CURVES ON THREEFOLDS AND A CONJECTURE OF GRIFFITHS-HARRIS

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ABSTRACT. We prove that any arithmetically Gorenstein curve on a smooth, general hypersurface $X \subset \mathbb{P}^4$ of degree at least 6, is a complete intersection. This gives a characterisation of complete intersection curves on general type hypersurfaces in $\mathbb{P}^4$. We also verify that certain 1-cycles on a general quintic hypersurface are non-trivial elements of the Griffiths group.

1. INTRODUCTION

We work over $\mathbb{C}$, the field of complex numbers. By a *general* point of a variety, we shall mean a point in a Zariski open subset and by a *very general* point we mean a point in the complement of a countable union of proper closed subvarieties.

For a very general hypersurface $X \subset \mathbb{P}^3$ of degree at least 4, the Noether-Lefschetz theorem (NLT) says that every curve $C \subset X$ is a complete intersection of X with a surface in $\mathbb{P}^3$ i.e., $C = X \cap S$ where $S \subset \mathbb{P}^3$ is a surface. Motivated by this, Griffiths and Harris [13] conjectured that the following analogue of NLT holds for curves in threefolds.

Conjecture 1 (Griffiths-Harris, [13]). *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$. Then any curve $C \subset X$ is of the form $C = X \cap S$, where S is a surface in $\mathbb{P}^4$.*

For the sake of brevity, we shall call curves $C \subset X$ which are not intersections of X with any surface as *special*. Voisin (see [24]) showed that a general threefold $X \subset \mathbb{P}^4$ always contains special curves $C \subset X$, thus proving that this conjecture is false. In fact, one can consider the analogous question for codimension two subvarieties in higher dimensional hypersurfaces; in [20], it is shown that there exists a large class of special codimension two subvarieties in smooth hypersurfaces of dimension at least three and degree at least two.

The aim of this note is to show that though NLT for curves in surfaces does not generalise to curves in threefolds, a restricted version of this theorem related to the non existence of certain special curves on very general hypersurfaces in $\mathbb{P}^3$ also holds for general hypersurfaces in $\mathbb{P}^4$. We shall make this precise now.

We start with a few definitions. A vector bundle E on X is said to be *arithmetically Cohen-Macaulay* (ACM for short) if $\mathrm{H}^i(X, E(\nu)) = 0$, $\forall\, \nu \in \mathbb{Z}$ and $0 < i < \dim X$. Similarly, a subscheme $Y \subset X$ with ideal sheaf $\mathcal{I}_{Y/X}$ is said to be ACM if $\mathrm{H}^i(X, \mathcal{I}_{Y/X}(\nu)) = 0$, $\forall\, \nu \in \mathbb{Z}$ and $1 \leq i \leq \dim Y$. In addition, if Y has codimension two in X, we shall say Y is *arithmetically Gorenstein* if Y is the zero scheme of a section of a rank two bundle E on X. It is not hard to see in this case that if X is a smooth projective hypersurface of dimension at least 3, then Y is a complete intersection if and only if E is a sum of line bundles and that Y is ACM if and only if E is ACM.

An equivalent formulation of NLT says that if $X \subset \mathbb{P}^3$ is a very general hypersurface of degree at least 4, then any line bundle L on X is $\mathcal{O}_X(m)$ for some $m \in \mathbb{Z}$; hence L is ACM. Rephrasing this, we may say that as a consequence of this theorem, any ACM line bundle on such an X is the restriction of a line bundle on $\mathbb{P}^3$.

One might now wonder that though the analogue of NLT for curves in threefolds $X \subset \mathbb{P}^4$ is false, is it still true that any ACM rank two vector bundle on X is the restriction of a rank two bundle on $\mathbb{P}^4$. In fact, one might even be tempted to formulate the "Noether-Lefschetz question" for higher rank ACM bundles as follows: Given an ACM rank r bundle E on a very general smooth hypersurface $X \subset \mathbb{P}^n$, is E the restriction of a vector bundle on $\mathbb{P}^n$? The case $(r, n) = (1, 3)$ is implied by the NLT. It is easy to see that any such extension, if it exists, is necessarily ACM on $\mathbb{P}^n$. By a theorem of Horrocks (see [14]), any ACM vector bundle on $\mathbb{P}^n$ is a sum of line bundles. Thus the Noether-Lefschetz question for higher rank ACM vector bundles can also be viewed as an extension of Horrocks' splitting criterion to bundles on hypersurfaces $X \subset \mathbb{P}^n$. Notice that the converse, namely that any sum of line bundles on X extends to $\mathbb{P}^n$ for $n \geq 4$ follows by the Grothendieck-Lefschetz theorem.

Buchweitz-Greuel-Schreyer have shown (see [5]) that there do exist non-trivial ACM bundles of sufficiently high rank on any hypersurface X. Conjecture B of *op. cit.* tells us precisely beyond what rank one might expect to get non-trivial ACM bundles. The main result of this paper is the following which can be viewed as a verification of the first non-trivial case of (a strengthening of) this conjecture.

Theorem 1. *Let X be a general hypersurface in $\mathbb{P}^4$ of degree $d \geq 6$. Any arithmetically Gorenstein curve $C \subset X$ is a complete intersection. Equivalently, any ACM bundle E of rank two on X is a sum of line bundles.*

By remark 3.1 in [7], it then follows that the above result is true for a general hypersurface of degree $d \geq 6$ in $\mathbb{P}^n$ for $n \geq 4$. However in *op. cit.*, it has been shown that the result is also true for $d = 3, 4$ and 5 when $n \geq 5$. Thus we recover the following theorem proved by us using completely different methods:

Corollary 1 (Mohan Kumar-Rao-Ravindra, [18]). *Any ACM bundle of rank two on a general hypersurface $X \subset \mathbb{P}^n$, $n \geq 5$, of degree at least 3 is a sum of line bundles.*

Soon after the main steps in this paper were carried out, we were able to extend the methods of *loc. cit.* to prove theorem 1 (see [19]). Partial results (see [18] for details) in this direction were also obtained by Chiantini and Madonna [6, 7, 8].

Theorem 1 is sharp: A smooth hypersurface in $\mathbb{P}^4$ of degree ≤ 5 contains a line. The corresponding rank two bundle, via Serre's construction, is ACM but not decomposable. Similarly, there are smooth hypersurfaces in $\mathbb{P}^4$ of degree ≥ 6 which contain a line. Hence the hypothesis of generality cannot be dropped.

We briefly outline the proof of Theorem 1. We may assume that the vector bundle E is normalised (i.e. $h^0(E(-1)) = 0$ and $h^0(E) \neq 0$). By Proposition 1, its first Chern class $\alpha := c_1(E)$ satisfies the inequality $3 - d \leq \alpha \leq d - 2$. Suppose on the contrary, a general hypersurface supports such a bundle which is indecomposable. This implies the following: Let S' be an open set of the parameter space of smooth hypersurfaces of degree d in $\mathbb{P}^4$ which support such a bundle and $\mathcal{X}' \to S'$ be the universal hypersurface. Then there exists a family of rank two vector bundles $\mathcal{E} \to \mathcal{X}'$ such that $\forall s \in S'$, $E_s := \mathcal{E}_{|X_s}$ is a normalised, indecomposable ACM bundle of rank two on X_s with $c_1(E_s) = \alpha$. Associated to this, there is a family of null-homologous 1-cycles $\mathcal{Z} \to S'$ whose fibre at any point $s \in S'$ is $\mathcal{Z}_s = dC_s - lD_s$ where $C_s \subset X_s$ is the zero locus of a section of E_s, $l = l(s) = \deg C_s$ and $D_s \subset X_s$ is a plane section. To such a family of cycles, one can associate a normal function $\nu_{\mathcal{Z}}$ and its infinitesimal invariant $\delta\nu_{\mathcal{Z}}$ (see section 2.2 for definitions). By a result of Mark Green [9] and Voisin (unpublished), $\delta\nu_{\mathcal{Z}} \equiv 0$ whenever $d \geq 6$. On the other hand, by refining a method of X. Wu (see [26]), we show that in the situation described above, $\delta\nu_{\mathcal{Z}} \not\equiv 0$ when $d \geq 5$. This is a contradiction when $d \geq 6$.

On hypersurfaces of degree $d \leq 5$, it is easy to construct (normalised) indecomposable ACM bundles of rank two (see [1]). The (refined) criterion of Wu has an interesting consequence for such bundles when $d = 5$. Recall that the Griffiths group of codimension k cycles is defined to be the group of homologically trivial codimension k cycles modulo the subgroup of cycles algebraically equivalent to zero. As a consequence of the non-degeneracy of the infinitesimal invariant, we have

Corollary 2. *Let E be a normalised, indecomposable ACM bundle of rank two on $X_5 \subset \mathbb{P}^4$, a smooth, general quintic hypersurface. If $Z = 5C - lD$ is as above, then Z defines a non-trivial element of $\mathrm{Griff}^2(X)$, the Griffiths group of codimension 2 cycles on X.*

The proof of the above corollary is identical to Griffiths' proof (see [10]) where he shows that the difference of two distinct lines defines a non-trivial element in the Griffiths group. Hence we shall only sketch the proof and refer the reader to *op. cit.* for details. Since $\delta\nu_{\mathcal{Z}} \neq 0$, this implies that $\nu_{\mathcal{Z}}$ is not locally constant (see [9]), hence Z has non-trivial Abel-Jacobi image. Now the subgroup of cycles algebraically equivalent to zero is contained in the kernel of the Abel-Jacobi map. Hence the corollary.

Comparison with Wu's results: In [27], using his criterion for the non-degeneracy of the infinitesimal invariant, Wu is able to prove the following:

Theorem 2. *Let $X \subset \mathbb{P}^4$ be a general, smooth hypersurface of degree $d \geq 6$, and let $C \subset X$ be a smooth curve with $\deg C \leq 2d - 1$. Then $C = X \cap \mathbb{P}^2$ is a plane section.*

Thus Theorem 1 may be viewed as a generalisation of this theorem of Wu. Though any characterisation of complete intersection curves cannot obviously have a constraint on their degrees as in the above theorem, it is interesting to note that the proof of Theorem 1 also follows by reducing to the case of bounded degree curves. To see this, let E be a rank two ACM bundle on X and assume that it has a non-zero section whose zero locus C is a curve. The Grothendieck-Riemann-Roch formula expresses $\chi(E)$ as a function of $c_1(E)$ and $c_2(E)$ (see [6] for a precise formula). Since E is ACM, $\chi(E(b)) = h^0(E(b)) - h^3(E(b)) = h^0(E(b)) - h^0(E(-c_1 + d - 5 - b))$. Choosing $b > 0$ so that $\chi(E(b)) \geq 0$, we see that $c_2(E(b))$ is bounded by a function of $c_1(E(b))$. From the outline of the proof given above, we may assume that $c_1(E)$ is bounded; hence it follows that $\deg C = c_2(E)$ is bounded in terms of the degree of X.

Remark 1. Theorem 1 has now been generalised to complete intersections subvarieties of sufficiently high multi-degree in projective space (see [2]).

2. Preliminaries

2.1. **Reductions.** Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. By the Grothendieck-Lefschetz theorem, we have $\mathrm{Pic}(X) \cong \mathbb{Z}$ with generator $\mathcal{O}_X(1)$. Also by the weak Lefschetz theorem, we have $\mathrm{H}^{2i}(X, \mathbb{Z}) \cong \mathrm{H}^{2i}(\mathbb{P}^4, \mathbb{Z}) \cong \mathbb{Z}$ for $i = 1$, 2. With these identifications, we may treat the first and second Chern classes of any vector bundle E on X as integers.

In this section, we shall show that it is enough to consider rank two ACM bundles whose first Chern class α satisfies the inequality $3 - d \leq \alpha \leq d - 2$. A useful result that we shall use is the following remark which can be found in [15].

Lemma 1. *Let E be a normalised, indecomposable ACM bundle of rank two on a smooth projective variety X with $\mathrm{Pic}(X) \cong \mathbb{Z}$. Then the zero scheme of any non-zero section of E has codimension 2 in X. In particular, if $\dim X \geq 2$, the zero scheme is non-empty.*

Let $X \subset \mathbb{P}^4$ be a smooth hypersurface of degree d, E an ACM bundle of rank two on X. If C is the zero scheme of a section of E (hence a curve by Lemma 1), one has a short exact sequence

(1) $$0 \to \mathcal{O}_X \to E \to \mathcal{I}_{C/X}(\alpha) \to 0,$$

where $\mathcal{I}_{C/X}$ denotes the ideal sheaf of C in X. It follows that such a C is sub-canonical i.e., $\omega_C \cong \mathcal{O}_C(m)$ for some $m \in \mathbb{Z}$. To determine m, note that from the above exact sequence, $E(-\alpha) \otimes \mathcal{O}_C \cong \mathcal{I}_{C/X}/\mathcal{I}_{C/X}^2$. Taking determinants on both sides and using adjunction, we have $K_X \otimes \omega_C^{-1} = \det E \otimes \mathcal{O}_C(-2\alpha)$. Rewriting, we get $m = \alpha + d - 5$.

The inclusions $C \subset X \subset \mathbb{P}^4$ yield the following short exact sequences:

(2) $$0 \to \mathcal{I}_{C/\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_C \to 0,$$

(3) $$0 \to \mathcal{I}_{C/X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0,$$

(4) $$0 \to \mathcal{O}_{\mathbb{P}^4}(-d) \to \mathcal{I}_{C/\mathbb{P}^4} \to \mathcal{I}_{C/X} \to 0.$$

Finally, E has a length one resolution by a sum of line bundles on $\mathbb{P}^4$:

(5) $$0 \to F_1 \xrightarrow{\Phi} F_0 \to E \to 0$$

where $F_0 = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^4}(-a_i)$, $a_i \geq 0$ for all i, $F_1 = F_0^\vee(\alpha - d)$, and Φ is a skew-symmetric matrix. Details may be found in [4, 18].

Recall (see [21]) that a coherent sheaf $\mathcal{F}$ on X is said to be m-regular in the sense of Castelnuovo-Mumford if $\mathrm{H}^i(X, \mathcal{F}(m - i)) = 0$ for $i > 0$. When $m = 0$, we say that $\mathcal{F}$ is regular.

Lemma 2. *With notation as above, $E(d-\alpha-1)$ is regular in the sense of Castelnuovo-Mumford.*

Proof. We need to check that $\mathrm{H}^i(X, E(d - \alpha - 1 - i)) = 0$ for $i = 1, 2, 3$. The vanishings for $i = 1, 2$ follow from the fact that E is ACM. For $i = 3$, note that $\mathrm{H}^3(E(d - \alpha - 1 - 3)) \cong \mathrm{H}^0(E^\vee(\alpha - d + 4 + d - 5)) \cong \mathrm{H}^0(E(-1)) = 0$ where the first isomorphism is by Serre duality and the second follows from the fact that $E^\vee \cong E(-\alpha)$. $\qquad\square$

Proposition 1. *Let E be a normalised, indecomposable ACM bundle of rank two on $X \subset \mathbb{P}^4$, a general hypersurface of degree d at least 6. Then its first Chern class satisfies the inequality, $3 - d \leq \alpha \leq d - 2$.*

Proof. $E(d - \alpha - 1)$ is regular implies that it is globally generated (see [21], page 99). Since $h^0(E(-1)) = 0$, we must have $d - \alpha - 1 > -1$ and so $\alpha < d$. When $\alpha = d - 1$, E is regular and so all its (minimal) generators are in degree 0 and we have a resolution

$$0 \to F_1 = \mathcal{O}_{\mathbb{P}^4}(-1)^{2d} \xrightarrow{\Phi} F_0 = \mathcal{O}_{\mathbb{P}^4}^{2d} \to E \to 0.$$

This implies in particular that X is defined by $\mathrm{pf}(\Phi) = 0$ where for any skew-symmetric matrix M, $\mathrm{pf}(M) := \sqrt{\det M}$. By an easy dimension count (or see Corollary 2.4 in [4]), a general hypersurface of degree at least 6 is not a linear Pfaffian and hence X cannot support such an E.

To see the lower bound, we reproduce the argument from [18]. Let C be a the zero-scheme of a section of E and let $\pi : C \to \mathbb{P}^1$ be a general projection so that π is finite. Then

$$\pi_* \omega_C \cong \mathcal{H}om(\pi_* \mathcal{O}_C, \mathcal{O}_{\mathbb{P}^1}(-2)).$$

Since $\mathcal{O}_{\mathbb{P}^1} \subset \pi_* \mathcal{O}_C$ is a direct summand, this implies that $\mathrm{H}^0(\pi_* \omega_C(2)) \cong \mathrm{H}^0(\omega_C(2)) \neq 0$. On the other hand, since C is ACM, it is clear from the cohomology sequence associated to sequence (3) that $\mathrm{H}^0(\mathcal{O}_C(l)) = 0$ if $l < 0$. Putting these together, we get $\alpha + d - 5 + 2 \geq 0$ or equivalently $\alpha \geq 3 - d$. $\qquad\square$

Corollary 3. *With notation as above,*

    a) $F_0 = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^4}(-a_i)$, $0 \le a_i < d - \alpha$.
    b) $F_1 \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^4}(-b_i)$, $b_i = d - a_i - \alpha > 0$.
    c) $\mathrm{H}^0(F_1) = 0$ *and* $\mathrm{H}^0(F_0) \cong \mathrm{H}^0(E)$.

Proof. These facts follow immediately from the regularity of $E(d - \alpha - 1)$ and the fact that $F_1 \cong F_0^\vee(\alpha - d)$. $\hfill\square$

2.2. **Griffiths' infinitesimal invariant and a result of Green and Voisin.** The main object of study is an invariant defined by Griffiths [11, 12] which we briefly discuss now. For the details, we refer the reader to *op. cit.* or Chapter 7 of Voisin's book [25].

Let $\mathcal{X} \to S$ be the universal family of smooth, degree d hypersurfaces in $\mathbb{P}^{2m}$. Let $\mathcal{C} \subset \mathcal{X}$ be a family of codimension m subvarieties over S. For a point $s \in S$, let $X = X_s$ and $C := \mathcal{C}_s \subset X$. If l is the degree of C and D_s is a codimension m linear section, then the family of cycles $\mathcal{Z}$ with fibre $\mathcal{Z}_s := d\,\mathcal{C}_s - lD_s$ for $s \in S$ defines a (fibre-wise null-homologous) cycle in $\mathrm{CH}^m(\mathcal{X}/S)_{hom}$. Let $\mathcal{J} := \{J(X_s)\}_{s \in S}$ be the family of intermediate Jacobians. In such a situation, Griffiths defines a holomorphic function $\nu_{\mathcal{Z}} : S \to \mathcal{J}$, called the *normal function*, by $\nu_{\mathcal{Z}}(s) = \mu_s(Z_s)$ where $\mu_s : \mathrm{CH}^m(X_s)_{hom} \to J(X_s)$ is the *Abel-Jacobi* map from the group of null-homologous cycles to the intermediate Jacobian. This normal function satisfies a "quasi-horizontal" condition (see [25], Definition 7.4). Associated to the normal function $\nu_{\mathcal{Z}}$ above, Griffiths (see [11] or [25] Definition 7.8) has defined the infinitesimal invariant $\delta\nu_{\mathcal{Z}}$. Later Green [9] generalised this definition and showed that Griffiths' original infinitesimal invariant is just one of the many infinitesimal invariants that one can associate to a normal function. He also showed that in particular $\delta\nu_{\mathcal{Z}}(s)$ is an element of the dual of the middle cohomology of the following complex

$$\overset{2}{\wedge} \mathrm{H}^1(X, T_X) \otimes \mathrm{H}^{m+1,m-2}(X) \to \mathrm{H}^1(X, T_X) \otimes \mathrm{H}^{m,m-1}(X) \to \mathrm{H}^{m-1,m}(X).$$

We now specialise to the case $m = 2$ where $X \subset \mathbb{P}^4$ is a smooth hypersurface and $C \subset X$ is a curve of degree l. Then $Z := dC - lD$ is a nullhomologous 1-cycle with support $W := C \cup D$. At a point $s \in S$, this infinitesimal invariant is a functional

$$\delta\nu_{\mathcal{Z}}(s) : \ker\left(\mathrm{H}^1(X, T_X) \otimes \mathrm{H}^1(X, \Omega_X^2) \to \mathrm{H}^2(X, \Omega_X^1)\right) \to \mathbb{C}.$$

The following result of Griffiths gives an explicit formula for computing the infinitesimal invariant associated to the family $\mathcal{Z}$ at a point $s \in S$ when restricted to

$$\ker\left(\mathrm{H}^1(X, T_X) \otimes \mathrm{H}^1(X, \mathcal{I}_{W/X} \otimes \Omega_X^2) \to \mathrm{H}^2(X, \Omega_X^1)\right).$$

Theorem 3 (Griffiths [11, 12]). *Let $\nu_{\mathcal{Z}}$ be the normal function as described above. Consider the following diagram:*
(6)

$$\mathrm{H}^1(X, T_X) \otimes \mathrm{H}^1(X, \mathcal{I}_{W/X} \otimes \Omega_X^2)$$
$$\downarrow \beta \qquad\qquad\qquad \searrow \gamma$$
$$0 \to \mathrm{H}^1(W, \Omega_X^1 \otimes \mathcal{O}_W)/\mathrm{H}^1(X, \Omega_X^1) \overset{\lambda}{\to} \quad \mathrm{H}^2(X, \mathcal{I}_{W/X} \otimes \Omega_X^1) \qquad \to \ \mathrm{H}^2(X, \Omega_X^1) \to 0$$
$$\downarrow \chi$$
$$\mathbb{C}$$

where χ is given by integration over the cycle Z. Then $\delta\nu_{\mathcal{Z}}(s)$, the infinitesimal invariant evaluated at a point $s \in S$, is the composite

$$\ker \gamma \to \frac{\mathrm{H}^1(W, \Omega_X^1 \otimes \mathcal{O}_W)}{\mathrm{H}^1(X, \Omega_X^1)} \overset{\chi}{\to} \mathbb{C},$$

where the map

$$\ker \gamma \to \frac{\mathrm{H}^1(W, \Omega_X^1 \otimes \mathcal{O}_W)}{\mathrm{H}^1(X, \Omega_X^1)}$$

is induced by the map β and the above short exact sequence.

The map χ can be understood as follows. Since D is a general plane section of X, by Bertini $C \cap D = \varnothing$. Thus $\mathcal{O}_W \cong \mathcal{O}_C \oplus \mathcal{O}_D$ and so

$$\mathrm{H}^1(W, \Omega_X^1 \otimes \mathcal{O}_W) \cong \mathrm{H}^1(C, \Omega_X^1 \otimes \mathcal{O}_C) \oplus \mathrm{H}^1(D, \Omega_X^1 \otimes \mathcal{O}_D).$$

For any irreducible curve $T \subset X$, let

$$r_T : \mathrm{H}^1(T, \Omega_X^1 \otimes \mathcal{O}_T) \to \mathrm{H}^1(T, \Omega_T^1) \cong \mathbb{C}$$

denote the natural restriction map. For any element $(a, b) \in \mathrm{H}^1(W, \Omega_X^1 \otimes \mathcal{O}_W)$, we define

$$(7) \qquad\qquad\qquad \chi(a, b) := dr_C(a) - lr_D(b) \in \mathbb{C}.$$

It is clear that this map factors via the quotient $\mathrm{H}^1(W, \Omega_X^1 \otimes \mathcal{O}_W)/\mathrm{H}^1(X, \Omega_X^1)$.

The following result is due to Green [9] and Voisin (unpublished).

Theorem 4. *Let $X \subset \mathbb{P}^4$ be a general hypersurface of degree at least 6. Then the infinitesimal invariant $\delta\nu$ of any quasi-horizontal normal function ν, is zero.*

2.3. **Wu's criterion.** Now we are ready to prove the final step i.e. that there are no non-trivial normalised ACM bundles E of rank two on a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$ such that $3 - d \leq \alpha \leq d - 2$. We shall suppose the contrary: that such an E exists on a general hypersurface X as above. In such a situation, (see section 3 of [18] for details) there exists a rank two bundle $\mathcal{E}$ on the universal hypersurface $\mathcal{X} \subset \mathbb{P}^4 \times S'$ where S' is a Zariski open subset of S, the moduli space of smooth, degree d hypersurfaces of $\mathbb{P}^4$, such that for a general point $s \in S'$, $\mathcal{E}_{|X_s}$ is normalised, indecomposable, ACM with first Chern class α. Furthermore, from the construction of this family, one sees that there exists a family of curves $\mathcal{C} \to S'$ such that $\mathcal{C}_s$ is the zero locus of a section of $\mathcal{E}_{|X_s}$. Let $\mathcal{Z}$ be a family of 1-cycles with fibre $\mathcal{Z}_s := d\mathcal{C}_s - lD_s$ where, as before, D_s is plane section of X_s and $l = l(s)$ is the degree of $\mathcal{C}_s$.

We shall show that under the hypotheses above, $\delta\nu_{\mathcal{Z}} \not\equiv 0$. The non-degeneracy of the infinitesimal invariant is shown by refining Xian Wu's proof in [26]. The proof has three main steps, which we describe now.

Let $\partial_f : \Omega_{\mathbb{P}^4}^3(2d) \to K_{\mathbb{P}^4}(3d)$ be the exterior differential between sheaves of meromorphic differential forms, where $\Omega_{\mathbb{P}^4}^3(2d)$ is identified with the sheaf of meromorphic 3-forms with poles of order at most 2 along X and $K_{\mathbb{P}^4}(3d)$ is identified with the sheaf of meromorphic 4-forms with poles of order at most 3 along X. Composing with the natural map $K_{\mathbb{P}^4}(3d) \twoheadrightarrow K_{\mathbb{P}^4}(3d)/K_{\mathbb{P}^4}(2d)$, we get a map $\bar{\partial}_f : \Omega_{\mathbb{P}^4}^3(2d) \to K_{\mathbb{P}^4}(3d)/K_{\mathbb{P}^4}(2d)$. Using the identification $\Omega_{\mathbb{P}^4}^3 \cong T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}$, and taking cohomology, we get

$$\mathrm{H}^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \xrightarrow{\partial_f} \mathrm{H}^0(\mathbb{P}^4, K_{\mathbb{P}^4}(3d)) \to \frac{\mathrm{H}^0(\mathbb{P}^4, K_{\mathbb{P}^4}(3d))}{\mathrm{H}^0(\mathbb{P}^4, K_{\mathbb{P}^4}(2d))}.$$

The cokernel of the composite map above can be identified with $\mathrm{H}^2(X, \Omega_X^1)$ (see [3], Page 174 or [16], Chapter 9 for details). Let $\bar{U} \subset \mathrm{H}^0(\mathbb{P}^4, K_{\mathbb{P}^4}(3d))$ be the subspace defined as follows:

$$(8) \qquad\qquad \bar{U} := \partial_f \mathrm{H}^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \cap \mathrm{H}^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(3d)).$$

The key ingredient in the proof is the following commutative diagram (see *op. cit.*):

(9)
$$\begin{array}{ccc} H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \otimes H^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) & \xrightarrow{\gamma'} & \dfrac{H^0(\mathbb{P}^4, K_{\mathbb{P}^4}(3d))}{\partial_f \, H^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d))} \\ \downarrow & & \downarrow \\ H^1(X, T_X) \otimes H^1(X, \mathcal{I}_{W/X} \otimes \Omega_X^2) & \xrightarrow{\gamma} & H^2(X, \Omega_X^1). \end{array}$$

Here the right vertical map is as explained above. The horizontal maps γ and γ' are (essentially) cup product maps. The vertical map on the left is a tensor product of two maps. The first factor is the composite $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \to H^0(X, \mathcal{O}_X(d)) \to H^1(X, T_X)$. The normal bundle of $X \subset \mathbb{P}^4$ is $\mathcal{O}_X(d)$ and $H^0(X, \mathcal{O}_X(d)) \to H^1(X, T_X)$ is the natural coboundary map in the cohomology sequence of the tangent bundle sequence for this inclusion. The second factor is the composite

$$H^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \to H^0(X, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(d) \otimes \mathcal{O}_X) \to H^1(X, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(d) \otimes T_X).$$

Here the first map is the natural restriction map and the second is obtained as above by first tensoring the tangent bundle sequence with $K_{\mathbb{P}^4}(d) \otimes \mathcal{I}_{W/\mathbb{P}^4}$ and observing that (i) $\mathcal{I}_{W/\mathbb{P}^4} \otimes \mathcal{O}_X \cong \mathcal{I}_{W/X}$ and, (ii) $T_X \otimes K_{\mathbb{P}^4}(d) \cong \Omega_X^2$.

This diagram yields a map $\ker \gamma' \to \ker \gamma$. To show that the infinitesimal invariant $\delta\nu_{\mathcal{Z}}(s) : \ker \gamma \to \mathbb{C}$ is non-zero, we shall show that the composite map

(10)
$$\ker \gamma' \to \ker \gamma \to \mathbb{C}$$

is non-zero (= surjective).

This is done as follows: consider the exact sequence

$$0 \to \mathcal{O}_X(-d) \to \Omega_{\mathbb{P}^4|X}^1 \to \Omega_X^1 \to 0.$$

Taking second exterior powers and tensoring the resulting sequence by $\mathcal{O}_X(d)$, we get a short exact sequence

(11)
$$0 \to \Omega_X^1 \to \Omega_{\mathbb{P}^4|X}^2(d) \to \Omega_X^2(d) \to 0.$$

The inclusion $\Omega_{X|W}^1 \hookrightarrow \Omega_{\mathbb{P}^4}^2(d)_{|W}$ induces a map of cohomologies and we let

$$V := \ker[H^1(W, \Omega_{X|W}^1) \to H^1(W, \Omega_{\mathbb{P}^4}^2(d)_{|W})].$$

The surjectivity of the composite map in equation (10) in turn is accomplished by constructing a surjection from $\ker \gamma'$ to the vector space $\bar{U}$ (defined in equation (8)) such that this fits into a commutative diagram

(12)
$$\begin{array}{ccccc} \ker \gamma' & \to & \ker \gamma & & \\ \downarrow & & & \searrow & \\ \bar{U} & \twoheadrightarrow & V & \twoheadrightarrow & \mathbb{C} \end{array}$$

where the map $\ker \gamma \to \mathbb{C}$ is the infinitesimal invariant evaluated at the point $s \in S$. In the next section, we shall carry out the three steps viz,

Step 1. There exists a surjection $\chi : V \twoheadrightarrow \mathbb{C}$.
Step 2. There exists a surjection $\bar{U} \twoheadrightarrow V$.
Step 3. There exists a surjection $\ker \gamma' \twoheadrightarrow \bar{U}$.

3. Proof of Theorem 1

3.1. **Step 1: The surjection $\chi : V \twoheadrightarrow \mathbb{C}$.** The main result of this section is the following

Proposition 2. *The restriction map*

$$\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4}(d)|_X) \to \mathrm{H}^1(C, \Omega^2_{\mathbb{P}^4}(d)|_C)$$

is zero.

Let us see how this Proposition implies Step 1. The natural map $\Omega^1_{X|C} \to \Omega^1_C$ yields a push-out diagram for sequence (11) (see [23] pages 41–43 for a definition):

(13)
$$
\begin{array}{ccccccccc}
0 & \to & \Omega^1_{X|C} & \to & \Omega^2_{\mathbb{P}^4}(d)_{|C} & \to & \Omega^2_X(d)_{|C} & \to & 0 \\
  &   & \downarrow   &   & \downarrow   &   & \| &   &   \\
0 & \to & \Omega^1_C   & \to & \mathcal{F}   & \to & \Omega^2_X(d)_{|C} & \to & 0.
\end{array}
$$

Lemma 3. *The map* $\mathrm{H}^1(C, \Omega^1_C) \to \mathrm{H}^1(C, \mathcal{F})$ *in the associated cohomology sequence of the bottom row in diagram (13) is zero. Thus we have a surjection*

$$\mathrm{V}_C := \ker[\mathrm{H}^1(C, \Omega^1_{X_{|C}}) \to \mathrm{H}^1(C, \Omega^2_{\mathbb{P}^4}(d)_{|C})] \twoheadrightarrow \mathrm{H}^1(C, \Omega^1_C).$$

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^1(X, \Omega^1_X) & \to & \mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4}(d)_{|X}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1(C, \Omega^1_{X_{|C}}) & \to & \mathrm{H}^1(C, \Omega^2_{\mathbb{P}^4}(d)_{|C}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1(C, \Omega^1_C) & \to & \mathrm{H}^1(C, \mathcal{F}).
\end{array}
$$

The composite of the vertical maps on the left is the natural restriction map $\mathrm{H}^1(X, \Omega^1_X) \to \mathrm{H}^1(C, \Omega^1_C)$ which maps $h_X \mapsto h_C$ where h_A is the hyperplane class for any scheme A. Since both these cohomologies are one-dimensional with h_X and h_C as the respective generators, this map is an isomorphism. Now $\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4}(d)_{|X}) \to \mathrm{H}^1(C, \Omega^2_{\mathbb{P}^4}(d)_{|C})$ is the zero map by Proposition 2, and so this implies that the map $\mathrm{H}^1(C, \Omega^1_C) \to \mathrm{H}^1(C, \mathcal{F})$ is zero. Thus we have a surjection $\mathrm{V}_C \twoheadrightarrow \mathrm{H}^1(C, \Omega^1_C)$. $\square$

Corollary 4 (Step 1)**.** *The composite map*

$$\mathrm{V}_C \hookrightarrow \mathrm{V} = \ker[\mathrm{H}^1(W, \Omega^1_{X_{|W}}) \to \mathrm{H}^1(W, \Omega^2_{\mathbb{P}^4}(d)_{|W})] \overset{\chi}{\to} \mathbb{C}$$

is a surjection. Hence χ is a surjection.

Proof. This first inclusion follows from the fact that $\mathcal{O}_W \cong \mathcal{O}_C \oplus \mathcal{O}_D$. The surjectivity of the composite follows from the definition of χ (see equation (7)) and the above lemma. $\square$

To prove Proposition 2, we shall need a few more results which we prove now.

Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^4}}(-, \mathcal{O}_{\mathbb{P}^4})$ to sequence (5), we get (see [18])

(14) $$0 \to F_0^\vee \overset{\Psi}{\to} F_1^\vee \to E^\vee(d) \to 0.$$

Let $\phi : F_0^\vee \to \mathcal{O}_{\mathbb{P}^4}$ be any morphism (equivalently a section $\phi \in \mathrm{H}^0(F_0)$). Associated to any such morphism, we have a push-out diagram:

$$
\begin{array}{ccccccccc}
0 & \to & F_0^\vee & \overset{\Psi}{\to} & F_1^\vee & \to & E^\vee(d) & \to & 0 \\
  &   & \downarrow \phi &   & \downarrow &   & \| &   &   \\
0 & \to & \mathcal{O}_{\mathbb{P}^4} & \to & G & \to & E^\vee(d) & \to & 0.
\end{array}
$$

Conversely, we have

Lemma 4. *Any exact sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^4} \to G \to E^{\vee}(d) \to 0,$$

arises as a push-out diagram above.

Proof. Any exact sequence as above corresponds to an element of $\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^4}}(E^{\vee}(d), \mathcal{O}_{\mathbb{P}^4})$. Applying the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(-, \mathcal{O}_{\mathbb{P}^4})$ to sequence (14) we get,

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(F_1^{\vee}, \mathcal{O}_{\mathbb{P}^4}) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^4}}(F_0^{\vee}, \mathcal{O}_{\mathbb{P}^4}) \to \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^4}}(E^{\vee}(d), \mathcal{O}_{\mathbb{P}^4}) \to 0.$$

The first term is $\mathrm{H}^0(F_1)$ which is zero by Corollary 3 and thus we have an isomorphism $\mathrm{H}^0(F_0) \cong \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^4}}(E^{\vee}(d), \mathcal{O}_{\mathbb{P}^4})$. $\qquad\square$

Putting these together, we get the following

Corollary 5. *Let $E^{\vee} \xrightarrow{s^{\vee}} \mathcal{O}_X$ be the map induced by a section $s \in \mathrm{H}^0(E)$. Consider the pull-back diagram (see [23] pages 51-53 for definition)*

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_{\mathbb{P}^4} & \to & G & \to & E^{\vee}(d) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow s^{\vee} & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^4} & \to & \mathcal{O}_{\mathbb{P}^4}(d) & \to & \mathcal{O}_X(d) & \to & 0.
\end{array}
$$

By Lemma 4, there is a section $\phi \in \mathrm{H}^0(F_0)$ such that the following diagram commutes:

(15)
$$
\begin{array}{ccccccccc}
0 & \to & F_0^{\vee} & \xrightarrow{\Psi} & F_1^{\vee} & \to & E^{\vee}(d) & \to & 0 \\
 & & \downarrow \phi & & \downarrow & & \downarrow s^{\vee} & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^4} & \to & \mathcal{O}_{\mathbb{P}^4}(d) & \to & \mathcal{O}_X(d) & \to & 0.
\end{array}
$$

In fact, under the isomorphism $\mathrm{H}^0(\mathbb{P}^4, F_0) \cong \mathrm{H}^0(X, E)$ (Corollary 3 (c)), ϕ maps to s.

Remark 2. Since $F_0^{\vee} = \bigoplus \mathcal{O}_{\mathbb{P}^4}(a_i)$ where $a_i \geq 0$, the map ϕ restricted to a summand $\mathcal{O}_{\mathbb{P}^4}(a_i)$ with $a_i > 0$ is zero. Hence ϕ is a split surjection.

Proof of Proposition 2. We remark that $\mathrm{H}^1(C, \Omega^2_{\mathbb{P}^4}(d)|_C) = 0$ when $\alpha < 2$, and so the lemma is obvious in these cases. The proof for all values $\alpha < d - 1$ is as follows. From Corollary 3, $F_1^{\vee} = \bigoplus_i \mathcal{O}_{\mathbb{P}^4}(b_i)$, $b_i > 0$ and so by Bott's formula (see for instance, [22] Page 8) $\mathrm{H}^i(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4} \otimes F_1^{\vee}) = 0$ for $i = 1, 2$. This implies that the boundary map $\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4} \otimes E^{\vee}(d)) \to \mathrm{H}^2(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4} \otimes F_0^{\vee})$ in the cohomology sequence associated to sequence (14) $\otimes \Omega^2_{\mathbb{P}^4}$ is an isomorphism.

Next, we tensor diagram (15) by $\Omega^2_{\mathbb{P}^4}$ and take cohomology to get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4} \otimes E^{\vee}(d)) & \cong & \mathrm{H}^2(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4} \otimes F_0^{\vee}) \\
\downarrow & & \downarrow \\
\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4} \otimes \mathcal{O}_X(d)) & \cong & \mathrm{H}^2(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4}),
\end{array}
$$

where the isomorphism in the bottom row follows again from Bott's formula (*op. cit.*). By Remark 2, the right vertical map above is onto, and this implies that the map

(16)
$$\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4} \otimes E^{\vee}(d)) \to \mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4}(d)|_X)$$

is onto. The map $E^{\vee}(d) \to \mathcal{O}_X(d)$ in diagram (15) is induced by a section $s \in \mathrm{H}^0(X, E)$ and hence has image $\mathcal{I}_{C/X}(d)$, where $C = Z(s)$. Thus the map in equation (16) factors via $\mathrm{H}^1(X, \mathcal{I}_{C/X} \otimes \Omega^2_{\mathbb{P}^4}(d)|_X)$ and so the map

$$\mathrm{H}^1(X, \mathcal{I}_{C/X} \otimes \Omega^2_{\mathbb{P}^4}(d)|_X) \to \mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4}(d)|_X)$$

is surjective. Thus we are done. $\qquad\square$

Remark 3. Proposition 2 is a crucial refinement of Wu's criterion. Wu actually requires that $\mathrm{H}^1(C, \Omega^2_{\mathbb{P}^4}(d)_{|C}) = 0$. Since $\mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4}(d)_{|X})$ is 1-dimensional, we were hopeful that the weaker statement that the map is zero, which is really what we need, might hold with our hypotheses.

3.2. **Step 2: The surjection $\bar{U} \to \mathrm{V}$.** We first describe the map $\bar{U} \to \mathrm{V}$.

Tensoring the short exact sequence

$$0 \to T_X \to T_{\mathbb{P}^4|X} \to \mathcal{O}_X(d) \to 0$$

with $K_{\mathbb{P}^4}(2d)_{|W}$ and taking cohomology, we get

$$0 \to \mathrm{H}^0(W, T_X \otimes K_{\mathbb{P}^4}(2d)_{|W}) \to \mathrm{H}^0(W, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)_{|W}) \to \mathrm{H}^0(W, K_{\mathbb{P}^4}(3d)_{|W}).$$

Since $T_X \otimes K_{\mathbb{P}^4}(d) \cong \Omega^2_X$, we have the following commutative diagram:

(17)
$$
\begin{array}{ccccccc}
0 & \to & \mathrm{U} & \to & \mathrm{H}^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) & \to & \mathrm{H}^0(W, K_{\mathbb{P}^4}(3d)_{|W}) \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & \mathrm{H}^0(W, \Omega^2_X(d)_{|W}) & \to & \mathrm{H}^0(W, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)_{|W}) & \to & \mathrm{H}^0(W, K_{\mathbb{P}^4}(3d)_{|W}).
\end{array}
$$

Here U is defined so that the top row is left exact. From the exactness of the cohomology sequence associated to sequence (11), we get

$$\mathrm{Image}[\mathrm{H}^0(W, \Omega^2_X(d)_{|W}) \to \mathrm{H}^1(W, \Omega^1_{X|W})] = \ker[\mathrm{H}^1(W, \Omega^1_{X|W}) \to \mathrm{H}^1(W, \Omega^2_{\mathbb{P}^4}(d)_{|W})] = \mathrm{V},$$

and hence a surjective map $\mathrm{H}^0(W, \Omega^2_X(d)_{|W}) \twoheadrightarrow \mathrm{V}$. Consider the composite

$$\mathrm{U} \to \mathrm{H}^0(W, \Omega^2_X(d)_{|W}) \twoheadrightarrow \mathrm{V}.$$

Corollary 6. *The map $U \to \mathrm{V}$ factors as $U \to \bar{U} \to \mathrm{V}$.*

Proof. Let $\widetilde{U}$ be the kernel of the map $\mathrm{H}^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \to \mathrm{H}^0(X, K_{\mathbb{P}^4}(3d)_{|X})$. Looking at the diagram analogous to (17) obtained by replacing W by X, we see that there is a map $\widetilde{U} \to \mathrm{H}^0(X, \Omega^2_X(d))$. The boundary map $\mathrm{H}^0(X, \Omega^2_X(d)) \to \mathrm{H}^1(X, \Omega^1_X)$ in the cohomology sequence associated to diagram (11) is the zero map (this is because the composite map $\mathrm{H}^1(X, \Omega^1_X) \to \mathrm{H}^1(X, \Omega^2_{\mathbb{P}^4|X}(d)) \cong \mathrm{H}^2(\mathbb{P}^4, \Omega^2_{\mathbb{P}^4})$ is the Gysin isomorphism). This implies that the map $U \to V$ above factors as $U \twoheadrightarrow U/\widetilde{U} \to V$. Next we claim that the map $U \twoheadrightarrow U/\widetilde{U}$ factors as $U \to \bar{U} \to U/\widetilde{U}$. For this we define $U_P := \ker[\mathrm{H}^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \xrightarrow{\partial_f} \mathrm{H}^0(\mathbb{P}^4, K_{\mathbb{P}^4}(3d))]$. It is enough to check the following:

(1) $U_P \subset \widetilde{U} \subset U$.
(2) ∂_f restricts to a surjective map $U \to \bar{U}$ which induces an isomorphism $U/U_P \cong \bar{U}$.

These follow easily from the definitions of U_P, $\bar{U}$ and $\tilde{U}$. $\qquad\square$

To show that the map $\bar{U} \to \mathrm{V}$ defined above is a surjection, it is enough to prove that the map $U \to \mathrm{V}$ is surjective. For this, we shall need some vanishings which we prove now.

Lemma 5. *For $d \geq 3$, $\mathrm{H}^1(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-4)) = 0 = \mathrm{H}^2(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-5))$.*

Proof. Tensoring the exact sequence

(18)
$$0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^{\oplus 2} \to \mathcal{I}_{D/X} \to 0,$$

by $\mathcal{I}_{C/X}$, we get the exact sequence

$$0 \to \mathcal{I}_{C/X}(-2) \to \mathcal{I}_{C/X}(-1)^{\oplus 2} \to \mathcal{I}_{W/X} \to 0.$$

Left exactness here can be checked at the level of stalks using the fact that $C \cap D = \varnothing$.

For the first vanishing, since C is ACM, taking cohomology of the above sequence we see that the boundary map $H^1(X, \mathcal{I}_{W/X}(2d-4)) \to H^2(X, \mathcal{I}_{C/X}(2d-6))$ in the cohomology sequence associated to the above exact sequence is an injection. Using the exact sequence $0 \to \mathcal{I}_{C/X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$, we see that $H^2(X, \mathcal{I}_{C/X}(2d-6)) \cong H^1(C, \mathcal{O}_C(2d-6))$ which in turn is Serre dual to $H^0(C, \mathcal{O}_C(\alpha-d+1))$. Since C is ACM and $\alpha < d-1$, we have $H^0(C, \mathcal{O}_C(\alpha-d+1)) = 0$, and so $0 = H^1(X, \mathcal{I}_{W/X}(2d-4)) = H^1(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-4))$ (for the last equality, use sequence (4) with C replaced by W).

For the second vanishing, consider the short exact sequence

$$0 \to \mathcal{I}_{W/\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_W \to 0.$$

Taking cohomology, it is easy to see that there are isomorphisms

$$H^2(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-5)) \cong H^1(W, \mathcal{O}_W(2d-5)) \cong H^1(C, \mathcal{O}_C(2d-5)) \oplus H^1(D, \mathcal{O}_D(2d-5)).$$

The first term is Serre dual to $H^0(C, \mathcal{O}_C(\alpha-d))$ and the second term to $H^0(D, \mathcal{O}_D(2-d))$. Since C, D are ACM, $\alpha < d-1$ and $d \geq 3$, it follows that $H^0(C, \mathcal{O}_C(\alpha-d))$ and $H^0(D, \mathcal{O}_D(2-d))$ are both zero. This finishes the proof. $\square$

Lemma 6. *For $d \geq 5$, $H^1(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes \mathcal{I}_{W/\mathbb{P}^4}(2d-5)) = 0$.*

Proof. Tensoring the Euler sequence by $\mathcal{I}_{W/\mathbb{P}^4}(2d-5)$, we get a short exact sequence (see [27] for left exactness)

$$0 \to \mathcal{I}_{W/\mathbb{P}^4}(2d-5) \to \mathcal{I}_{W/\mathbb{P}^4}(2d-4)^{\oplus 5} \to \mathcal{I}_{W/\mathbb{P}^4}(2d-5) \otimes T_{\mathbb{P}^4} \to 0.$$

This gives rise to a part of a long exact sequence of cohomology

$$\to H^1(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-4))^{\oplus 5} \to H^1(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes \mathcal{I}_{W/\mathbb{P}^4}(2d-5)) \to H^2(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-5)) \to$$

By Lemma 5, the extreme terms vanish and so we are done. $\square$

Proposition 3. *For $d \geq 5$, the natural map $U \to V$ is a surjection.*

Proof. The middle vertical arrow in diagram (17) can be seen to be a surjection by using the fact that the cokernel of this map injects into $H^1(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4} \otimes \mathcal{I}_{W/\mathbb{P}^4}(2d))$ which vanishes by Lemma 6. By snake lemma, the first map is also a surjection. Thus the map $U \to H^0(W, \Omega_X^2(d)_{|W})$ is a surjection. This finishes the proof. $\square$

Thus we have the required

Corollary 7 (Step 2). *For $d \geq 5$, the map $\bar{U} \to V$ is also a surjection.*

3.3. **Step 3: The surjection** $\ker \gamma' \to \bar{U}$**.** We first describe the map $\ker \gamma' \to \bar{U}$.

Recall from section 2.3 that γ' is the natural map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \otimes H^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \to \frac{H^0(\mathbb{P}^4, K_{\mathbb{P}^4}(3d))}{\partial_f H^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d))}.$$

Consider the multiplication map

$$(19) \qquad H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \otimes H^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \to H^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(3d)).$$

Restricting this map to $\ker \gamma'$, we get a map

$$\ker \gamma' \to \bar{U} = \partial_f H^0(\mathbb{P}^4, T_{\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(2d)) \cap H^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4} \otimes K_{\mathbb{P}^4}(3d)).$$

Proposition 4 (Step 3). *For $d \geq 5$, the map $\ker \gamma' \to \bar{U}$ is surjective.*

Proof. To prove surjectivity of the above map, it is enough to prove that for $d \geq 5$, the multiplication map in equation (19) i.e., the map

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \otimes \mathrm{H}^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-5)) \to \mathrm{H}^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(3d-5))$$

is surjective.

To see this, we first tensor the exact sequence

$$(20) \qquad 0 \to \mathcal{O}_X(-2) \to \mathcal{O}_X(-1)^{\oplus 2} \to \mathcal{I}_{D/X} \to 0,$$

by E to get

$$(21) \qquad 0 \to E(-2) \to E(-1)^{\oplus 2} \to \mathcal{I}_{D/X} E \to 0.$$

Let $T_m \coloneqq \mathrm{H}^0(X, \mathcal{O}_X(m))$. The exact sequence above gives rise to a diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \to & \mathrm{H}^0(X, E(n-2)) \otimes T_m & \to & \mathrm{H}^0(X, E(n-1))^{\oplus 2} \otimes T_m & \to & \mathrm{H}^0(X, \mathcal{I}_{D/X} E(n)) \otimes T_m & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{H}^0(X, E(m+n-2)) & \to & \mathrm{H}^0(X, E(m+n-1))^{\oplus 2} & \to & \mathrm{H}^0(X, \mathcal{I}_{D/X} E(m+n)) & \to & 0.
\end{array}
$$

Here the vertical arrows are all multiplication maps and the exactness on the right is because E is ACM.

Since E is $(d-\alpha-1)$–regular, the middle vertical arrow is a surjection for $n \geq d-\alpha$ and $m \geq 0$. It follows that the multiplication map

$$\mathrm{H}^0(X, \mathcal{I}_{D/X} E(n)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(m)) \to \mathrm{H}^0(X, \mathcal{I}_{D/X} E(m+n))$$

is surjective for $n \geq (d-\alpha)$ and $m \geq 0$. Next consider the exact sequence

$$0 \to \mathcal{I}_{D/X} \to \mathcal{I}_{D/X} E \to \mathcal{I}_{W/X}(\alpha) \to 0$$

obtained by tensoring sequence (1) by $\mathcal{I}_{D/X}$. As before, left exactness here can be checked at the level of stalks using the fact that $C \cap D = \varnothing$. Repeating the previous argument, it is easy to check that the multiplication map

$$\mathrm{H}^0(X, \mathcal{I}_{W/X}(n)) \otimes \mathrm{H}^0(X, \mathcal{O}_X(m)) \to \mathrm{H}^0(X, \mathcal{I}_{W/X}(m+n))$$

is surjective for $n \geq d$ and $m \geq 0$. In particular, if $d \geq 5$, the map is surjective for $n = 2d-5$. Also the multiplication map

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m)) \otimes \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(n)) \to \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(m+n))$$

is surjective for $m, n \geq 0$. Now using the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-d) \to \mathcal{I}_{W/\mathbb{P}^4} \to \mathcal{I}_{W/X} \to 0,$$

and repeating the argument above, we can conclude using snake lemma, that the multiplication map

$$\mathrm{H}^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(2d-5)) \otimes \mathrm{H}^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \to \mathrm{H}^0(\mathbb{P}^4, \mathcal{I}_{W/\mathbb{P}^4}(3d-5))$$

is surjective (again $d \geq 5$ is needed here). $\qquad\square$

3.4. **Non-degeneracy of the infinitesimal invariant.**

Proposition 5. *In the situation above, $\delta\nu_{\mathcal{Z}} \not\equiv 0$.*

Proof. We shall show that $\delta\nu_{\mathcal{Z}}(s) \neq 0$ at any point $s \in S$ parametrising a smooth hypersurface $X \subset \mathbb{P}^4$. From steps 1–3, we have surjections $\ker \gamma' \twoheadrightarrow \bar{U} \twoheadrightarrow V \overset{\chi}{\twoheadrightarrow} \mathbb{C}$. By the compatibility of these maps (see [26]) with the map $\ker \gamma' \to \ker \gamma$ and those in diagram (6), we conclude (using Griffiths' formula) that $\delta\nu_{\mathcal{Z}}(s) \neq 0$. $\qquad\square$

Proof of Theorem 1. Assume that a general hypersurface X supports an indecomposable, ACM vector bundle E of rank two. As seen earlier, we may assume that E is normalised, with $3 - d \le \alpha \le d - 2$. Let $\mathcal{Z}$ be the family of degree zero 1-cycles defined earlier. By the refined Wu's criterion $\delta\nu_{\mathcal{Z}} \not\equiv 0$: this contradicts Green's theorem. Thus we are done. $\square$

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