HODGE STYLE CHERN CLASSES FOR VECTOR BUNDLES ON SCHEMES

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1. Introduction

In this note, we develop the formalism of Hodge style chern classes of vector bundles over arbitrary quasi-projective schemes defined over K, a field of characteristic zero. The theory of Chern classes is well known by now and without any restriction on the characteristic, can be defined in many theories with rational coefficients, like for example the Chow ring. Atiyah [1] developed the theory of Chern classes of vector bundles with values in the Hodge ring $\oplus \mathrm{H}^p(X, \Omega^p_X)$ for smooth complex varieties. Grothendieck [2] remarked that Atiyah's constructions could be transposed (*"sans difficulté"*) to the case of any S-scheme X and referred to a future paper where it would appear. To the best of our knowledge, this has not occured.

The primary purpose of this note is to satisfy ourselves that indeed the formalism extends to the case of arbitrary schemes and at the same time to fill a gap in the existing literature. Everything in this paper is "known" to experts and has indeed been written down many times in the case of smooth varieties and has been generalised in various directions. Our purpose is to provide a suitable reference for our own use of this theory on schemes as well as to provide a self-contained exposition that would be suitable for a new-comer to the subject.

Our personal motivation for making sure that the theory was valid for arbitrary schemes was in understanding intersection theory on non-reduced hypersurfaces in projective spaces (see [7]). Unlike in the theory of Chern classes with values in the Chow ring, there is a closer connection between the Hodge cohomology of these hypersurfaces and that of the projective space.

The goal of this paper is to verify the following: let X be a quasi-projective scheme over a field K of characteristic zero. For any vector bundle $\mathcal{E}$ of rank r on X, there is an element $\mathrm{c}(\mathcal{E}) = 1+\sum_{i=1}^{r} \mathrm{c}_i(\mathcal{E})$ in the graded commutative K-algebra $\oplus \mathrm{H}^i(X, \Omega^i_X)$ satisfying the following two properties:

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(1) If $f : Y \to X$ is a morphism, then $f^*(\mathrm{c}_i(\mathcal{E})) = \mathrm{c}_i(f^*\mathcal{E})$.
(2) If $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0$ is an exact sequence of vector bundles on X, then $\mathrm{c}(\mathcal{E}) = \mathrm{c}(\mathcal{F})\,\mathrm{c}(\mathcal{G})$.

In order to do so, we approach it at the level of a standard textbook like [4] sketching the various theories required. One of the remarks we make is that in order to get a theory which satisfies the Whitney sum formula (2) above, it is necessary that the field have characteristic zero or larger than the ranks of the bundles involved.

2. Cup products in cohomology of schemes

Let X be a quasi-projective scheme defined over a base ring R, $\mathcal{F}$, $\mathcal{G}$, coherent sheaves of $\mathcal{O}_X$-modules over X. We will review the theory of cup-products where given $\alpha \in \mathrm{H}^p(X, \mathcal{F})$ and $\beta \in \mathrm{H}^q(X, \mathcal{G})$, $\alpha \cup \beta$ is defined in $\mathrm{H}^{p+q}(X, \mathcal{F} \otimes \mathcal{G})$. This theory is well known going back to [3] where it is shown that cup product is a homomorphism

$$\mathrm{H}^p(X; \mathcal{F}) \otimes_{\mathrm{H}^0(X, \mathcal{O}_X)} \mathrm{H}^q(X, \mathcal{G}) \to \mathrm{H}^{p+q}(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

2.1. **Cohomology.** Fix an affine open cover $\mathfrak{U} := \{U_i\}$ of X. It is well known that sheaf cohomology of a coherent sheaf $\mathcal{F}$ (defined using injective resolutions) on X can equivalently be computed using Cech cocycles relative to the cover $\mathfrak{U}$ (see for instance [4]).

The Cech cohomology groups of $\mathcal{F}$ are defined as follows:

Given an affine open cover $\mathfrak{U}$ of X, define α to be a k-cochain with values in $\mathcal{F}$ if for each $(k+1)$-tuple $(i_0, \cdots, i_k)$, $i_0 < \cdots < i_k$, $\alpha(i_0, \cdots, i_k) \in \Gamma(U_{i_0 \cdots i_k}, \mathcal{F})$ where $U_{i_0 \cdots i_k} := \cap_{j=0}^{k} U_{i_j}$.

Define a boundary map

$$\partial : \oplus\Gamma(U_{i_0 \cdots i_k}, \mathcal{F}) \to \oplus\Gamma(U_{i_0 \cdots i_{k+1}}, \mathcal{F})$$

where $\partial\alpha(i_0, \cdots, i_{k+1}) = \Sigma_{j=0}^{k+1} (-1)^j \alpha(i_0, \cdots, \hat{i_j}, \cdots, i_{k+1})$

It is standard that $\partial^2 = 0$. Thus we get a complex which we denote by $\mathcal{C}^\bullet(X, \mathfrak{U}, \mathcal{F})$ and refer to as the Cech complex.

Definition 1. (1) A *k-cocycle* is a k-cochain α such that $\partial\alpha = 0$
(2) A *k-coboundary* is a k-cochain of the form $\partial\alpha$ where α is a $(k-1)$-cochain.
(3) The k-th Cech cohomology group of $\mathcal{F}$ with respect to the affine open cover $\mathfrak{U}$ is defined as

$$\overset{\vee}{\mathrm{H}}{}^k_{\mathfrak{U}}(X, \mathcal{F}) := \frac{k - \text{cocycles}}{k - \text{coboundaries}}$$

For convenience of notation, in the ensuing discussion we shall drop the subscript $\mathfrak{U}$ and simply denote the above cohomology group by $\mathrm{H}^k(X,\mathcal{F})$.

Remark 1. In the usual theory of Cech cohomology, one looks not at just one open cover but also refinements of a given cover. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, then there are induced maps of Cech complexes $\mathcal{C}^\bullet(X,\mathfrak{U},\mathcal{F}) \to \mathcal{C}^\bullet(X,\mathfrak{V},\mathcal{F})$ which descends to a map of cohomologies. It is a fact then that the sheaf cohomology is a direct limit over all such coverings.

Given a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

there exists a long exact sequence of Cech cohomology groups

$$\cdots \to \mathrm{H}^k(X,\mathcal{F}') \to \mathrm{H}^k(X,\mathcal{F}) \to \mathrm{H}^k(X,\mathcal{F}'') \xrightarrow{\delta_k} \mathrm{H}^{k+1}(X,\mathcal{F}')$$

where the *connecting homomorphism* δ_i is defined as follows:

Locally $\alpha(i_0,\cdots,i_k) \in \Gamma(U_{i_0\cdots i_k},\mathcal{F}'')$ can be lifted to an element $\tilde{\alpha}(i_0,\cdots,i_k) \in \Gamma(U_{i_0\cdots i_k},\mathcal{F})$. Since the image of $\partial\tilde{\alpha}$ in $\mathcal{F}''$ is zero, therefore $\partial\tilde{\alpha}(i_0,\cdots,i_{k+1}) \in \Gamma(U_{i_0\cdots i_{k+1}},\mathcal{F}')$. We now define $\delta\alpha = [\partial\tilde{\alpha}] \in \mathrm{H}^{k+1}(\mathcal{F}')$. It can be shown that this is independent of the choices.

2.2. **Yoneda Extensions.** For any coherent sheaves $\mathcal{F}$ and $\mathcal{K}$, elements in the group $\mathrm{Ext}^p(\mathcal{K},\mathcal{F})$ for $p \geq 1$ can be interpreted as Yoneda extensions of length p of the form

$$0 \to \mathcal{F} \to \mathcal{P}_1' \to \cdots \to \mathcal{P}_p' \to \mathcal{K} \to 0 \tag{1}$$

For our purposes, we will be interested in extensions where $\mathcal{K}$ is a vector bundle. In this situation, we will show that we can restrict our attention to extensions where the $\mathcal{P}_i'$ are vector bundles for $i = 2,\cdots,p$.

We use induction on p. Let $\mathcal{K}$ be any vector bundle. The base case $p = 1$ is obvious. Assume it is true for $p-1$. Since X is quasi-projective, $\mathcal{K}(a)$ is generated by global sections for $a >> 0$. Thus we get a short exact sequence of vector bundles

$$0 \to \mathcal{K}' \to \mathcal{P}_p \to \mathcal{K} \to 0$$

where $\mathcal{P}_p$ is sum of very negative line bundles. As a result we have $\mathrm{Ext}^j(\mathcal{P}_p,\mathcal{F}) = 0$ for $j \geq 1$. Via the associated long exact sequence of the Ext groups, one has $\mathrm{Ext}^{p-1}(\mathcal{K}',\mathcal{F}) \xrightarrow{\cong} \mathrm{Ext}^p(\mathcal{K},\mathcal{F})$ where the image of an element

$$0 \to \mathcal{F} \to \mathcal{P}_1 \to \cdots \to \mathcal{P}_{p-1} \to \mathcal{K}' \to 0$$

in $\mathrm{Ext}^{p-1}(\mathcal{K}, \mathcal{F})$ is obtained by splicing as

$$0 \to \mathcal{F} \to \mathcal{P}_1 \to \cdots \to \mathcal{P}_p \to \mathcal{K} \to 0$$

By induction, $\mathcal{P}_i$ is a vector bundle for $i = 2, \cdots, p-1$ and hence for $i = 2, \cdots, p$.

Lemma 1. *Given $\alpha \in \mathrm{H}^p(X, \mathcal{F})$ for $p \geq 1$, there exists a short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{P}_\alpha \to \mathcal{Q}_\alpha \to 0 \tag{2}$$

satisying

(1) *There exists an element $\alpha' \in \mathrm{H}^{p-1}(X, \mathcal{Q}_\alpha)$ which maps to $\alpha \in \mathrm{H}^p(X, \mathcal{F})$*
(2) *For any coherent sheaf $\mathcal{G}$, the sequence above remains exact on tensoring with $\mathcal{G}$.*

Proof. There is an isomorphism $\mathrm{H}^p(X, \mathcal{F}) \to \mathrm{Ext}^p(\mathcal{O}_X, \mathcal{F})$ under which the element $\alpha \in \mathrm{H}^p(X, \mathcal{F})$ is associated to an extension

$$0 \to \mathcal{F} \to \mathcal{P}_1 \to \cdots \to \mathcal{P}_p \to \mathcal{O}_X \to 0$$

Further, the element $1 \in \mathrm{H}^0(X, \mathcal{O}_X)$ maps to $\alpha \in \mathrm{H}^p(X, \mathcal{F})$ via the various connecting homomorphisms. Since we may choose $\mathcal{P}_i$ for $i = 2, \cdots, p$ to be vector bundles, we get a short exact sequence by choosing $\mathcal{P}_\alpha = \mathcal{P}_1$ and $\mathcal{Q}_\alpha$ as the kernel of the map $\mathcal{P}_2 \to \mathcal{P}_3$. It is clear that $1 \in \mathrm{H}^0(X, \mathcal{O}_X)$ maps to an element $\alpha' \in \mathrm{H}^{p-1}(X, \mathcal{Q}_\alpha)$ which in turn maps to $\alpha \in \mathrm{H}^p(X, \mathcal{F})$. This proves (1) above. For (2), we note that since $\mathcal{Q}_\alpha$ is a vector bundle, this implies that the sequence in (1) is locally split and so remains exact on tensoring with $\mathcal{G}$. □

2.3. **Cup products.** Let $\alpha \in \mathrm{H}^p(X, \mathcal{F})$ and $\beta \in \mathrm{H}^q(X, \mathcal{G})$. We define $\alpha \cup \beta \in \mathrm{H}^{p+q}(X, \mathcal{F} \otimes \mathcal{G})$ as follows:

Let $\tilde{\alpha}$ and $\tilde{\beta}$ be cocycle representatives of α and β repectively. Then define $\tilde{\alpha} \cup \tilde{\beta}(i_0, \cdots, i_p, i_{p+1}, \cdots, i_{p+q}) := \tilde{\alpha}(i_0, \cdots, i_p) \otimes \tilde{\beta}(i_p, \cdots, i_{p+q})$

Lemma 2. *$\tilde{\alpha} \cup \tilde{\beta}$ is a cocycle.*

Proof.

$$\partial(\tilde{\alpha} \cup \tilde{\beta})(i_0, \cdots, i_{p+q+1}) = \sum_{j=0}^{p+q+1} (-1)^j \tilde{\alpha} \cup \tilde{\beta}(i_0, \cdots, \hat{i_j}, \cdots, i_{p+q+1})$$

$$= \left(\sum_{j=0}^{p} (-1)^j \tilde{\alpha}(i_0, \cdots, \hat{i_j}, \cdots, i_{p+1}) \right) \otimes \tilde{\beta}(i_{p+1}, \cdots, i_{p+q+1})$$

$$+\tilde{\alpha}(i_0,\cdots,i_p)\left(\sum_{j=p+1}^{p+q+1}(-1)^j\tilde{\beta}(i_p,\cdots,\hat{i_j},\cdots,i_{p+q+1})\right)$$
$$=(-1)^p\tilde{\alpha}(i_0,\cdots,i_p)\otimes\tilde{\beta}(i_{p+1},\cdots,i_{p+q+1})$$
$$+(-1)^{p+1}\tilde{\alpha}(i_0,\cdots,i_p)\otimes\tilde{\beta}(i_{p+1},\cdots,i_{p+q+1})=0$$

□

To check that $\alpha\cup\beta$ is well defined is standard and we leave it to the reader. Furthermore, for morphisms $\mathcal{F}\to\mathcal{F}'$ and $\mathcal{G}\to\mathcal{G}'$, it is obvious that if α,β are elements in the cohomologies of $\mathcal{F}$ and $\mathcal{G}$ with images α' and β', then $\alpha'\cup\beta'$ is the image of $\alpha\cup\beta$.

The isomorphism $(\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H}\cong\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H})$ which defines associativity of tensor products induces an isomorphism

$$\mathrm{H}^{p+q+r}(X,(\mathcal{F}\otimes\mathcal{G})\otimes\mathcal{H})\cong\mathrm{H}^{p+q+r}(X,\mathcal{F}\otimes(\mathcal{G}\otimes\mathcal{H}))$$

under which it is obvious that $(\alpha\cup\beta)\cup\gamma\mapsto\alpha\cup(\beta\cup\gamma)$. In other words, cup product is an associative operation.

Now let $\alpha\in\mathrm{H}^p(X,\mathcal{F})$ and $\beta\in\mathrm{H}^q(X,\mathcal{G})$. Then one has the following two sequences from Lemma (1):

$$0\to\mathcal{F}\otimes\mathcal{G}\to\mathcal{P}_\alpha\otimes\mathcal{G}\to\mathcal{Q}_\alpha\otimes\mathcal{G}\to0 \tag{3}$$

$$0\to\mathcal{F}\otimes\mathcal{G}\to\mathcal{F}\otimes\mathcal{P}_\beta\to\mathcal{F}\otimes\mathcal{Q}_\beta\to0 \tag{4}$$

Lemma 3. *Let $\alpha_0\in\mathrm{H}^{p-1}(X,\mathcal{Q}_\alpha)$ be a pre-image of α i.e., $\delta\alpha_0=\alpha$. Then $\alpha\cup\beta=\delta(\alpha_0\cup\beta)$ where δ is the connecting homomorphism in (3).*

Proof. By definition,

$$\alpha_0\cup\beta=\alpha_0(i_0,\cdots,i_p)\otimes\beta(i_p,\cdots,i_{p+q})=\tilde{\alpha}_0(i_0,\cdots,i_p)\otimes\beta(i_p,\cdots,i_{p+q})$$

On the other hand,
$\delta(\alpha_0\cup\beta)(i_0,\cdots,i_{p+q+1})$

$$=\begin{array}{c}[\delta\alpha_0(i_0,\cdots,i_{p+1})+(-1)^p\tilde{\alpha}_0(i_0,\cdots,i_p)]\otimes\beta(i_{p+1},\cdots,i_{p+q+1})\\+\\(-1)^{p+1}\tilde{\alpha}_0(i_0,\cdots,i_p)\otimes\beta(i_{p+1},\cdots,i_{p+q+1})\end{array}$$

$$=\partial\tilde{\alpha}_0(i_0,\cdots,i_{p+1})\otimes\beta(i_{p+1},\cdots,i_{p+q+1})$$

$$=\delta\alpha\cup\beta(i_0,\cdots,i_{p+q+1})$$

□

Lemma 4. *Let $\beta_0 \in \mathrm{H}^{q-1}(X, \mathcal{Q}_\beta)$ such that $\delta\beta_0 = \beta$. Then $\alpha \cup \beta = (-1)^p \delta(\alpha \cup \beta_0)$ where δ is the connecting homomorphism in (4).*

Proof. Let $\tilde{\beta}_0$ be a local lift to the sheaf $\mathcal{P}_\beta$. Then $\delta(\alpha \cup \beta_0)$

$$= \begin{array}{c} (-1)^p \alpha(i_0, \cdots, i_p) \otimes \tilde{\beta}(i_{p+1}, \cdots, i_{p+q+1}) \\ + \\ (-1)^p \alpha(i_0, \cdots, i_p) \otimes [\tilde{\beta}_0(i_{p+1}, \cdots, i_{p+q+1}) - \delta\tilde{\beta}_0(i_p, \cdots, i_{p+q+1})] \end{array}$$

$$= (-1)^p \alpha(i_0, \cdots, i_p) \otimes \partial\tilde{\beta}_0(i_p, \cdots, i_{p+q+1})$$

$$= (-1)^p (\alpha \cup \delta\beta)(i_0, \cdots, i_{p+q+1})$$

□

Lemma 5. *Let $\psi : \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \otimes \mathcal{F}$ be the obvious map. For $\alpha \in \mathrm{H}^p(\mathcal{F})$ and $\beta \in \mathrm{H}^q(\mathcal{G})$,*

$$\psi(\beta \cup \alpha) = (-1)^{pq} \alpha \cup \beta$$

Proof. The proof is by induction on q. When $q = 0$, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{F} \otimes \mathrm{H}^0(X, \mathcal{G}) & \xrightarrow{\psi} & \mathrm{H}^0(X, \mathcal{G}) \otimes \mathcal{F} \\ \downarrow \mathrm{id} \otimes \mathrm{ev} & & \mathrm{ev} \otimes \mathrm{id} \downarrow \\ \mathcal{F} \otimes \mathcal{G} & \xrightarrow{\psi} & \mathcal{G} \otimes \mathcal{F} \end{array}$$

Here $\mathrm{ev} : \mathrm{H}^0(X, \mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G}$ is the evaluation map and id is the identity map. Taking cohomology we see that the image $\mathrm{ev} \otimes \mathrm{id}(\alpha \otimes \beta) = \alpha \cup \beta$. The commutativity of the diagram then yields $\psi(\alpha \cup \beta) = \beta \cup \alpha$.

We now prove the statement for arbitrary $\beta \in \mathrm{H}^q(X, \mathcal{G})$. For each such β, we have a commuting diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{G} \otimes \mathcal{F} & \to & \mathcal{P}_\beta \otimes \mathcal{F} & \to & \mathcal{Q}_\beta \otimes \mathcal{F} & \to & 0 \\ & & \downarrow \psi & & \downarrow \psi & & \downarrow \psi & & \\ 0 & \to & \mathcal{F} \otimes \mathcal{G} & \to & \mathcal{F} \otimes \mathcal{P}_\beta & \to & \mathcal{F} \otimes \mathcal{Q}_\beta & \to & 0 \end{array}$$

Assume statement is true for $q' = q - 1$ i.e. $\psi(\beta' \cup \alpha) = (-1)^{pq'} \alpha \cup \beta'$. Let $\beta' \in \mathrm{H}^{q-1}(X, \mathcal{Q}_\beta)$ such that $\delta(\beta') = \beta$. Then we have $\delta\psi(\beta' \cup \alpha) = (-1)^{pq'} \delta(\alpha \cup \beta')$. Since δ commutes with ψ, the left hand side in the statement of the Lemma is

$$\psi(\beta \cup \alpha) = \psi(\delta(\beta') \cup \alpha) = \psi\delta(\beta' \cup \alpha) = \delta\psi(\beta' \cup \alpha) = \delta((-1)^{pq'} \alpha \cup \beta'))$$
$$= (-1)^{pq'} (-1)^p (\alpha \cup \beta) = (-1)^{pq} (\alpha \cup \beta)$$

□

2.4. **Leray maps.** Let $f : Y \to X$ be a morphism of quasi-projective schemes. There are homomorphisms called Leray maps

$$\ell^p : \mathrm{H}^p(X, \mathcal{F}) \to \mathrm{H}^p(Y, f^*\mathcal{F})$$

which are defined as follows. Let $\mathfrak{U} = \{U_i\}$ and $\mathfrak{V} = \{V_{ij}\}$ be affine covers of X and Y respectively where $f^{-1}U_i := \bigcup_j V_{ij}$ so that $\mathfrak{V}$ is a refinement of $f^{-1}\mathfrak{U}$ and is given the lexicographic ordering induced by the ordering on $\mathfrak{U}$. There are maps f^*

$$\Gamma(U_i, \mathcal{F}) \to \Gamma(f^{-1}U_i, f^*\mathcal{F}) \to \Gamma(V_{ij}, f^*\mathcal{F})$$

which induce a map f^* of Cech complexes

$$\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^\bullet(f^{-1}\mathfrak{U}, f^*\mathcal{F}) \to \mathcal{C}^\bullet(\mathfrak{V}, f^*\mathcal{F})$$

These in turn induce maps

$$\mathrm{H}^p(X, \mathcal{F}) \cong \mathrm{H}^p_{\mathfrak{U}}(X, \mathcal{F}) \to \mathrm{H}^p_{f^{-1}\mathfrak{U}}(Y, f^*\mathcal{F}) \to \mathrm{H}^p_{\mathfrak{V}}(Y, f^*\mathcal{F}) \cong \mathrm{H}^p(Y, f^*\mathcal{F})$$

which are the Leray maps ℓ^p.

Lemma 6 (Commutativity of the Leray maps with cup products)**.** *Let $f : Y \to X$ be a morphism and $\mathcal{F}$, $\mathcal{G}$ be sheaves on X. Then given $\alpha \in \mathrm{H}^p(X, \mathcal{F})$ and $\beta \in \mathrm{H}^q(X, \mathcal{G})$, we have*

$$\ell^p(\alpha) \cup \ell^q(\beta) = \ell^{p+q}(\alpha \cup \beta) \in \mathrm{H}^{p+q}(Y, f^*\mathcal{F} \otimes f^*\mathcal{G})$$

Proof. Let $\tilde{\alpha}$ (resp. $\tilde{\beta}$) be a cocycle representing the class α (resp. β). Then $f^*\tilde{\alpha}$ is the image of $\tilde{\alpha}$ under the map $\mathcal{C}^p(\mathfrak{U}, \mathcal{F}) \to \mathcal{C}^p(\mathfrak{V}, f^*\mathcal{F})$. For any affine open set U which is an appropriate intersection of U_i's, let V be an appropriate intersection V_{ij}'s. Under the natural map

$$\Gamma(U, \mathcal{F} \otimes \mathcal{G}) \to \Gamma(f^{-1}U, f^*\mathcal{F} \otimes f^*\mathcal{G}) \to \Gamma(V, f^*\mathcal{F} \otimes f^*\mathcal{G})$$

$$\tilde{\alpha} \otimes \tilde{\beta} \mapsto f^*(\tilde{\alpha} \otimes \tilde{\beta}) = f^*(\tilde{\alpha}) \otimes f^*(\tilde{\beta})$$

The result now follows from the definition of cup products. □

2.5. **Exterior powers.** Let $\alpha \in \mathrm{H}^p(X, \overset{\mathrm{p}}{\wedge} \mathcal{A})$ and $\beta \in \mathrm{H}^q(X, \overset{\mathrm{q}}{\wedge} \mathcal{A})$. The morphism

$$\psi : \overset{\mathrm{p}}{\wedge} \mathcal{A} \otimes \overset{\mathrm{q}}{\wedge} \mathcal{A} \to \overset{\mathrm{q}}{\wedge} \mathcal{A} \otimes \overset{\mathrm{p}}{\wedge} \mathcal{A}$$

descends to the map

$$\overline{\psi} : \overset{\mathrm{p+q}}{\wedge} \mathcal{A} \to \overset{\mathrm{p+q}}{\wedge} \mathcal{A}$$

which sends $a_1 \wedge \cdots \wedge a_{p+q} \mapsto a_{p+1} \wedge \cdots \wedge a_{p+q} \wedge a_1 \wedge \cdots \wedge a_p$

Clearly $\overline{\psi} = (-1)^{pq}$

Define $\alpha \wedge \beta$ to be $\pi(\alpha \cup \beta)$ where π is the projection

$$\mathrm{H}^{p+q}(X, \overset{\mathrm{p}}{\wedge} \mathcal{A} \otimes \overset{\mathrm{q}}{\wedge} \mathcal{A}) \xrightarrow{\pi} \mathrm{H}^{p+q}(X, \overset{\mathrm{p+q}}{\wedge} \mathcal{A})$$

Lemma 7. *For α, β as above, $\alpha \wedge \beta = \beta \wedge \alpha$.*

Proof. $\pi\psi(\alpha \cup \beta) = \pi((-1)^{pq}\beta \cup \alpha) = (-1)^{pq}\beta \wedge \alpha$. Since $\pi\psi = \overline{\psi}\pi$, and $\overline{\psi}\pi(\alpha \cup \beta) = (-1)^{pq}\alpha \wedge \beta$ it follows then that $\alpha \wedge \beta = \beta \wedge \alpha$. □

Remark 2. More generally, if $\sigma \in S_n$, the symmetric group and $\alpha_i \in \mathrm{H}^{p_i}(X, \mathcal{A}_i)$, one has $\overset{\mathrm{n}}{\wedge}_{i=1} \alpha_i = \overset{\mathrm{n}}{\wedge}_{i=1} \alpha_{\sigma(i)}$

Taking $\mathcal{A}$ to be Ω^1_X, the sheaf of Kähler differentials, we have

Proposition 1. $\oplus_{i=1}^n \mathrm{H}^i(X, \Omega^i_X)$ *is a graded commutative ring with wedge product for multiplication.*

3. Chern Classes

3.1. **The Atiyah sequence.** Let X be a quasi-projective scheme over K and $\mathcal{E} \to X$ be a vector bundle. Let $\Delta_X \hookrightarrow X \times X$ be the inclusion of the "diagonal" subscheme and let $\mathcal{I}$ denote the ideal sheaf of Δ_X in $X \times X$ and $\mathcal{X}$ denote the thickened diagonal given by $\mathcal{I}^2$. Then $I := \mathcal{I}/\mathcal{I}^2$ is the ideal of Δ_X in $\mathcal{X}$. Let $p, q : \mathcal{X} \to X$ denote the two projections. One has the following natural exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules:

$$0 \to I \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\Delta_X} \to 0 \tag{5}$$

The push forward sequence

$$0 \to p_*I \to p_*\mathcal{O}_{\mathcal{X}} \to p_*\mathcal{O}_{\Delta_X} \to 0 \tag{6}$$

splits. We remind the reader that the first term in the sequence p_*I is the sheaf of differentials Ω^1_X. Note that the above sequence is also a sequence of $\mathcal{O}_X$-modules.

Tensoring (5) with $q^*\mathcal{E}$, we get

$$0 \to I \otimes q^*\mathcal{E} \to \mathcal{O}_{\mathcal{X}} \otimes q^*\mathcal{E} \to \mathcal{O}_{\Delta} \otimes q^*\mathcal{E} \to 0 \tag{7}$$

On taking direct image p_*, we get

$$0 \to p_*(I \otimes q^*\mathcal{E}) \to p_*\mathcal{O}_{\mathcal{X}} \otimes q^*\mathcal{E} \to p_*\mathcal{O}_{\Delta} \otimes q^*\mathcal{E} \to 0 \tag{8}$$

Here the right exactness follows from the fact that p is a finite map and hence has no higher direct images. Rewritten, (since $I \otimes q^*\mathcal{E} = I \otimes p^*\mathcal{E}$) this is the sequence

$$0 \to \Omega^1_X \otimes \mathcal{E} \to p_*q^*\mathcal{E} \to \mathcal{E} \to 0 \tag{9}$$

which we shall refer to as the *Atiyah sequence* of $\mathcal{E}$. This sequence is locally split since $\mathcal{E}$ is a vector bundle.

3.2. **Naturality.** Let $\mathcal{E} \to X$ be as above and $f : Y \to X$ be a morphism of quasi-projective schemes. The Atiyah sequence of $\mathcal{E}$ pulls back to a sequence on Y

$$0 \to f^*\Omega^1_X \otimes f^*\mathcal{E} \to f^*p_*q^*\mathcal{E} \to f^*\mathcal{E} \to 0$$

which remains exact since the Atiyah sequence is locally split.

The natural map $\mathrm{df} : f^*\Omega^1_X \to \Omega^1_Y$ which we define below induces a push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & f^*\Omega^1_X \otimes f^*\mathcal{E} & \to & f^*p_*q^*\mathcal{E} & \to & f^*\mathcal{E} & \to & 0 \\ & & \downarrow \mathrm{df} \otimes 1 & & \downarrow & & || & & \\ 0 & \to & \Omega^1_Y \otimes f^*\mathcal{E} & \to & \mathcal{P} & \to & f^*\mathcal{E} & \to & 0 \end{array} \tag{10}$$

The claim of naturality is that the bottom horizontal sequence in (10) is indeed the Atiyah sequence of $f^*\mathcal{E}$.

To see this, consider $f \times f : Y \times Y \to X \times X$ which induces a map $F : \mathcal{Y} \to \mathcal{X}$ where $\mathcal{Y}$ is defined as a thickening of Δ_Y and let p', q' denote the projections of $\mathcal{Y} \to Y$. One has a diagram with commuting squares:

$$\begin{array}{ccccc} \Delta_Y & \hookrightarrow & \mathcal{Y} & \overset{p',\, q'}{\twoheadrightarrow} & Y \\ \downarrow f & & \downarrow F & & \downarrow f \\ \Delta_X & \hookrightarrow & \mathcal{X} & \overset{p,\, q}{\twoheadrightarrow} & X \end{array}$$

The commutativity of the above diagram defines a morphism of functors

$$\alpha : f^*p_* \to p'_*F^*$$

Let $\tilde{I}$ denote the ideal sheaf of $\Delta_Y \hookrightarrow \mathcal{Y}$. Then the composite

$$f^*p_*I \xrightarrow{\alpha} p'_*F^*I \to p'_*\tilde{I}$$

defines the natural (differentiation) map $\mathrm{df} : f^*\Omega^1_X \to \Omega^1_Y$.

The morphism α when applied to the sequence (7) induces a morphism of sequences

$$\begin{array}{ccccccccc} 0 & \to & f^*p_*(I \otimes q^*\mathcal{E}) & \to & f^*p_*q^*\mathcal{E} & \to & f^*p_*(\mathcal{O}_{\Delta_X}) \otimes q^*\mathcal{E} & \to & 0 \\ & & \downarrow \xi & & \downarrow & & || & & \\ & \to & p'_*F^*(I \otimes q^*\mathcal{E}) & \to & p'_*F^*q^*\mathcal{E} & \to & p'_*F^*(\mathcal{O}_{\Delta_X} \otimes q^*\mathcal{E}) & \to & 0 \\ & & \downarrow \kappa & & || & & \downarrow \omega & & \\ 0 & \to & p'_*(\tilde{I} \otimes q'^*f^*\mathcal{E}) & \to & p'_*F^*q^*\mathcal{E} & \to & p'_*(\mathcal{O}_{\Delta_Y}) \otimes F^*q^*\mathcal{E} & \to & 0 \end{array} \tag{11}$$

Here the map ω above is a surjection induced by the surjection $F^*\mathcal{O}_{\Delta_X} \to \mathcal{O}_{\Delta_Y}$.

We first explain the composition $\kappa \circ \xi$:

$$\begin{array}{ccccc} f^*p_*(I \otimes_{\mathcal{O}_\mathcal{X}} q^*\mathcal{E}) & = & f^*p_*(I \otimes_{\mathcal{O}_\mathcal{X}} p^*\mathcal{E}) & = & f^*p_*I \otimes_{\mathcal{O}_Y} f^*\mathcal{E} \\ \downarrow \xi & & \downarrow & & \downarrow \alpha \otimes 1 \\ p'_*F^*(I \otimes_{\mathcal{O}_\mathcal{X}} q^*\mathcal{E}) & = & p'_*(F^*I \otimes_{\mathcal{O}_\mathcal{X}} F^*q^*\mathcal{E}) & = & p'_*F^*I \otimes_{\mathcal{O}_Y} f^*\mathcal{E} \\ \downarrow \kappa & & \downarrow & & \downarrow \\ p'_*(\tilde{I} \otimes_{\mathcal{O}_\mathcal{Y}} q'^*f^*\mathcal{E}) & = & p'_*(\tilde{I} \otimes_{\mathcal{O}_\mathcal{Y}} p'^*f^*\mathcal{E}) & = & p'_*\tilde{I} \otimes_{\mathcal{O}_Y} f^*\mathcal{E} \end{array}$$

Thus $\kappa \circ \xi$ is just the map $\mathrm{df} \otimes 1$. This completes a proof of the claim.

3.3. **Atiyah class.** The Atiyah sequence (7) of $\mathcal{E}$ defines an element in the extension group $\mathrm{Ext}^1(\mathcal{E}, \Omega^1_X \otimes \mathcal{E}) \cong \mathrm{H}^1(X, \Omega^1_X \otimes \mathcal{E}nd\, \mathcal{E})$. We refer to this class as the *Atiyah class* $\mathrm{at}(\mathcal{E})$.

Equivalently, the class $\mathrm{at}(\mathcal{E})$ can be defined as follows:
Tensoring the Atiyah sequence (7) with $\mathcal{E}^\vee$ yields

$$0 \to \Omega^1_X \otimes \mathcal{E} \otimes \mathcal{E}^\vee \to p_*q^*\mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{E} \otimes \mathcal{E}^\vee \to 0 \tag{12}$$

The natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{E} \otimes \mathcal{E}^\vee$ gives rise to a pull-back sequence:

$$0 \to \Omega^1_X \otimes \mathcal{E} \otimes \mathcal{E}^\vee \to \mathcal{B} \to \mathcal{O}_X \to 0 \tag{13}$$

Then under the coboundary map

$$\begin{array}{ccc} \mathrm{H}^0(X, \mathcal{O}_X) & \to & \mathrm{H}^1(X, \Omega^1_X \otimes \mathcal{E} \otimes \mathcal{E}^\vee) \\ 1 & \mapsto & \mathrm{at}(\mathcal{E}) \end{array}$$

It is convenient to have the following definition: Let $f : Y \to X$ be a morphism and $\mathcal{F}$ be any sheaf on X. Then there are maps

$$f^* : \mathrm{H}^p(X, \mathcal{F} \otimes {\Omega^1_X}^{\otimes q}) \to \mathrm{H}^p(Y, f^*\mathcal{F} \otimes {\Omega^1_Y}^{\otimes q})$$

(defined as the composite of the two maps $\ell^p : \mathrm{H}^p(X, \mathcal{F} \otimes {\Omega^1_X}^{\otimes q}) \to \mathrm{H}^p(Y, f^*\mathcal{F} \otimes f^*{\Omega^1_X}^{\otimes q})$ and $\overset{q}{\otimes}\, df : \mathrm{H}^p(Y, f^*\mathcal{F} \otimes f^*{\Omega^1_X}^{\otimes q}) \to \mathrm{H}^p(Y, f^*\mathcal{F} \otimes {\Omega^1_Y}^{\otimes q}))$

$$\text{and} \qquad f^* : \mathrm{H}^p(X, \mathcal{F} \otimes \Omega^q_X) \to \mathrm{H}^p(Y, f^*\mathcal{F} \otimes \Omega^q_Y)$$

(defined similarly).

With these definitions, we have

Lemma 8. $f^*\,\mathrm{at}(\mathcal{E}) = \mathrm{at}(f^*\mathcal{E}) \in \mathrm{H}^1(Y, \mathcal{E}nd\, f^*\mathcal{E} \otimes \Omega^1_Y)$

Proof. From the discussion in section 3.2 and diagram (13), we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & f^*\mathcal{E} \otimes f^*\mathcal{E}^\vee \otimes f^*\Omega^1_X & \to & f^*\mathcal{B} & \to & \mathcal{O}_Y & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & f^*\mathcal{E} \otimes f^*\mathcal{E}^\vee \otimes \Omega^1_Y & \to & \mathcal{B} & \to & \mathcal{O}_Y & \to & 0 \end{array}$$

This induces the following diagram at the level of cohomology:

$$\begin{array}{ccc}
\mathrm{H}^0(\mathcal{O}_X) & \to & \mathrm{H}^1(X,\Omega^1_X\otimes\mathcal{E}\otimes\mathcal{E}^\vee) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathcal{O}_Y) & \to & \mathrm{H}^1(Y,f^*\Omega^1_X\otimes f^*\mathcal{E}\otimes f^*\mathcal{E}^\vee) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(\mathcal{O}_Y) & \to & \mathrm{H}^1(Y,\Omega^1_Y\otimes f^*\mathcal{E}\otimes f^*\mathcal{E}^\vee)
\end{array}$$

Here the horizontal arrows are boundary maps which map the identity element to the Atiyah class of the respective vector bundles and the vertical arrows between the cohomology on X and Y are the Leray maps. It is elementary to check that the commutativity of the diagram implies $f^*\,\mathrm{at}(\mathcal{E}) = \mathrm{at}(f^*\mathcal{E})$. □

Proposition 2. *With notation as above,*

$$f^*(\mathrm{at}(\mathcal{E})^{\cup k}) = \mathrm{at}(f^*\mathcal{E})^{\cup k}$$

Proof. This follows since cup products commute with morphisms and Leray maps (Lemma 6). □

3.4. **Atiyah Class of a Line bundle.** Let $\mathcal{L}$ be a line bundle on X. We shall now explicitly describe the Atiyah class $\mathrm{at}(\mathcal{L})$.

Let $\{U_i\}$ be a trivialising affine open cover for $\mathcal{L}$. Then $\mathcal{L}_{U_i} \cong \mathcal{O}_{U_i}e_i$ where e_i is a basis vector. On the intersection $U_{ij} := U_i \cap U_j$, $e_i = \alpha_{ij}e_j$, $\alpha_{ij} \in \Gamma(U_{ij}, \mathcal{O}^*_X)$ and this defines an element $\eta := \{\alpha_{ij}\} \in \mathrm{H}^1(X, \mathcal{O}^*_X)$. Similary one has $\mathcal{L}^{-1}_{U_i} \cong \mathcal{O}_{U_i}f_i$ where $f_i = \alpha^{-1}_{ij}f_j$ Furthermore, the isomorphism $\mathcal{L}\otimes\mathcal{L}^{-1} \cong \mathcal{O}_X$ is given locally by sending by $e_i \otimes f_i \mapsto 1$. Let $\{\mathcal{U}_i\}$ be a trivialising cover for $p^*\mathcal{L}$ with basis $\{e_{ij}\}$ and transition function $\{p(\alpha_{ij})\}$. Then on each $\mathcal{U}_i$, the element 1 has local lifts given by $e_i \otimes f_i$ in $p^*\mathcal{L}\otimes_{\mathcal{O}_\mathcal{X}} q^*\mathcal{L}^{-1}$. Thus on the intersection $\mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$, the element

$$e_i\otimes f_i - e_j\otimes f_j = p(\alpha_{ij})e_j\otimes q(\alpha^{-1}_{ij})f_j - e_j\otimes f_j = p(\alpha_{ij})q(\alpha^{-1}_{ij})e_j\otimes f_j - e_j\otimes f_j$$

$= [p(\alpha_{ij}).q(\alpha^{-1}_{ij}) - 1]e_j\otimes f_j$ defines a 1-cocycle for $I\otimes_{\mathcal{O}_\mathcal{X}} p^*\mathcal{L}\otimes q^*\mathcal{L}^{-1} \cong I\otimes_{\mathcal{O}_\Delta} p^*\mathcal{L}\otimes q^*\mathcal{L}^{-1} \cong I$. Here the last isomorphism is effected by sending $e_j \otimes f_j \mapsto 1$ on U_j.

On the other hand, $[p(\alpha_{ij}).q(\alpha^{-1}_{ij}) - 1] = q(\alpha^{-1}_{ij})[p(\alpha_{ij}) - q(\alpha_{ij})] =: -q(\alpha^{-1}_{ij})d(\alpha_{ij})$ (following the definition of $d(\alpha_{ij})$ as in [4] Prop. 8.1A page 173). Viewing $I \cong \Omega^1_X$ as an $\mathcal{O}_X$-module, we see that the last term is just $\alpha^{-1}_{ij}d(\alpha_{ij}) = \mathrm{dlog}(\alpha_{ij})$.

Summarizing, we see that $1 \in \mathrm{H}^0(X, \mathcal{O}_X)$ maps onto the cocycle

$$1\otimes 1 - \alpha^{-1}_{ij}\otimes\alpha_{ij} = \alpha^{-1}_{ij}(\alpha_{ij}\otimes 1 - 1\otimes\alpha_{ij}) = -\alpha^{-1}_{ij}d\alpha_{ij} = -\,\mathrm{dlog}(\alpha_{ij})$$

in $\Omega^1_{X_{|U_{ij}}}$ and thus we see that the $1 \mapsto \mathrm{at}(\mathcal{L}) = -[\mathrm{dlog}\,\alpha_{ij}] \in \mathrm{H}^1(X, \Omega^1_X)$

3.5. **Definitions of Chern Classes.** Henceforth we shall restrict our attention to schemes defined over a field K. We will assume char $K = 0$ or arbitrarily large.

As seen in the previous section, cup product yields elements

$$(\mathrm{at}\,\mathcal{E})^{\cup k} \in \mathrm{H}^k(X, \big(\mathcal{E}nd\,\mathcal{E} \otimes \Omega^1_X\big)^{\otimes k})$$

There exists a map $\phi_k = \phi_k(\mathcal{E}, \Omega^1_X) : (\mathcal{E}nd\,\mathcal{E} \otimes \Omega^1_X)^{\otimes k} \to \Omega^k_X$ which we describe in detail below. Let $\pi : \mathcal{E}^{\otimes k} \to \overset{\mathrm{k}}{\wedge}\mathcal{E}$ be the projection. Since $(\overset{\mathrm{k}}{\wedge}\mathcal{E})^\vee \cong \overset{\mathrm{k}}{\wedge}\mathcal{E}^\vee$, there is a standard inclusion $j : \overset{\mathrm{k}}{\wedge}\mathcal{E} \to \mathcal{E}^{\otimes k}$ which when composed with π is multiplication by $k!$ on $\overset{\mathrm{k}}{\wedge}\mathcal{E}$. Define the map λ_k from

$$(\mathcal{E}nd\,\mathcal{E} \otimes \Omega^1_X)^{\otimes k} = \mathcal{H}om\,(\mathcal{E}^{\otimes k}, \mathcal{E}^{\otimes k} \otimes {\Omega^1_X}^{\otimes k}) \to \mathcal{H}om\,(\overset{\mathrm{k}}{\wedge}\mathcal{E}, \overset{\mathrm{k}}{\wedge}\mathcal{E} \otimes \Omega^{1\otimes k}_X)$$

given by $\lambda_k(f) = (\pi \otimes 1) \circ f \circ j$.

The symmetric group S_k acts on $(\mathcal{E}nd\,\mathcal{E} \otimes \Omega^1_X)^{\otimes k}$ via $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$ where σ is a permutation of the k copies and on $\mathcal{H}om\,(\overset{\mathrm{k}}{\wedge}\mathcal{E}, \overset{\mathrm{k}}{\wedge}\mathcal{E} \otimes \Omega^{1\otimes k}_X)$ via $\sigma(g) = (1 \otimes \sigma) \circ g$. For these actions λ_k is equivariant.

The trace map $tr_k : \mathcal{H}om\,(\overset{\mathrm{k}}{\wedge}\mathcal{E}, \overset{\mathrm{k}}{\wedge}\mathcal{E}) \to \mathcal{O}_X$ and the projection $p : \Omega^{1\otimes k}_X \to \Omega^k_X$ combine to give $\phi_k = (tr_k \otimes p) \circ \lambda_k$.

More generally, given the bundle $\mathcal{E}$ and any sheaf $\mathcal{F}$ we can similarly define $\phi_k(\mathcal{E}, \mathcal{F}) : (\mathcal{E}nd\,(\mathcal{E}) \otimes \mathcal{F})^{\otimes k} \to \overset{\mathrm{k}}{\wedge}\mathcal{F}$. It is easy to see that for any morphism $\mu : \mathcal{F} \to \mathcal{G}$, there is a commutative diagram

$$\begin{array}{ccc} (\mathcal{E}nd\,(\mathcal{E}) \otimes \mathcal{F})^{\otimes k} & \xrightarrow{\phi_k(\mathcal{E},\mathcal{F})} & \overset{\mathrm{k}}{\wedge}\mathcal{F} \\ \downarrow (1 \otimes \mu)^{\otimes k} & & \downarrow \overset{\mathrm{k}}{\wedge}\mu \\ (\mathcal{E}nd\,(\mathcal{E}) \otimes \mathcal{G})^{\otimes k} & \xrightarrow{\phi_k(\mathcal{E},\mathcal{G})} & \overset{\mathrm{k}}{\wedge}\mathcal{G} \end{array}$$

In the particular case when $\mathcal{F} = \mathcal{O}_X$, the map $\phi_k(\mathcal{E}, \mathcal{O}_X)$ factors through $\mathrm{Sym}^k(\mathcal{E}nd\,(\mathcal{E}))$ and can be viewed as a symmetric multilinear form giving the "k-th coefficient of the characteristic polynomail" (see [2, 1]).Our description of ϕ_k can be found in [5].

Definition 2. The k-th Chern class of $\mathcal{E}$

$$\mathrm{c}_k(\mathcal{E}) := \frac{(-1)^k}{k!}\phi_k(\mathrm{at}(\mathcal{E})^{\cup k}) \in \mathrm{H}^k(X, \Omega^k_X)$$

We make the following comments:

(1) We define $\mathrm{c}_0(\mathcal{E}) := 1$.
(2) For $k > \dim X$, we have $\mathrm{c}_k(\mathcal{E}) = 0$.
(3) For $k > \mathrm{rank}(\mathcal{E})$, we have $\mathrm{c}_k(\mathcal{E}) = 0$.

Proposition 3. *For $f : Y \to X$ and $\mathcal{E}$ a bundle on X, $f^* \mathrm{c}_k(\mathcal{E}) = \mathrm{c}_k(f^*\mathcal{E})$.*

Proof. Recall that we defined the map f^* in section 3.3. The first step is to notice that $f^*\phi_k(\mathcal{E}, \Omega^1_X) = \phi_k(f^*\mathcal{E}, f^*\Omega^1_X)$. Next, using the fact that Leray maps are functorial and that $\phi_k(f^*\mathcal{E}, -)$ is functorial for the map $df : f^*\Omega^1_X \to \Omega^1_Y$, we conclude that

$$f^*\phi_k(\mathcal{E}, \Omega^1_X)(\mathrm{at}(\mathcal{E})^{\cup k}) = \phi_k(f^*\mathcal{E}, \Omega^1_Y)(f^* \mathrm{at}(\mathcal{E})^{\cup k}).$$

Since $f^*(\mathrm{at}(\mathcal{E})^{\cup k}) = \mathrm{at}(f^*\mathcal{E})^{\cup k}$, we are done. □

3.6. **Whitney Sum formulae.** We will now assume that the vector bundle $\mathcal{E}$ is a direct sum; $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$. Then $\mathcal{E}nd\,(\mathcal{E}) \otimes \Omega^1_X$ has $\mathcal{E}nd\,(\mathcal{F}) \otimes \Omega^1_X$ and $\mathcal{E}nd\,(\mathcal{G}) \otimes \Omega^1_X$ as pieces of its direct sum decomposition. Let $\delta = \mathrm{at}(\mathcal{E})$, $\delta_1 = \mathrm{at}(\mathcal{F})$ and $\delta_2 = \mathrm{at}(\mathcal{G})$. Via the natural inclusions, we can view δ, δ_1 and δ_2 as elements in $\mathrm{H}^1(X, \mathcal{E}nd\,\mathcal{E} \otimes \Omega^1_X)$.

Lemma 9. $\mathrm{at}(\mathcal{E}) = \mathrm{at}(\mathcal{F}) + \mathrm{at}(\mathcal{G})$

Proof. This follows easily from the observation that $p_*q^*\mathcal{E} = p_*q^*\mathcal{F} \oplus p_*q^*\mathcal{G}$ and hence the Atiyah sequence of $\mathcal{E}$ is the direct sum of the Atiyah sequences of $\mathcal{F}$ and $\mathcal{G}$. □

Lemma 10. $\mathrm{c}_k(\mathcal{E}) = \sum_{r=0}^{k} \mathrm{c}_r(\mathcal{F})\,\mathrm{c}_{k-r}(\mathcal{G})$

Proof. Suppose $r, s \geq 0$ such that $r + s = k$. In the expansion of $\delta^{\cup k} = (\delta_1 + \delta_2)^{\cup k}$, there are $\binom{k}{r}$ terms involving r δ_1's and s δ_2's. Let d be one such cup product. Let σ be a permutation of $(\mathcal{E}nd\,(\mathcal{E}) \otimes \Omega^1_X)^{\otimes k}$ which moves the r δ_1's of d to the first r positions. Let ψ_σ be the induced action on $\mathrm{H}^k(X, \mathcal{E}nd\,(\mathcal{E}) \otimes \Omega^1_X)^{\otimes k})$. It follows from Lemma 5 that $\psi_\sigma(d) = sgn(\sigma)(\delta_1^{\cup r} \cup \delta_2^{\cup s})$.

We will now focus on the term $\delta_1^{\cup r} \cup \delta_2^{\cup s}$ which arises when we look at the obvious inclusion

$$\left(\mathcal{E}nd\,\mathcal{F} \otimes \Omega^1_X\right)^{\otimes r} \otimes \left(\mathcal{E}nd\,\mathcal{G} \otimes \Omega^1_X\right)^{\otimes s} \hookrightarrow \left(\mathcal{E}nd\,\mathcal{E} \otimes \Omega^1_X\right)^{\otimes k}$$

Consider the diagram

$$\begin{array}{ccc}
\mathcal{E}nd\,(\mathcal{F}^{\otimes r})\otimes {\Omega^1_X}^{\otimes r}\otimes \mathcal{E}nd\,(\mathcal{G}^{\otimes s})\otimes {\Omega^1_X}^{\otimes s} & \to & \mathcal{E}nd\,(\mathcal{E}^{\otimes k})\otimes {\Omega^1_X}^{\otimes k} \\
\downarrow \lambda_r\otimes\lambda_s & & \downarrow \lambda_k \\
\mathcal{E}nd\,(\overset{\mathrm{r}}{\wedge}\mathcal{F})\otimes {\Omega^1_X}^{\otimes r}\otimes \mathcal{E}nd\,(\overset{\mathrm{s}}{\wedge}\mathcal{G})\otimes {\Omega^1_X}^{\otimes s} & \to & \mathcal{E}nd\,(\overset{\mathrm{k}}{\wedge}\mathcal{E})\otimes {\Omega^1_X}^{\otimes k} \\
tr_r\otimes 1\downarrow\otimes tr_s\otimes 1 & & \downarrow tr_k\otimes 1 \\
{\Omega^1_X}^{\otimes r}\otimes {\Omega^1_X}^{\otimes s} & \to & {\Omega^1_X}^{\otimes k} \\
\downarrow p\otimes p & & \downarrow p \\
\Omega^r_X\otimes\Omega^s_X & \to & \Omega^k_X
\end{array}$$

The top square commutes because we have commuting diagrams

$$\begin{array}{ccc}
\mathcal{F}^{\otimes r}\otimes\mathcal{G}^{\otimes s} & \hookrightarrow & \mathcal{E}^{\otimes k} \\
\downarrow \pi\otimes\pi & & \downarrow \pi \\
\overset{\mathrm{r}}{\wedge}\mathcal{F}\otimes\overset{\mathrm{s}}{\wedge}\mathcal{G} & \hookrightarrow & \overset{\mathrm{k}}{\wedge}\mathcal{E}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{F}^{\otimes r}\otimes\mathcal{G}^{\otimes s} & \twoheadleftarrow & \mathcal{E}^{\otimes k} \\
\uparrow j\otimes j & & \uparrow j \\
\overset{\mathrm{r}}{\wedge}\mathcal{F}\otimes\overset{\mathrm{s}}{\wedge}\mathcal{G} & \twoheadleftarrow & \overset{\mathrm{k}}{\wedge}\mathcal{E}
\end{array}$$

The middle square commutes because

$$\mathcal{E}nd\,(\overset{\mathrm{r}}{\wedge}\mathcal{F})\otimes{\Omega^1_X}^{\otimes r}\otimes\mathcal{E}nd\,(\overset{\mathrm{s}}{\wedge}\mathcal{G})\otimes{\Omega^1_X}^{\otimes s}\cong\mathcal{E}nd\,(\overset{\mathrm{r}}{\wedge}\mathcal{F}\otimes\overset{\mathrm{s}}{\wedge}\mathcal{G})\otimes{\Omega^1_X}^{\otimes r}\otimes{\Omega^1_X}^{\otimes s}$$

where the ordered pair of two matrices maps to the tensor (or Kröneker) product of the two matrices and the trace of the Kröneker product of two matrices is the tensor product of the traces.

The commutativity of the third square is obvious.

Now consider the two elements $\delta_1^{\cup r}\in \mathrm{H}^r(X,(\mathcal{E}nd\,\mathcal{F}\otimes\Omega^1_X)^{\otimes r})$ and $\delta_2^{\cup s}\in \mathrm{H}^s(X,(\mathcal{E}nd\,\mathcal{G}\otimes\Omega^1_X)^{\otimes s})$. Since the image of the cup product equals the cup product of the images for the left vertical map of sheaves, $\delta_1^{\cup r}\cup\delta_2^{\cup s}$ maps to $\phi_r(\delta_1^{\cup r})\cup\phi_s(\delta_2^{\cup s})$. Therefore viewing $\delta_1^{\cup r}\cup\delta_2^{\cup s}$ as an element in $\mathrm{H}^k(X,(\mathcal{E}nd\,\mathcal{E}\otimes\Omega^1_X)^{\otimes k})$, $\phi_k(\delta_1^{\cup r}\cup\delta_2^{\cup s})=\phi_r(\delta_1^{\cup r})\wedge\phi_s(\delta_2^{\cup s})$.

The same argument as in Lemma 7 tells us that for any term d as above, $\phi_k(d)=\phi_k(\delta_1^{\cup r}\cup\delta_2^{\cup s})$ and therefore one has $\phi_k(\delta^{\cup k})=\sum_{r=0}^k\binom{k}{r}\phi_r(\delta_1^{\cup r})\wedge\phi_s(\delta_2^{\cup s})$. Thus we have $\mathrm{c}_k(\mathcal{E})=\sum_{r=0}^k\mathrm{c}_r(\mathcal{F})\,\mathrm{c}_{k-r}(\mathcal{G})$ where the product in the expression is the commutative wedge product. □

Proposition 4 (Whitney sum formula for exact sequences)**.** *Let $0\to\mathcal{E}\to\mathcal{V}\to\mathcal{F}\to 0$ be a sequence of bundles on X. Then*

$$\mathrm{c}_k(\mathcal{V})=\mathrm{c}_k(\mathcal{E}\oplus\mathcal{F})$$

Proof. Let $\eta \in \mathrm{Ext}^1(\mathcal{F},\mathcal{E})$ denote the class of the exact sequence above and consider the affine line corresponding to the 1-dimensional vector subspace generated by η. Let $1 \in \mathbb{A}^1$ correspond to the extension η above and $0 \in \mathbb{A}^1$ correspond to the trivial extension

$$0 \to \mathcal{E} \to \mathcal{E} \oplus \mathcal{F} \to \mathcal{F} \to 0$$

Let $X \times \mathbb{A}^1 \xrightarrow{p} X$ and $X \times \mathbb{A}^1 \xrightarrow{q} \mathbb{A}^1$ be the projection maps and $X \times \{x\} \xrightarrow{\iota_x} X \times \mathbb{A}^1$ be the natural inclusion.

We have

$$\Omega^1_{X\times\mathbb{A}^1} \cong p^*\Omega^1_X \oplus q^*\Omega^1_{\mathbb{A}^1} \cong p^*\Omega^1_X \oplus q^*\mathcal{O}_{\mathbb{A}^1}.dt$$

which on taking exterior powers yields

$$\Omega^k_{X\times\mathbb{A}^1} \cong p^*\Omega^k_X \oplus \big(p^*\Omega^{k-1}_X \otimes q^*\mathcal{O}_{\mathbb{A}^1}.dt\big)$$

and so we have

$$\mathrm{H}^k(\Omega^k_{X\times\mathbb{A}^1}) \cong \big(\mathrm{H}^k(\Omega^k_X)\otimes k[t]\big) \oplus \big(\mathrm{H}^k(\Omega^{k-1}_X)\otimes k[t]dt\big) \tag{14}$$

The composite

$$\mathrm{H}^k(\Omega^k_X) \xrightarrow{p^*} \mathrm{H}^k(\Omega^k_{X\times\mathbb{A}^1}) \xrightarrow{\iota_x^*} \mathrm{H}^k(\Omega^k_X)$$

is the identity.

On $\mathbb{A}^1$, we have the natural sequence

$$0 \to \mathcal{O}_{\mathbb{A}^1} \xrightarrow{\lambda} \mathcal{O}_{\mathbb{A}^1} \to \mathcal{O}_0 \to 0$$

which in turn induces a map

$$p^*\mathcal{F} \otimes q^*\mathcal{O}_{\mathbb{A}^1} \xrightarrow{id\otimes q^*\lambda} p^*\mathcal{F} \otimes q^*\mathcal{O}_{\mathbb{A}^1}$$

We rewrite this as

$$p^*\mathcal{F} \xrightarrow{\tilde{\lambda}} p^*\mathcal{F}$$

Let $\mathbb{V}$ be the universal extension defined by the pull back diagram

$$\begin{array}{ccccccccc} 0 & \to & p^*\mathcal{E} & \to & \mathbb{V} & \to & p^*\mathcal{F} & \to & 0 \\ & & || & & \downarrow & & \downarrow \lambda & & \\ 0 & \to & p^*\mathcal{E} & \to & p^*\mathcal{V} & \to & p^*\mathcal{F} & \to & 0 \end{array}$$

Further, $\iota_1^*\mathbb{V} \cong \mathcal{V}$ and $\iota_0^*\mathbb{V} \cong \mathcal{E}\oplus\mathcal{F}$.

Let $j : X \times (\mathbb{A}^1 \setminus \{0\}) \hookrightarrow X \times \mathbb{A}^1$ be the natural inclusion. Since λ is an isomorphism on $\mathbb{A}^1 \setminus \{0\}$, $j^*\mathbb{V} \cong j^*p^*\mathcal{V}$. This implies

$$j^*\,\mathrm{c}_k(\mathbb{V}) = \mathrm{c}_k(j^*\mathbb{V}) = \mathrm{c}_k(j^*p^*\mathcal{V}) = j^*p^*\,\mathrm{c}_k(\mathcal{V}) = \mathrm{c}_k(\mathcal{V})\otimes 1$$

Since $j^* : \mathrm{H}^k(\Omega^k_X)\otimes k[x] \to \mathrm{H}^k(\Omega^k_X)\otimes k[x,x^{-1}]$ is an inclusion,

$$\mathrm{c}_k(\mathbb{V}) = \mathrm{c}_k(\mathcal{V})\otimes 1 = p^*\,\mathrm{c}_k(\mathcal{V}) \in \mathrm{H}^k(\Omega^k_X)\otimes k[x]$$

Now for any $x \in \mathbb{A}^1$, $\iota_x^* p^* \mathcal{V} \cong \mathcal{V}$ and so

$$\mathrm{c}_k(\iota_x^* \mathbb{V}) = \iota_x^* p^* \,\mathrm{c}_k(\mathcal{V}) = \mathrm{c}_k(\mathcal{V})$$

which is independent of $x \in \mathbb{A}^1$.

□

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