idea of the proof in both the theorems is the same: first one proves the so-called *infinitesimal Noether-Lefschetz theorem* (INLT), as stated in [14] (see also [5]), and then one uses this and a standard "spreading out" argument to prove the *global* Noether-Lefschetz theorem. The difference in the two proofs is in the proof of the infinitesimal Noether-Lefschetz theorem (INLT): while in *op. cit.* this is achieved by showing that if the first Chern class of a line bundle L, on a smooth fibre X, $\mathrm{c}_1(L) \in \mathrm{H}^1(X, \Omega^1_X)$, in the universal family $\mathcal{X} \to S$ deforms to a neighbouring fibre X', then it lifts to the first Chern class of a line bundle $\mathcal{L}$ on the ambient 3-fold Y. This is where both the hypotheses are used. While we further break up the proof of INLT into two steps: in the first step, we show that if a line bundle L on X deforms to a neighbouring fibre, then it lifts to a line bundle L_1 on the first order thickening $X_1 \subset Y$. This is where hypothesis (ii) in Theorem 3 is used. Next using hypothesis (i), we show that the first chern class of L_1 lifts to the first chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ on Y. Moreover our proof of the INLT is a consequence of the general theory of obstruction classes and deformation theory developed in §2 for arbitrary rank bundles.

The first algebraic proof of the Noether-Lefshetz theorem was Hodge-theoretic, based on the theory of infiinitesimal variations of Hodge structures introduced by Griffiths (see [3]). The theory was greatly developed by Green (see [7, 8]) and Voisin in an unpublished article as well as in [19, 20]. Related results may also be found in [13] and [2].

2. Acknowlegements

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3. Some results on deformation theory and obstruction classes for bundles

3.1. **Notation and set-up.** Let Y be a smooth, projective variety over an algebraically closed field of characteristic 0 of dimension $n+1$. Let $\mathcal{O}_Y(1)$ be an ample line bundle, and assume that, for a positive integer d, $V \subset \mathrm{H}^0(Y, \mathcal{O}_Y(d))$ is a base point free linear system. Let $S \subset \mathbb{P}(V^*)$ denote a Zariski open set parametrising smooth hypersurfaces, $\mathcal{X} := \{(y, s) | s(y) = 0\} \subset Y \times S$ denote the universal hypersurface, and $p : \mathcal{X} \to Y$, and $q : \mathcal{X} \to S$ denote the two projections. For a point $s_0 \in S$, let $\mathfrak{m} \subset \mathcal{O}_S$ denote its (maximal) ideal sheaf and let $X \subset Y$ be the smooth hypersurface parametrized by $s_0 \in S$.

Let X_k be the k-th infinitesimal neighbourhood of X in Y: so if X is the zero locus of a section $s_0 \in \mathrm{H}^0(Y, \mathcal{O}_Y(d))$, then X_k is the zero locus of the section $s_0^{k+1} \in \mathrm{H}^0(Y, \mathcal{O}_Y((k+1)d)$. We will let $\widehat{Y}$ denote the formal completion of Y along X. Similarly, let $\mathcal{X}_k := \mathcal{X} \times_S \mathcal{O}_S/\mathfrak{m}^{k+1}$ be the k-th order infinitesimal neighbourhood of $X = \mathcal{X}_s$ in $\mathcal{X}$. Note that since $\mathcal{X}_0 \to X_0 = X$ is an isomorphism, we have an inclusion of ideals $\mathcal{I}\mathcal{O}_{\mathcal{X}} \subset \mathfrak{m}\mathcal{O}_{\mathcal{X}}$ where $\mathcal{I} \cong \mathcal{O}_Y(-d)$ is the ideal sheaf of X in Y. This implies that there is a morphism of schemes $p_k : \mathcal{X}_k \to X_k \ \forall \ k \geq 0$, compatible with the morphism $p : \mathcal{X} \to Y$, and with the isomorphism $\mathcal{X}_0 \to X_0 = X$.

By a 1-*step resolution* of a sheaf E on a projective scheme T, we shall mean a sequence

$$0 \to F_1 \to F_0 \to E \to 0,$$

where $F_0 := \oplus_{i=1}^r \mathcal{O}_T(a_i)$ is a sum of line bundles such that the map $F_0 \to E$ is given by a set of generators of the graded module $\oplus \mathrm{H}^0(T, E \otimes \mathcal{O}_T(\nu))$. In particular, the map at the level of global sections

$$\mathrm{H}^0(T, F_0(\nu)) \to \mathrm{H}^0(T, E(\nu))$$

is a surjection for all $\nu \in \mathbb{Z}$, and the sheaf F_1 is the kernel of the map $F_0 \to E$. Furthermore, if $\mathrm{H}^1(T, F_0(\nu)) = 0$ for all $\nu \in \mathbb{Z}$, then $\mathrm{H}^1(T, F_1(\nu)) = 0$ for all $\nu \in \mathbb{Z}$.

For any sheaf $\mathcal{F}$ on Y, we will let $\overline{\mathcal{F}}$ denote its restriction to X.

3.2. **The basic spectral sequence.** Let E be a bundle on X_{k-1}. Any lift of E to a coherent sheaf $\mathcal{E}$ on X_k, is an element of $\mathrm{Ext}^1_{X_k}(E, \overline{E}(-kd))$ i.e., $\mathcal{E}$ sits in an exact sequence of $\mathcal{O}_{X_k}-$sheaves

$$0 \to \overline{E}(-kd) \to \mathcal{E} \to E \to 0. \tag{2}$$

It is a standard fact from deformation theory, that *the obstruction for E to lift to a coherent sheaf on X_k* is an element

$$\eta_E \in \mathrm{H}^2(X, \mathcal{E}nd\, \overline{E}(-kd)).$$

Similarly, if E is a bundle on $\mathcal{X}_{k-1}$, then any lift of E to a coherent sheaf $\mathcal{E}$ on $\mathcal{X}_k$ is an element of $\mathrm{Ext}^1_{\mathcal{X}_k}(E, \overline{E} \otimes S^kV^*)$ i.e., the lift $\mathcal{E}$ sits in an exact sequence of $\mathcal{O}_{\mathcal{X}_k}$-sheaves

$$0 \to \overline{E} \otimes S^kV^* \to \mathcal{E} \to E \to 0. \tag{3}$$

Here we have used the identification of the ideal sheaf of $\mathcal{X}_{k-1} \subset \mathcal{X}_k$:

$$\mathcal{I}_{\mathcal{X}_{k-1}/\mathcal{X}_k} \cong S^kV^* \otimes \mathcal{O}_X.$$

It is a standard fact from deformation theory, that *the obstruction for E to lift to a coherent sheaf on $\mathcal{X}_k$* is an element

$$\eta_E \in \mathrm{H}^2(X, \mathcal{E}nd\, \overline{E} \otimes S^kV^*).$$

The following *local criterion for flatness* (Proposition 2.2, [10]) tells us that such lifts are in fact vector bundles on X_k and $\mathcal{X}_k$ respectively.

Proposition 1. *Let $A' \to A$ be a surjective homomorphism of noetherian rings whose kernel J has square zero. Then an $A'-$module M' is flat over A' if and only if*

1. *$M := M' \otimes_A A'$ is at over A, and*
2. *The natural map $M \otimes_A J \to M'$ is injective.*

The following result is due to A.P.Rao.

Proposition 2. *Let A be a bundle on X_{k-1} and $\bar{B}$ be a bundle on X.*

(i) *Then there is an exact sequence*

$$0 \to \mathrm{H}^1(X, \bar{A}^\vee \otimes \bar{B}(-kd)) \to \mathrm{Ext}^1_{X_k}(A, \bar{B}(-kd)) \to \mathrm{H}^0(X, \bar{A}^\vee \otimes \bar{B}) \to \mathrm{H}^2(\bar{A}^\vee \otimes \bar{B}(-kd)).$$

(ii) *If A lifts to a bundle $\mathcal{A}$ on X_k, then*

$$0 \to \mathrm{H}^1(X, \bar{A}^\vee \otimes \bar{B}(-kd)) \to \mathrm{Ext}^1_{X_k}(A, \bar{B}(-kd)) \to \mathrm{H}^0(X, \bar{A}^\vee \otimes \bar{B}) \to 0$$

forms a split exact sequence.

Proof. (i) By the local-to-global *Ext* spectral sequence, we have a 4-term sequence

$$\begin{aligned} 0 \;\to\; & \mathrm{H}^1(X, \mathcal{H}om_{X_k}(A, \overline{B}(-kd))) \to \mathrm{Ext}^1_{X_k}(A, \overline{B}(-kd)) \to \mathrm{H}^0(X, \mathcal{E}xt^1_{X_k}(A, \overline{B}(-kd))) \\ & \to \mathrm{H}^2(X, \mathcal{H}om_{X_k}(A, \overline{B}(-kd))). \end{aligned} \tag{4}$$

Since $\mathcal{H}om_{X_k}(A, \overline{B}(-kd)) \cong \bar{A}^\vee \otimes \overline{B}(-kd)$, it is enough to prove

$$\mathcal{E}xt^1_{X_k}(A, \overline{B}(-kd)) \cong \bar{A}^\vee \otimes \overline{B}.$$

The inclusion $X_{k-1} \subset X_k$ yields an exact sequence

$$0 \to \mathcal{O}_X(-kd) \to \mathcal{O}_{X_k} \to \mathcal{O}_{X_{k-1}} \to 0. \tag{5}$$

We will first compute $\mathcal{E}xt^1_{X_k}(\mathcal{O}_{X_{k-1}}(-a), F)$ for any bundle F on X. To do this, we tensor (5) by $\mathcal{O}_{X_k}(-a)$ and apply $\mathcal{H}om_{X_k}(-, F)$ to get a long exact sequence

$$0\to\mathcal{H}om_{X_k}(\mathcal{O}_{X_{k-1}}(-a),F)\to\mathcal{H}om_{X_k}(\mathcal{O}_{X_k}(-a),F)\to\mathcal{H}om_{X_k}(\mathcal{O}_X(-a-kd),F)\to\mathcal{E}xt^1_{X_k}(\mathcal{O}_{X_{k-1}}(-a),F)\to 0$$

The first two terms are isomorphic to $F(a)$, and the third term is isomorphic to $F(a + kd)$. Thus we see that

$$\mathcal{E}xt^1_{X_k}(\mathcal{O}_{X_{k-1}}(-a), F) \cong F(a + kd). \tag{6}$$

Next, we consider a 1-step resolution of A on X_{k-1}:

$$0 \to G \to L_0 \to A \to 0, \tag{7}$$

Tensoring this sequence by $\mathcal{H}om_{X_k}(-, \overline{B}(-kd))$, we get a long exact sequence

$$\begin{array}{ccccccc} 0 \to & \mathcal{H}om_{X_k}(A, \overline{B}(-kd)) & \to & \mathcal{H}om_{X_k}(L_0, \overline{B}(-kd)) & \to & \mathcal{H}om_{X_k}(G, \overline{B}(-kd)) & \to \\ & \mathcal{E}xt^1_{X_k}(A, \overline{B}(-kd)) & \to & \mathcal{E}xt^1_{X_k}(L_0, \overline{B}(-kd)) & \to & \mathcal{E}xt^1_{X_k}(G, \overline{B}(-kd)) & \to \end{array} \tag{8}$$

Now the first three terms form the exact sequence

$$0 \to \bar{A}^\vee \otimes \overline{B}(-kd) \to \bar{L}_0^\vee \otimes \overline{B}(-kd) \to \overline{G}^\vee \otimes \overline{B}(-kd) \to 0. \tag{9}$$

Hence we have an exact sequence

$$0 \to \mathcal{E}xt^1_{X_k}(A, \overline{B}(-kd)) \to \mathcal{E}xt^1_{X_k}(L_0, \overline{B}(-kd)) \to \mathcal{E}xt^1_{X_k}(G, \overline{B}(-kd)). \tag{10}$$

where the middle term is isomorphic to $\bar{L}_0^\vee(kd) \otimes \overline{B}(-kd) = \bar{L}_0^\vee \otimes \overline{B}$ by (6).

We carry out the same steps for G by considering the following 1-step resolution on X_{k-1}:

$$0 \to G' \to L_1 \to G \to 0. \tag{11}$$

In this case we get an exact sequence

$$0 \to \mathcal{E}xt^1_{X_k}(G, \overline{B}(-kd)) \to \mathcal{E}xt^1_{X_k}(L_1, \overline{B}(-kd)) \cong \bar{L}_1^\vee \otimes \overline{B} \to \mathcal{E}xt^1_{X_k}(G', \overline{B}(-kd)). \tag{12}$$

From (10) and (12), we see that

$$\mathcal{E}xt^1_{X_k}(A, \overline{B}(-kd)) = \ker[\bar{L}_0^\vee \otimes \overline{B} \to \bar{L}_1^\vee \otimes \overline{B}]. \tag{13}$$

On the other hand, the resolutions of A and G above when put together yield a sequence

$$L_1 \to L_0 \to A \to 0$$

which is a resolution for A on X_{k-1}. On applying the functor $\mathcal{H}om_{X_k}(-, \overline{B}(-kd))$ to this sequence we get

$$0 \to \bar{A}^\vee \otimes \overline{B} \to L_0^\vee \otimes \overline{B} \to L_1^\vee \otimes \overline{B}$$

Thus we have

$$\mathcal{E}xt^1_{X_k}(A, \overline{B}(-kd)) \cong \bar{A}^\vee \otimes \overline{B}. \tag{14}$$

(ii) Suppose A lifts to a bundle $\mathcal{A}$ on X_k. Then we have an exact sequence of $\mathcal{O}_{X_k}$-sheaves

$$0 \to \bar{A}(-kd) \to \mathcal{A} \to A \to 0.$$

Applying $\mathrm{Hom}_{X_k}(\cdot, \bar{B}(-kd))$ to the above sequence, we get a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{X_k}(A, \bar{B}(-kd)) \xrightarrow{\cong} \mathrm{Hom}_{X_k}(\mathcal{A}, \bar{B}(-kd)) \to \mathrm{Hom}_{X_k}(\bar{A}(-kd), \bar{B}(-kd)) \to \\ \mathrm{Ext}^1_{X_k}(A, \bar{B}(-kd)) \to \mathrm{Ext}^1_{X_k}(\bar{A}(-kd), \bar{B}(-kd)) \to \cdots . \end{aligned} \tag{15}$$

The last three terms yield an exact sequence

$$0 \to \mathrm{H}^0(X, \bar{A}^\vee \otimes \bar{B}) \to \mathrm{Ext}^1_{X_k}(A, \bar{B}(-kd)) \to \mathrm{H}^1(X, \bar{A}^\vee \otimes \bar{B}).$$

It is easy to check that either of the two maps in the sequence above, provide a splitting to the spectral sequence in (i).

□

Remark 1. The map

$$\mathrm{Ext}^1_{X_k}(A, \bar{B}(-kd)) \to \mathrm{H}^0(X, \bar{A}^\vee \otimes \bar{B})$$

in the above exact sequence can be described as follows: Let

$$0 \to \bar{B}(-kd) \to \mathcal{C} \to A \to 0$$

be a sequence of $\mathcal{O}_{X_k}-$modules. On restricting it to X, we get

$$\mathrm{Tor}^1_{X_k}(A, \mathcal{O}_X) \to \bar{B}(-kd) \to \mathcal{C} \otimes \mathcal{O}_X \to \bar{A} \to 0.$$

Since $\mathrm{Tor}^1_{X_k}(A, \mathcal{O}_X) \cong \bar{A}(-kd)$, the map $\bar{A}(-kd) \to \bar{B}(-kd)$ gives an element in $\mathrm{H}^0(X, \bar{A}^\vee \otimes \bar{B})$. Furthermore, if this map is the zero map, then we see that

$$0 \to \bar{B}(-kd) \to \mathcal{C} \otimes \mathcal{O}_X \to \bar{A} \to 0$$

is an exact sequence of $\mathcal{O}_X-$modules, and hence we get an element in $\mathrm{Ext}^1_X(A, \bar{B}(-kd)) \cong \mathrm{H}^1(X, \bar{A}^\vee \otimes \bar{B}(-kd))$.

3.3. **An explicit description of η_E via projective resolutions.** Let $X \subset Y$ be as above and E be a bundle on X_{k-1} for some $k \geq 1$. Let

$$0 \to \widetilde{F}_1 \xrightarrow{\Phi} \widetilde{F}_0 \to E \to 0 \tag{16}$$

be a 1-step resolution of E on Y.

On restricting this to X_{k-1}, we get the following 4-term sequence

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_Y}(E, \mathcal{O}_{X_{k-1}}) \to F_1 \to F_0 \to E \to 0, \tag{17}$$

where $F_i := \widetilde{F}_i \otimes \mathcal{O}_{X_{k-1}}$ for $i = 1, 2$.

The first term can be computed from the short exact sequence

$$0 \to \mathcal{O}_Y(-kd) \to \mathcal{O}_Y \to \mathcal{O}_{X_{k-1}} \to 0,$$

by tensoring with E. Doing so, yields a sequence

$$0 \to \mathrm{Tor}^1_{\mathcal{O}_Y}(E, \mathcal{O}_{X_{k-1}}) \to E(-kd) \to E \to E \to 0.$$

Hence we have an isomorphism

$$\mathrm{Tor}^1_{\mathcal{O}_Y}(E, \mathcal{O}_{X_{k-1}}) \cong E(-kd).$$

Thus (17) is the *sequence of vector bundles* on X_{k-1}

$$0 \to E(-kd) \to F_1 \to F_0 \to E \to 0. \tag{18}$$

This 4-term sequence defines an element

$$\widetilde{\eta}_E \in \mathrm{H}^2(X_{k-1}, \mathcal{E}nd\, E(-kd)). \tag{19}$$

Breaking this up, we get the following two short exact sequences of bundles on X_{k-1}:

$$0 \to E(-kd) \to F_1 \to G \to 0. \tag{20}$$

$$0 \to G \to F_0 \to E \to 0. \tag{21}$$

We will also need their dual sequences:

$$0 \to G^\vee \to F_1^\vee \to E^\vee(kd) \to 0. \tag{22}$$

$$0 \to E^\vee \to F_0^\vee \to G^\vee \to 0. \tag{23}$$

The following two results will come in handy later.

Lemma 1. *There is an exact sequence*

$$0 \to \widetilde{F}_0(-kd) \to \widetilde{F}_1 \to G \to 0. \tag{24}$$

Proof. Let $\widetilde{F}_0(-kd) \hookrightarrow \widetilde{F}_0$ be multiplication by the section $s_0^k \in \mathrm{H}^0(Y, \mathcal{O}_Y(kd))$. Clearly this map factors via $\widetilde{F}_1$ and has G as its cokernel. □

Corollary 1. *The boundary map* $\mathrm{H}^0(X, \overline{G}(\nu) \to \mathrm{H}^1(X, \overline{E}(\nu - kd))$ *in the cohomology sequence associated to* (20) *is the zero map* $\forall\, \nu \in \mathbb{Z}$*, provided that we have* $\mathrm{H}^1(X, \mathcal{O}_X(\nu)) = 0$ *for all* $\nu \in \mathbb{Z}$.

Proof. It is enough to show that the map $\mathrm{H}^0(X, \overline{F_1}(\nu)) \to \mathrm{H}^0(X, \overline{G}(\nu))$ is surjective $\forall\, \nu \in \mathbb{Z}$. But from the cohomology sequence of (24), we see that the map $\mathrm{H}^0(X, \widetilde{F_1}(\nu)) \to \mathrm{H}^0(X, \overline{G}(\nu))$ is surjective $\forall\, \nu \in \mathbb{Z}$. This map factors via $\mathrm{H}^0(X, \overline{F_1}(\nu))$ and so we are done. □

We recall some general facts from homological algebra.

(a) The short exact sequences (21) and (20) give two elements,
  (i) $\alpha \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathrm{E}, \mathrm{G}) \cong \mathrm{H}^1(\mathrm{X}, \mathrm{E}^\vee \otimes \mathrm{G})$, and
  (ii) $\beta \in \mathrm{Ext}^1_{\mathcal{O}_X}(\mathrm{G}, \mathrm{E}(-\mathrm{kd})) \cong \mathrm{H}^1(\mathrm{X}, \mathrm{G}^\vee \otimes \mathrm{E}(-\mathrm{kd}))$.

(b) Via the Yoneda correspondence, the 4-term sequence (18) on restriction to X, yields an element
$$\bar{\eta}_E \in \mathrm{H}^2(X, \mathcal{E}nd\, \overline{E}(-d)).$$

(c) Consider the composite maps
$$\mathrm{H}^0(X, \mathcal{E}nd\, E) \xrightarrow{\partial_1} \mathrm{H}^1(X, G \otimes E^\vee) \xrightarrow{\partial_2} \mathrm{H}^2(X, (\mathcal{E}nd\, E)(-kd)), \text{ and}$$
$$\mathrm{H}^0(X, \mathcal{E}nd\, E) \xrightarrow{\partial_2^\vee} \mathrm{H}^1(X, G^\vee \otimes E(-kd)) \xrightarrow{\partial_1^\vee} \mathrm{H}^2(X, (\mathcal{E}nd\, E)(-kd))$$
Here ∂_1 and ∂_2 are the (co)boundary maps in the long exact sequences of cohomology associated to (21) and (20) respectively, and $\partial_1^\vee$ and $\partial_2^\vee$ are the (co)boundary maps in the long exact sequences of cohomology associated to the dual sequences (23) and (22) respectively. If $1 \in \mathrm{H}^0(X, \mathcal{E}nd\, \overline{E})$ denotes the identity endomorphism $id : \overline{E} \to \overline{E}$, then one has
$$\partial_1(1) = \alpha,\ \ \partial_2(\alpha) = \bar{\eta}_E,\ \ \partial_2^\vee(1) = \beta, \text{ and } \partial_1^\vee(\beta) = \bar{\eta}_{\mathrm{E}}.$$

The following result gives a more explicit way of checking when the obstruction class $\bar{\eta}_E$ vanishes.

Proposition 3. *Let* $1 \in \mathrm{H}^0(X, \mathcal{E}nd\, \overline{E})$ *denote the identity endomorphism* $id : \overline{E} \to \overline{E}$*. Then one has*
$$\bar{\eta}_E = 0 \iff \partial_1^\vee(\partial_2^\vee(1)) = \partial_2(\partial_1(1)) = 0.$$

Proof. ($\Longrightarrow$) Suppose that $\bar{\eta}_E = 0$. This is equivalent to saying that E lifts to X_k i.e. there exists a sequence of $\mathcal{O}_{X_k}$−modules

$$0 \to \overline{E}(-kd) \to \mathcal{E} \to E \to 0, \tag{25}$$

The surjection in the above sequence, together with (16), gives rise to a pull-back diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \overline{E}(-kd) & = & \overline{E}(-kd) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{F}_1 & \to & \widetilde{P} & \to & \mathcal{E} & \to & 0 \\
 & & || & & \downarrow & & \downarrow & & \\
0 & \to & \widetilde{F}_1 & \to & \widetilde{F}_0 & \to & E & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{26}
$$

Restricting the above diagram to X, we get

$$
\begin{array}{cccccccccc}
 & & & & & & \downarrow & & \downarrow & \\
 & & & & & & \overline{E}(-kd) & = & \overline{E}(-kd) & \\
 & & & & & \swarrow & \downarrow & & \downarrow 0 & \\
 & \to & \mathrm{Tor}^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X)=\overline{E}(-(k+1)d) & \to & \overline{F}_1 & \to & \overline{P} & \to & \overline{E} & \to 0 \\
 & & \downarrow & & || & & \downarrow & & \downarrow & \\
0 & \to & \overline{E}(-kd) & \to & \overline{F}_1 & \to & \overline{F}_0 & \to & \overline{E} & \to 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & & 0 & & 0 & & 0 &
\end{array} \tag{27}
$$

Since the leftmost vertical map is the zero map, we get a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \overline{E}(-kd) & = & \overline{E}(-kd) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \overline{F}_1 & \to & \overline{P} & \to & \overline{E} & \to & 0 \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \to & \overline{G} & \to & \overline{F}_0 & \to & \overline{E} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{28}
$$

The first two columns imply that

$$
\begin{array}{rcl}
\beta = [\,0 \to \overline{E}(-kd) \to \overline{F}_1 \to \overline{G} \to 0\,] & \in & \mathrm{Image}\left(\mathrm{H}^1(X, \overline{F}_0^{\vee} \otimes \overline{E}(-kd)) \to \mathrm{H}^1(X, \overline{G}^{\vee} \otimes \overline{E}(-kd))\right) \\
 & & || \\
 & & \mathrm{Ker}\left(\mathrm{H}^1(X, \overline{G}^{\vee} \otimes \overline{E}(-kd)) \xrightarrow{\partial_1^{\vee}} \mathrm{H}^2(X, \overline{E}^{\vee} \otimes \overline{E}(-kd))\right)
\end{array} \tag{29}
$$

Thus we have $\partial_1^{\vee}(\partial_2^{\vee}(1)) = \partial_1^{\vee}(\beta) = 0$.

Similarly, the middle and the bottom rows imply that

$$
\begin{array}{rcl}
\alpha = [\,0 \to \overline{G} \to \overline{F}_0 \to \overline{E} \to 0\,] & \in & \mathrm{Image}\left(\mathrm{H}^1(X, \overline{E}^{\vee} \otimes \overline{F}_1) \to \mathrm{H}^1(X, \overline{E}^{\vee} \otimes \overline{G})\right) \\
 & & || \\
 & & \mathrm{Ker}\left(\mathrm{H}^1(X, \overline{E}^{\vee} \otimes \overline{G}) \xrightarrow{\partial_2} \mathrm{H}^2(X, \overline{E}^{\vee} \otimes \overline{E}(-kd))\right)
\end{array} \tag{30}
$$

Thus we have $\partial_2(\partial_1(1)) = \partial_2(\alpha) = 0$.

($\Longleftarrow$) We next prove the converse. So assume that $\partial_2(\partial_1(1)) = 0 = \partial_1^\vee(\partial_2^\vee(1))$. We shall show that E lifts to X_k. Now $\partial_2(\partial_1(1)) = 0$ implies that

$$\partial_1(1) \in \operatorname{Ker}\left(\mathrm{H}^1(X, \overline{E}^\vee \otimes \overline{G}) \xrightarrow{\partial_2} \mathrm{H}^2(X, \overline{E}^\vee \otimes \overline{E}(-kd))\right) = \operatorname{Im}\left(\mathrm{H}^1(X, \overline{E}^\vee \otimes \overline{F}_1) \to \mathrm{H}^1(X, \overline{E}^\vee \otimes \overline{G})\right).$$

Hence there exists a push-forward diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \overline{E}(-kd) & = & \overline{E}(-kd) & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \overline{F}_1 & \to & \overline{P} & \to & \overline{E} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \overline{G} & \to & \overline{F}_0 & \to & \overline{E} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{31}$$

As before, notice that the first two columns imply that $\partial_2^\vee(\partial_1^\vee(1)) = 0$.

Using the middle column, we obtain a pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & \overline{E}(-kd) & \to & \widetilde{P} & \to & \widetilde{F}_0 & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{E}(-kd) & \to & \overline{P} & \to & \overline{F}_0 & \to & 0 \end{array} \tag{32}$$

Let $\Phi : \widetilde{F}_1 \to \widetilde{F}_0$ be the map in (16) and $\phi : \overline{F}_1 \to \overline{F}_0$ denote its restriction to X. We claim that Φ factors as $\widetilde{F}_1 \to \widetilde{P} \to \widetilde{F}_0$. To see this, we apply the functor $\mathrm{Hom}_Y(\widetilde{F}_1, \cdot)$ to (32), to get a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^0(Y, \widetilde{F}_1^\vee \otimes \widetilde{P}) & \to & \mathrm{H}^0(Y, \widetilde{F}_1^\vee \otimes \widetilde{F}_0) & \to & \mathrm{H}^1(Y, \widetilde{F}_1^\vee \otimes \overline{E}(-kd)) \\ \downarrow & & \downarrow & & \| \\ \mathrm{H}^0(X, \widetilde{F}_1^\vee \otimes \overline{P}) & \to & \mathrm{H}^0(X, \widetilde{F}_1^\vee \otimes \overline{F}_0) & \to & \mathrm{H}^1(Y, \widetilde{F}_1^\vee \otimes \overline{E}(-kd)) \end{array} \tag{33}$$

Under the composite (of the middle vertical arrow followed by the right arrow in the bottom row)

$$\mathrm{H}^0(Y, \widetilde{F}_1^\vee \otimes \widetilde{F}_0) \to \mathrm{H}^0(X, \widetilde{F}_1^\vee \otimes \overline{F}_0) \to \mathrm{H}^1(Y, \widetilde{F}_1^\vee \otimes \overline{E}(-kd)),$$

we are given that $\Phi \mapsto \phi \mapsto 0$: this is because from (31), we see that the map $\phi : \overline{F}_1 \to \overline{F}_0$ factors as $\overline{F}_1 \to \overline{P} \to \overline{F}_0$; this implies that in (33), we have

$$\phi \in \mathrm{Image}\left(\mathrm{H}^0(X, \widetilde{F}_1^\vee \otimes \overline{P}) \to \mathrm{H}^0(X, \widetilde{F}_1^\vee \otimes \overline{F}_0)\right),$$

and hence $\Phi \mapsto 0$ in the top row which proves our claim. Thus we have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \overline{E}(-kd) & = & \overline{E}(-kd) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{F}_1 & \to & \widetilde{P} & \to & \mathcal{E} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \widetilde{F}_1 & \to & \widetilde{F}_0 & \to & E & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{34}$$

where $\mathcal{E}$, in the rightmost column, is defined so that the diagram has exact rows and columns.

On restricting the right most column to X_{k-1}, we get

$$\cdots \to \mathrm{Tor}^1_{\mathcal{O}_Y}(E, \mathcal{O}_{X_{k-1}}) = E(-kd) \to \overline{E}(-kd) \to \mathcal{E} \otimes \mathcal{O}_{X_{k-1}} \to E \to 0.$$

Since the map $E(-kd) \to \overline{E}(-kd)$ is a surjection, we see that $\mathcal{E} \otimes \mathcal{O}_{X_{k-1}} \cong E$. The fact that $\mathcal{E}$ is a bundle on X_k follows from the *local criterion for flatness* stated in Proposition 1. □

Remark 2. For our purposes, we may (and do) assume that the map

$$\mathrm{H}^0(X, \mathcal{E}nd\,\overline{E}) \to \mathrm{H}^2(X, \mathcal{E}nd\,\overline{E}(-kd))$$

in the spectral sequence in Proposition 2 is the map $\partial_1 \circ \partial_2 = \partial_2^\vee \circ \partial_1^\vee$ and that $1 \mapsto \bar{\eta}_E = \eta_E$.

Remark 3. One can also define the obstruction class η_E via its *Atiyah class*. Let

$$\mathfrak{a}_E \in \mathrm{H}^1(X_{k-1}, \Omega^1_{X_{k-1}} \otimes \mathcal{E}nd\,E)$$

denote the *Atiyah class* of E and consider the cotangent sheaf sequence for the inclusion $X_{k-1} \subset Y$:

$$0 \to \mathcal{O}_X(-kd) \xrightarrow{d(s_0^k)} \Omega^1_Y \otimes \mathcal{O}_{X_{k-1}} \to \Omega^1_{X_{k-1}} \to 0.$$

Tensoring this sequence by $\mathcal{E}nd\,E$ and taking cohomology, we get a boundary map

$$\mathrm{H}^1(X_{k-1}, \Omega^1_{X_{k-1}} \otimes \mathcal{E}nd\,E) \to \mathrm{H}^2(X, \mathcal{E}nd\,E(-kd)).$$

By a result in [11], $\mathfrak{a}_E \mapsto \eta_E$ under this map.

4. Proofs of the main theorem

4.1. **Grothendieck-Lefschetz theory.** We first recall the *Lefschetz conditions* of Grothendieck (see [9]).

Definition 1. Let Y be a scheme and $X \subset Y$ be a subscheme. Let $\widehat{\ }$ denote the completion of Y along X. We say that the pair (Y, X) satisfies the *Lefschetz condition*, written $\mathrm{Lef(Y,X)}$, if for every open set $U \subset Y$, and every vector bundle $\mathcal{E}$ on U, there exists an open set U' with $X \subset U' \subset U$ such that the natural map

$$\mathrm{H}^0(U', \mathcal{E}_{|_{U'}}) \to \mathrm{H}^0(\widehat{Y}, \widehat{\mathcal{E}})$$

is an isomorphism.

Definition 2. Let Y be a scheme and $X \subset Y$ be a subscheme. Let $\widehat{\ }$ denote the completion of Y along X. We say that the pair (Y, X) satisfies the *effective Lefschetz condition*, written $\mathrm{Leff(Y,X)}$, if $\mathrm{Lef(Y,X)}$ is satisfied, and in addition, for every formal vector bundle $\mathfrak{E}$ on $\widehat{Y}$, there exists an open set $U \supset X$, and a vector bundle $\mathcal{E}$ on U, such that $\widehat{\mathcal{E}} \cong \mathfrak{E}$.

Theorem 4 (Grothendieck, [9])**.** *Let Y be a smooth, projective variety and X be a smooth, ample hyperplane section of Y. Then* $\mathrm{Leff(Y,X)}$ *holds.*

Proof of Theorem 1. The vanishing of $\mathrm{H}^2(X, \mathcal{E}nd\,E \otimes \mathcal{O}_Y(-kX)_{|X})$ for all $k \in \mathbb{Z}_{>0}$, implies that E extends to a bundle E_k on each thickening X_k. Thus we get a formal vector bundle $\widehat{E} := \varprojlim_k E_k$ on $\widehat{Y}$. By $\mathrm{Leff(Y,X)}$, we are done. □

Proof of Theorem 2. We first make a remark about the hypothesis that E is a bundle on a general, hyperplane section of Y. By this we mean that if $\mathcal{X} \to S$ is the universal family of hypersurfaces in Y, then there exists an open set $S' \subset S$, and a bundle $\mathcal{E} \to \mathcal{X} \times_S S'$, flat over S', such that for $s \in S'$, if $\mathcal{X}_s = X$, we have $\mathcal{E}_s \cong E$. Suppose that we have succeeded in extending E to a bundle E_{k-1} on X_{k-1}. Then one has a commutative diagram of 4-term exact sequences (35)

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^1(X, (\mathcal{E}nd\,E)(-kd)) & \to & \mathrm{Ext}^1_{X_k}(E_{k-1}, E(-kd)) & \to & \mathrm{H}^0(X, \mathcal{E}nd\,E) & \to & \mathrm{H}^2(X, (\mathcal{E}nd\,E)(-kd)). \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{H}^1(X, \mathcal{E}nd\,E) \otimes S^kV^* & \to & \mathrm{Ext}^1_{X_k}(E_{k-1}, E \otimes S^kV^*) & \to & \mathrm{H}^0(X, (\mathcal{E}nd\,E)(kd)) \otimes S^kV^* & \to & \mathrm{H}^2(X, \mathcal{E}nd\,E) \otimes S^kV^*.
\end{array}$$

Here both the rows are a consequence of the exact sequence from Proposition 2 (i), (the top row is obtained by taking $A = E_{k-1}$ and $\bar{B} = E$, and the bottom row is obtained by taking $A = E_{k-1}$ and $\bar{B} = E(kd) \otimes S^k V^*$). The vertical maps between the various cohomology groups are induced by the dual of the evaluation map $\mathcal{O}_X(-d) \to V^* \otimes \mathcal{O}_X$ and its symmetric powers.

To show that the bundle E lifts to X_k, we need to show that $1 \mapsto \eta_{E_{k-1}} = 0$ under the map

$$\mathrm{H}^0(X, \mathcal{E}nd\, E) \to \mathrm{H}^2(X, (\mathcal{E}nd\, E)(-kd))$$

in the top row of (35).

Since E lives on a general hypersurface, this implies that $\eta_{\mathcal{E}_i} = 0$ for all $i \geq 0$ where $\mathcal{E}_i := \mathcal{E} \otimes \mathcal{O}_{\mathcal{X}_i}$. Consider the exact sequence

$$0 \to E \otimes S^k V^* \to \mathcal{E}_k \to \mathcal{E}_{k-1} \to 0.$$

Applying the push-forward functor Rp_* for the projection map $p : \mathcal{X} \to X$, we get the short exact sequence

$$\theta_k : 0 \to E \otimes S^k V^* \to p_*(\mathcal{E}_k) \to p_*(\mathcal{E}_{k-1}) \to 0. \tag{36}$$

We first claim that there are maps $E_i \to p_*\mathcal{E}_i$ for $0 \leq i \leq k-1$. The proof is by induction on i. When $i = 0$, this is just the isomorphism $E \cong \mathcal{E} \otimes \mathcal{O}_X$. So assume that there is a map $E_{k-2} \to p_*\mathcal{E}_{k-2}$.

The pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & E \otimes S^{k-1}V^* & \to & E_{k-1} & \to & E_{k-2} & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & E \otimes S^{k-1}V^* & \to & p_*\mathcal{E}_{k-1} & \to & p_*\mathcal{E}_{k-2} & \to & 0. \end{array}$$

yields the desired map $E_{k-1} \to p_*\mathcal{E}_{k-1}$, and taking the pull-back of the sequence (36) under this map yields an element $\theta'_k \in Ext^1_{X_k}(E_{k-1}, E \otimes S^k V^*)$.

Since $1 \in H^0(X, \mathcal{E}nd\, E)$ and θ'_k map to the same element in $H^0(Y, (\mathcal{E}nd\, E)(kd)) \otimes S^k V^*$ in diagram (35), by the commutativity of the diagram, we see that $1 \mapsto 0$ under the composition

$$\mathrm{H}^0(X, \mathcal{E}nd(E)) \to H^2(X, \mathcal{E}nd\, E(-kd)) \to \mathrm{H}^2(X, \mathcal{E}nd\, E) \otimes S^k V^*.$$

Condition (1) is equivalent, by Serre duality, to the injectivity of the map

$$\mathrm{H}^2(X, (\mathcal{E}nd\, E)(-kd)) \to \mathrm{H}^2(X, (\mathcal{E}nd\, E)(kd)) \otimes S^k V^*,$$

which implies that

$$1 \in \mathrm{Image}\,[\,\mathrm{Ext}^1_{X_k}(E_{k-1}, E(-kd)) \to \mathrm{H}^0(X, \mathcal{E}nd\, E)\,].$$

Thus we see that the bundle E extends to a bundle E_k to the thickening X_k. Set $\widehat{E}$ to be the inverse limit of E_k's before. Then $\widehat{E}$ is the extension of E to the completion Y and so by the *effective Lefschetz condition* of Grothendieck, extends to a reflexive sheaf $\widetilde{E}$ whose singular locus is a finite set of points in the complement of X in Y.

□

5. The Noether-Lefschetz theorem revisited

5.1. **The infinitesimal Noether-Lefschetz theorem.** Let Y be a smooth, projective 3-fold and $\mathcal{O}_Y(1)$ be an ample line bundle defining a base point free linear system V. Let $X \subset Y$ be a smooth member of this linear system. We will further assume that the line bundle $\mathcal{O}_Y(1)$ satisfies the following *positivity* conditions.

(P1) The multiplication map
$$\mathrm{H}^0(Y,\mathcal{O}_Y(d))\otimes \mathrm{H}^0(Y,K_Y(d)) \to \mathrm{H}^0(Y,K_Y(2d))$$
is surjective.

(P2) The map
$$\mathrm{H}^2(Y,\Omega^1_Y\otimes\mathcal{O}_Y(-d)) \to \mathrm{H}^2(Y,\Omega^1_Y)$$
is an inclusion.

(P3) $\mathrm{H}^1(Y,K_Y(d))=0$.

Remark 4. By the adjunction formula, we have $K_Y\otimes\mathcal{O}_X(d)=K_X$; and thus an exact sequence
$$0\to K_Y\otimes\mathcal{O}_Y(d)\to K_Y\otimes\mathcal{O}_Y(2d)\to K_X\otimes\mathcal{O}_X(d)\to 0.$$
Consider the following diagram where the vertical maps are restriction maps:
$$\begin{array}{ccc} \mathrm{H}^0(Y,\mathcal{O}_Y(d))\otimes \mathrm{H}^0(Y,K_Y(d)) & \to & \mathrm{H}^0(Y,K_Y(2d)) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(X,\mathcal{O}_X(d))\otimes \mathrm{H}^0(X,K_Y(d)\otimes\mathcal{O}_X) & \to & \mathrm{H}^0(Y,K_Y(2d)\otimes\mathcal{O}_X) \end{array}$$
By (P3) above, the vertical map on the right is surjective. This together with (P1) implies that the map in the bottom row is also surjective.

Theorem 5 (The infinitesimal Noether-Lefschetz theorem)**.** *With hypothesis as above,*
$$\mathrm{Image[Pic}(\mathcal{X}_1)\to \mathrm{Pic(X)]} = \mathrm{Image[Pic(Y)\to Pic(X)]}.$$

Proof. Recall that for any smooth projective variety V, we have the *exponential* short exact sequence
$$0\to\mathbb{Z}\to\mathcal{O}_V\to\mathcal{O}_V^*\to 0.$$
On taking cohomology, we get a map $Pic(V)\to \mathrm{H}^2(V,\mathbb{Z})$, which associates to a line bundle A, its first Chern class $c_1(A)\in\mathrm{H}^2(V,\mathbb{Z})$.

We need to prove that if $A\in\mathrm{Pic}(X)$ extends to a line bundle $\mathcal{A}\in\mathrm{Pic}(\mathcal{X}_1)$, then under the hypothesis of the theorem, A is the restriction of a line bundle on Y. We will consider three cases:

Case (i): $c_1(A)=0$ i.e. $L\in\mathrm{Pic}^0(X)$. In this case, by the Weak Lefschetz theorem, we have $\mathrm{Pic}^0(Y)\cong\mathrm{Pic}^0(X)$, and so A is the restriction of a line bundle from Y.

Case (ii): $c_1(A)$ is a *non-zero torsion* element. In this case, one argues as follows: first, consider the exact sequence
$$0\to\mathbb{Z}\xrightarrow{\times m}\mathbb{Z}\to\mathbb{Z}/m\mathbb{Z}\to 0.$$
Taking cohomology, we get a commutative diagram of long exact sequences for X and Y:
$$\begin{array}{ccccccccc} \mathrm{H}^1(Y,\mathbb{Z}) & \xrightarrow{\times m} & \mathrm{H}^1(Y,\mathbb{Z}) & \to & \mathrm{H}^1(Y,\mathbb{Z}/m\mathbb{Z}) & \to & \mathrm{H}^2(Y,\mathbb{Z}) & \xrightarrow{\times m} & \mathrm{H}^2(Y,\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow\cong & & \downarrow & & \downarrow \\ \mathrm{H}^1(X,\mathbb{Z}) & \xrightarrow{\times m} & \mathrm{H}^1(X,\mathbb{Z}) & \to & \mathrm{H}^1(X,\mathbb{Z}/m\mathbb{Z}) & \to & \mathrm{H}^2(X,\mathbb{Z}) & \xrightarrow{\times m} & \mathrm{H}^2(X,\mathbb{Z}) \end{array}$$
Now if $m\cdot c_1(A)=0$, then this implies that
$$c_1(A)\in\mathrm{Image[H^1(X,\mathbb{Z}/m\mathbb{Z})\to H^2(X,\mathbb{Z})]}.$$
Now by the Weak Lefschetz theorem for finite coefficients, we have
$$\mathrm{H}^1(Y,\mathbb{Z}/m\mathbb{Z})\cong\mathrm{H}^1(X,\mathbb{Z}/m\mathbb{Z}).$$

Let c be the lift of $c_1(A)$ in $\mathrm{H}^1(Y,\mathbb{Z}/m\mathbb{Z})$; then c maps to a non-zero torsion element $\tilde{c} \in \mathrm{H}^2(Y,\mathbb{Z})$. Consider the following diagram :

$$\begin{array}{ccccccc} \mathrm{H}^1(Y,\mathcal{O}_Y) & \to & \mathrm{Pic}(Y) & \to & \mathrm{H}^2(Y,\mathbb{Z}) & \to & \mathrm{H}^2(Y,\mathcal{O}_Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^1(X,\mathcal{O}_X) & \to & \mathrm{Pic}(X) & \to & \mathrm{H}^2(X,\mathbb{Z}) & \to & \mathrm{H}^2(X,\mathcal{O}_X) \end{array}$$

Here the rows are the long exact sequences associated to the exponential sequences for Y and X, and the vertical arrows are restriction maps. From the top row, we see that $\tilde{c} = c_1(\tilde{A})$ for some $\tilde{A} \in \mathrm{Pic}(Y)$; this is because $\mathrm{H}^2(Y,\mathcal{O}_Y)$ is a $\mathbb{C}$-vector space and so $\tilde{c} \mapsto 0$ under the map $\mathrm{H}^2(Y,\mathbb{Z}) \to \mathrm{H}^2(Y,\mathcal{O}_Y)$. By the commutativity of the diagram, we see that $c_1(\tilde{A}\otimes\mathcal{O}_X) = c_1(A) \in \mathrm{H}^2(X,\mathbb{Z})$. This means that $\tilde{A}\otimes\mathcal{O}_X = A \otimes C$ for some line bundle $C \in Pic^0(X)$. Since C has a unique lift $\tilde{C}$ on Y, we see that $A \cong \tilde{A}\otimes\tilde{C}^{-1}\otimes\mathcal{O}_X$. Hence we are done.

Case (iii): $A \in Pic^0(X)$ and $c_1(A)$ is non-zero and non-torsion. In this case, one sees that

$$c_1(A) \in \mathrm{H}^{1,1}(X) \cap \mathrm{H}^2(X,\mathbb{Z}).$$

We need to prove that if A lifts to a line bundle $\mathcal{A}$ on $\mathcal{X}_1$, then A is the restriction of a bundle on Y. This is done by in two steps:

Step 1. $\mathrm{Image}[\,\mathrm{Pic}(\mathcal{X}_1) \to \mathrm{Pic(X)}\,] = \mathrm{Image}[\,\mathrm{Pic(X_1)} \to \mathrm{Pic(X)}\,]$.
Step 2. $\mathrm{Image}[\,\mathrm{Pic(X_1)} \to \mathrm{Pic(X)}\,] = \mathrm{Image}[\,\mathrm{Pic(Y)} \to \mathrm{Pic(X)}\,]$.

Proof of Step 1: Let $\mathcal{A} \mapsto A$ under the map $\mathrm{Pic}(\mathcal{X}_1) \to \mathrm{Pic}(X)$; then we have an exact sequence of $\mathcal{O}_{\mathcal{X}_1}$-sheaves:

$$0 \to A \otimes V^* \to \mathcal{A} \to A \to 0,$$

and hence an element of $\mathrm{Ext}^1_{\mathcal{X}_1}(A, A\otimes V^*)$.

Let $p_1 : \mathcal{X}_1 \to X_1$ be the natural map (recall that is restricts to an isomorphism $p : \mathcal{X}_0 \xrightarrow{=} X_0 := X$); applying $p_{1,*}$ to the exact sequence above, we get an exact sequence of $\mathcal{O}_{X_1}$-sheaves

$$0 \to A \otimes V^* \to p_{1*}\mathcal{A} \to A \to 0,$$

and hence an element $\mathrm{Ext}^1_{X_1}(A, A\otimes V^*)$.

As in the proof of Theorem 2 before, we start with the commutative diagram (35) applied in the case where $E = A$ is a line bundle:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}^1(X,\mathcal{O}_X(-d)) & \to & \mathrm{Ext}^1_{X_1}(A,A(-d)) & \to & \mathrm{H}^0(X,\mathcal{O}_X) & \to & \mathrm{H}^2(Y,\mathcal{O}_X(-d)) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{H}^1(X,\mathcal{O}_X)\otimes V^* & \to & \mathrm{Ext}^1_{X_1}(A,A\otimes V^*) & \to & \mathrm{H}^0(X,\mathcal{O}_X(d))\otimes V^* & \to & \mathrm{H}^2(X,\mathcal{O}_X)\otimes V^*. \end{array}$$

The fact that A lifts to $\mathcal{X}_1$ implies that the $1 \mapsto 0$ under the "diagonal" map

$$\mathrm{H}^0(X,\mathcal{O}_X) \to \mathrm{H}^2(X,\mathcal{O}_X)\otimes V^*.$$

The vertical map

$$\mathrm{H}^2(X,\mathcal{O}_X(-d)) \to \mathrm{H}^2(X,\mathcal{O}_X)\otimes V^*$$

is injective since its dual is surjective by condition (1) (and Remark 4). This in turn implies that

$$1 \in \mathrm{Ker}\,[\,\mathrm{H^0(X,\mathcal{O}_X) \to H^2(X,\mathcal{O}_X(-d)}\,].$$

Thus we see that A lifts a line bundle A_1 on X_1.

Proof of Step 2: We let $c_1(A_1) \in \mathrm{H}^1(X_1,\Omega^1_{X_1})$ denote the first Chern class of A_1 (see [15]). By functoriality, we note that under the natural restriction map

$$\mathrm{H}^1(X_1,\Omega^1_{X_1}) \to \mathrm{H}^1(X,\Omega^1_X), \quad c_1(A_1) \mapsto c_1(A).$$

This map in turn factors as

$$\mathrm{H}^1(X_1,\Omega^1_{X_1}) \to \mathrm{H}^1(X_1,\Omega^1_{X_1}\otimes\mathcal{O}_X) \to \mathrm{H}^1(X,\Omega^1_X).$$

Since $c_1(A) \neq 0$ by assumption, this in turn implies that, both $c_1(A_1)$ and its image in $\mathrm{H}^1(X_1, \Omega^1_{X_1} \otimes \mathcal{O}_X)$, are both non-zero.

Now consider the cotangent sheaf sequence for the inclusion $X_1 \subset Y$:

$$0 \to \mathcal{O}_Y(-2d) \xrightarrow{d(s^2)} \Omega^1_Y \otimes \mathcal{O}_{X_1} \to \Omega^1_{X_1} \to 0,$$

where $s \in \mathrm{H}^0(Y, \mathcal{O}_Y(d))$ is the section whose zero locus is X. Since $d(s^2) = s.ds$, this means that when we restrict the above sequence to X, we get an isomorphism

$$\Omega^1_Y \otimes \mathcal{O}_X \cong \Omega^1_{X_1} \otimes \mathcal{O}_X,$$

which in turn induces an isomorphism

$$\mathrm{H}^1(X, \Omega^1_Y \otimes \mathcal{O}_X) \cong \mathrm{H}^1(X, \Omega^1_{X_1} \otimes \mathcal{O}_X).$$

Now consider the short exact sequence,

$$0 \to \Omega^1_Y \otimes \mathcal{O}_Y(-d) \to \Omega^1_Y \to \Omega^1_Y \otimes \mathcal{O}_X \to 0.$$

On taking cohomology, we get a long exact sequence

$$\mathrm{H}^1(Y, \Omega^1_Y) \to \mathrm{H}^1(X, \Omega^1_Y \otimes \mathcal{O}_X) \to \mathrm{H}^2(Y, \Omega^1_Y \otimes \mathcal{O}_Y(-d)) \to \mathrm{H}^2(Y, \Omega^1_Y) \to .$$

From condition (3) above,

$$\mathrm{H}^2(Y, \Omega^1_Y \otimes \mathcal{O}_Y(-d)) \hookrightarrow \mathrm{H}^2(Y, \Omega^1_Y),$$

and hence we obtain a surjection

$$\mathrm{H}^1(Y, \Omega^1_Y) \twoheadrightarrow \mathrm{H}^1(X, \Omega^1_Y \otimes \mathcal{O}_X).$$

Thus we see that $c_1(A)$ lifts to $\mathrm{H}^1(Y, \Omega^1_Y)$. It is not hard to see that in fact $c_1(A)$ lifts to an element in $\mathrm{H}^{1,1}(Y) \cap \mathrm{H}^2(Y, \mathbb{Z})$. By the Lefschetz $(1,1)$-theorem, we see that this lift is $c_1(\tilde{A})$ for some line bundle $\tilde{A}$ on Y.

We have the following diagram where the rows come from the exponential sequence and the vertical maps are restriction maps. The left most vertical map is an isomorphism while the right most vertical maps is an injection by the Weak Lefschetz theorem,

$$\begin{array}{ccccccc} 0 & \to & Pic^0(Y) & \to & \mathrm{Pic}(Y) & \twoheadrightarrow & \mathrm{H}^2(Y, \mathbb{Z}) \cap \mathrm{H}^1(Y, \Omega^1_Y) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Pic}^0(X) & \to & \mathrm{Pic}(X) & \twoheadrightarrow & \mathrm{H}^2(X, \mathbb{Z}) \cap \mathrm{H}^1(X, \Omega^1_X) \end{array}$$

By a diagram chase (as in Case (ii) above), we see that $A \cong \tilde{A} \otimes \tilde{C} \otimes \mathcal{O}_X$ for some $\tilde{C} \in \mathrm{Pic}^0(Y)$. Thus we see that A lifts to a line bundle on Y. □

5.2. **A Noether-Lefschetz theorem for the divisor class group.** Let Y be a normal, projective threefold and $\mathcal{O}_Y(d)$ be an ample line bundle defining a base point free linear system V. Let $X \subset Y$ be a general member of this linear system; by Bertini's theorem, X is normal. Let $\pi : \widetilde{Y} \to Y$ be a desingularisation and let $\widetilde{X} := \widetilde{Y} \times_Y X$. Then $\widetilde{X}$ is a smooth member of the linear system $\pi^* V$, and $\pi : \widetilde{X} \to X$ is a desingularisation. The following result was proved in [16].

Theorem 6. *With notation as above, let $f : Y \to \mathbb{P}^n$ be the map defined by the linear system V and, $g : \widetilde{Y} \to \mathbb{P}^N$ denote the composite $\widetilde{Y} \to Y \to \mathbb{P}^N$. If $\widehat{Y}$ denotes the formal scheme obtained by completing $\widetilde{Y}$ along $\widetilde{X}$, then* $\mathrm{Pic}(\widehat{Y}) \to \mathrm{Pic}(\widetilde{X})$ *is injective, and there is an exact sequence*

$$0 \to A \to \mathrm{Pic}(\widetilde{Y}) \to \mathrm{Pic}(\widehat{Y}) \to B \to 0$$

where

(i) *A is freely generated by irreducible g-exceptional divisors in $\widetilde{Y}$ which have 0-dimensional image.*
(ii) *B is freely generated by the irreducible $g_{|\widetilde{X}}$-exceptional divisors in $\widetilde{X}$, which have 0-dimensional image under g, modulo the group generated by the classes of exceptional divisors of the form $E \cdot \widetilde{X}$, where E is an irreducible g-exceptional divisor on $\widetilde{Y}$ with* $\dim g(E) = 1$.

Proof of Theorem 3. For any *smooth* member X of the linear system V, we let $\tilde{\mathcal{X}}_1$ denote its first order thickening in the universal family $\tilde{\mathcal{X}} \to S := \mathbb{P}(V)$. By theorem 5 (Hypotheses (i) and (ii) are conditions (P2) and (P1) respectively and (P3) is true by Kawamata-Viehweg vanishing theorem since $\mathcal{O}_Y(1)$ is *big and nef*), we have

$$\mathrm{Image}[\,\mathrm{Pic}(\tilde{\mathcal{X}}_1) \to \mathrm{Pic}(\widetilde{\mathrm{X}})\,] = \mathrm{Image}[\,\mathrm{Pic}(\widetilde{\mathrm{Y}}) \to \mathrm{Pic}(\widetilde{\mathrm{X}})\,].$$

Let $K = K(S)$ be the function field of the parameter variety S, and $\overline{K}$ be its algebraic closure. Let $\widetilde{X}_{\overline{K}}$ and $\widetilde{Y}_{\overline{K}}$, denote the base change of $\widetilde{X}$ and $\widetilde{Y}$ to $\overline{K}$ respectively. By a standard spreading out argument (see [17], §3 for details), this means we have an exact sequence

$$0 \to A \to \mathrm{Pic}(\widetilde{Y}_{\overline{K}}) \to \mathrm{Pic}(\widetilde{X}_{\overline{K}}) \to B \to 0,$$

where A, B are as before (i.e., as in Theorem 6). Equivalently, for a *very general member* $\widetilde{X}$ of the linear system π^*V, there is an exact sequence

$$0 \to A \to \mathrm{Pic}(\widetilde{Y}) \to \mathrm{Pic}(\widetilde{X}) \to B \to 0.$$

For a normal projective variety V and a desingularisation $h : \widetilde{V} \to V$, there is a natural isomorphism (see [16], §1 for a more detailed explanation)

$$\mathrm{Cl}(V) \;\cong\; \frac{\mathrm{Pic}(\widetilde{V})}{(\text{subgroup generated by } h\text{-exceptional divisors})}.$$

Consequently, we have the following diagram, with exact rows and columns (see [16] for more details):

$$\begin{array}{ccccccccccc} & & & & 0 & & 0 & & & & \\ & & & & \downarrow & & \downarrow & & & & \\ 0 & \to & A & \to & \mathbb{Z}[E_Y] & \to & \mathbb{Z}[E_X] & \to & B & \to & 0 \\ & & || & & \downarrow & & \downarrow & & || & & \\ 0 & \to & A & \to & \mathrm{Pic}(\widetilde{Y}) & \to & \mathrm{Pic}(\widetilde{X}) & \to & B & \to & 0 \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & \mathrm{Cl}(Y) & \to & \mathrm{Cl}(X) & & & & \\ & & & & \downarrow & & \downarrow & & & & \\ & & & & 0 & & 0 & & & & \end{array}$$

Here $\mathbb{Z}[E_Y]$ and $\mathbb{Z}[E_X]$ are the subgroups in the respective Picard groups freely generated by the irreducible exceptional divisors in $\widetilde{Y}$ and $\widetilde{X}$. By a simple diagram chase, we now conclude that $\mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ is an isomorphism. □

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