HIGHER ABEL-JACOBI MAPS

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Introduction

We work over a subfield k of $\mathbb{C}$, the field of complex numbers. For a smooth variety V over k, the Chow group of cycles of codimension p is defined (see [5]) as

$$\mathrm{CH}^p(V) = \frac{Z^p(V)}{R^p(V)}$$

where the group of cycles $Z^p(V)$ is the free abelian group on scheme-theoretic points of V of codimension p and rational equivalence $R^p(V)$ is the subgroup generated by cycles of the form $\mathrm{div}_W(f)$ where W is a subvariety of V of codimension $(p-1)$ and f is a non-zero rational function on it. There is a natural cycle class map

$$\mathrm{cl}_p : \mathrm{CH}^p(V) \to \mathrm{H}^{2p}(V)$$

where the latter denotes the singular cohomology group $\mathrm{H}^{2p}(V(\mathbb{C}), \mathbb{Z})$ with the (mixed) Hodge structure given by Deligne (see [4]). The kernel of cl_p is denoted by $F^1 \mathrm{CH}^p(V)$. There is an Abel-Jacobi map (see [8]),

$$\Phi_p : F^1 \mathrm{CH}^p(V) \to \mathrm{IJ}^p(\mathrm{H}^{2p-1}(V))$$

where the latter is the intermediate Jacobian of a Hodge structure, defined as follows

$$\mathrm{IJ}^p(H) = \frac{H \otimes \mathbb{C}}{F^p(H \otimes \mathbb{C}) + H}.$$

We note for future reference that for a pure Hodge structure of weight $2p-1$ (such as the cohomology of a smooth *projective* variety) we have the natural isomorphism,

$$\frac{H \otimes \mathbb{R}}{H} \xrightarrow{\sim} \mathrm{IJ}^p(H)$$

The kernel of Φ_p is denoted by $F^2 \mathrm{CH}^p(V)$.

The conjecture of S. Bloch says (see [10]) that there is a filtration $F^\cdot$ on $\mathrm{CH}^p(V)$ which extends the F^1 and F^2 defined above. Moreover, the associated graded group $\mathrm{gr}_F^k \mathrm{CH}^p(V)$ is *governed* by the cohomology groups $\mathrm{H}^{2p-k}(V)$ for each integer k (upto torsion). More precisely, if $N^l \mathrm{H}^m(V)$ denotes the filtration by co-niveau (see [9]) which is generated by cohomology classes supported on subvarieties of codimension $\leq l$, then $\mathrm{gr}_F^k \mathrm{CH}^p(V)$ is actually governed by the quotient groups,

$$\mathrm{H}^{2p-k}(V)/N^{p-k+1}\,\mathrm{H}^{2p-k}(V).$$

Specifically, in the case when V is a smooth projective surface with geometric genus 0 (so that $\mathrm{H}^2(V) = N^1\,\mathrm{H}^2(V)$) this conjecture implies that $F^2\,\mathrm{CH}^2(V)$ is torsion (and thus 0 by a theorem of Roitman [14]).

The traditional Hodge-theoretic approach to study this problem is based on the fact that the intermediate Jacobian $\otimes\mathbb{Q}$ can be interpreted as the extension group $\mathrm{Ext}^1(\mathbb{Q}(-p), H)$ in the category of Hodge structures. One can then propose that the associated graded groups $\mathrm{gr}_F^k \mathrm{CH}^p(V) \otimes \mathbb{Q}$ should be interpreted as the higher extension groups $\mathrm{Ext}^k(\mathbb{Q}(-p), \mathrm{H}^{2p-k}(V))$ for $k \geq 2$. Unfortunately, there are no such extension groups in the category of Hodge structures. Thus it was proposed that all these extension groups be computed in a suitable category of mixed motives.[1]

Even if such a category is constructed a Hodge-theoretic interpretation of these extension groups would be useful. In section 2 we discuss M. Green's approach (see [7]) called the Higher Abel-Jacobi map. In section 3 we provide a counter-example to show that Green's approach does not work; a somewhat more complicated example was earlier obtained by C. Voisin (see [19]). In section 4 we introduce an alternative approach based on Deligne-Beilinson cohomology and its interpretation in terms of Morihiko Saito's theory of Hodge modules; such an approach has also been suggested earlier by M. Asakura and independently by M. Saito (see [1] and [15]). Following this approach it becomes possible to deduce Bloch's conjecture from some conjectures of Bloch and Beilinson for cycles and varieties defined over a number field (see [10]).

1. Green's Higher Abel-Jacobi Map

The fundamental idea behind M. Green's construction can be interpreted as follows (see [19]). One expects that the extension groups are *effaceable* in the abelian category of mixed motives. Thus the elements of Ext^k can be written in terms of k different elements in various Ext^1's. The latter groups can be understood in terms of Hodge theory, via the Intermediate Jacobians. So we can try to write the Ext^k as a sub of a quotient of a (sum of) tensor products of Intermediate Jacobians.

Specifically, consider the case of a surface S. Let C be a curve, then we have a product map (see [5]),

$$\mathrm{CH}^1(C) \times \mathrm{CH}^2(C \times S) \to \mathrm{CH}^2(S)$$

which in fact respects the filtration $F^{\cdot}$ (see [10]), so that we have

$$F^1\,\mathrm{CH}^1(C) \times F^1\,\mathrm{CH}^2(C \times S) \to F^2\,\mathrm{CH}^2(S).$$

Conversely, we can use an argument of Murre (see [11]) to show,

[1] Such a category has recently been constructed by M. V. Nori (unpublished).

Lemma 1. *Given any cycle class* ξ *in* $F^2\operatorname{CH}^2(S)$ *there is a curve* C *so that* ξ *is in the image of the map,*

$$F^1\operatorname{CH}^1(C)\times F^1\operatorname{CH}^2(C\times S)\to F^2\operatorname{CH}^2(S).$$

Proof. Let z be a cycle representing the class ξ. There is a smooth (see [12]) curve C on S that contains the support of z. Hence it is enough to show that there is a homologically trivial cycle Y on $C\times S$ so that $(z,Y)\mapsto z$ for every cycle z on C such that the image under $\operatorname{CH}^1(C)\to\operatorname{CH}^2(S)$ lies in $F^2\operatorname{CH}^2(S)$. Let Γ denote the graph of the inclusion $\iota: C\hookrightarrow S$. Then clearly $(z,\Gamma)\mapsto z$ but Γ is not homologically trivial.

Choose a point p on C. Now, by a result of Murre (see [11]), for some positive integer m we have an expresssion in $\operatorname{CH}^2(S\times S)$

$$m\Delta_S=m(p\times S+S\times p)+X_{2,2}+X_{1,3}+X_{3,1}$$

where Δ_S is the diagonal and $X_{i,j}$ is a cycle so that its cohomology class has non-zero Künneth component only in $\mathrm{H}^i(S)\otimes\mathrm{H}^j(S)$. In particular, $X_{1,3}$ gives a map $F^1\operatorname{CH}^2(S)\to F^1\operatorname{CH}^2(S)$ which induces multiplication by m on $\mathrm{IJ}^2(\mathrm{H}^3(S))$. Since $p\times S$ and $S\times p$ induce 0 on $F^1\operatorname{CH}^2(S)$ it follows that the correspondence $X_{2,2}+X_{3,1}$ induces multiplication by m on $F^2\operatorname{CH}^2(S)$.

Now, $\Gamma=(\iota\times 1_S)^*(\Delta_S)$ so we have an expression

$$m\Gamma=mC\times p+(\iota\times 1_S)^*X_{2,2}+(\iota\times 1_S)^*X_{1,3}$$

Let $D=p_{2*}((\iota\times 1_S)^*X_{2,2})$. Then the cohomology class of $Y=(\iota\times 1_S)^*X_{2,2}-p\times D$ is 0. Moreover, the map $F^1\operatorname{CH}^1(C)\to F^1\operatorname{CH}^2(S)$ induced by $p\times D$ is zero. Thus, by the above propery of $X_{2,2}+X_{3,1}$ we see that $mz=(z,m\Gamma)=(z,Y)$ for any z in $F^1\operatorname{CH}^1(C)$ whose image lies in $F^2\operatorname{CH}^2(S)\otimes\mathbb{Q}$. By Roitman's theorem (see [14]) the group $F^2\operatorname{CH}^2(S)$ is divisible. Hence, we conclude the result. □

We now use the Abel-Jacobi maps to interpret the two terms on the left-hand side in terms of Hodge theory.

Firstly, we have the classical Abel-Jacobi isomorphisms $F^1\operatorname{CH}^1(C)=J(C)=\mathrm{IJ}^1(\mathrm{H}^1(C))$. Let $\mathrm{H}^2(S)_{\mathrm{tr}}=\mathrm{H}^2(S)/N^1\mathrm{H}^2(S)$ denote the lattice of transcendental cycles on S. Consider the factor $\mathrm{IJ}^2(\mathrm{H}^1(C)\otimes\mathrm{H}^2(S)_{\mathrm{tr}})$ of the intermediate Jacobian $\mathrm{IJ}^2(\mathrm{H}^3(C\times S))$. We can compose the Abel-Jacobi map with the projection to this factor to obtain

$$F^1\operatorname{CH}^2(C\times S)\to\mathrm{IJ}^2(\mathrm{H}^1(C)\otimes\mathrm{H}^2(S)_{\mathrm{tr}}).$$

Using the identification $\mathrm{IJ}^p(H)=H\otimes(\mathbb{R}/\mathbb{Z})$ for a pure Hodge structure H of weight $2p-1$ we have

$$\mathrm{IJ}^1(\mathrm{H}^1(C))\otimes\mathrm{IJ}^2(\mathrm{H}^1(C)\otimes\mathrm{H}^2(S)_{\mathrm{tr}})=\mathrm{H}^1(C)^{\otimes 2}\otimes\mathrm{H}^2(S)\otimes(\mathbb{R}/\mathbb{Z})^{\otimes 2}$$

The pairing $\mathrm{H}^1(C)^{\otimes 2} \to \mathrm{H}^2(C) = \mathbb{Z}$, given by the cup product, can be used to further collapse the latter term. Thus, we obtain a diagram,

$$\begin{array}{ccc} F^1\,\mathrm{CH}^1(C) \times F^1\,\mathrm{CH}^2(C \times S) & \to & F^2\,\mathrm{CH}^2(S) \\ \downarrow & & \\ \mathrm{IJ}^1(\mathrm{H}^1(C)) \times \mathrm{IJ}^2(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)_{\mathrm{tr}}) & \to & \mathrm{H}^2(S)_{\mathrm{tr}} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} \end{array}$$

Definition 1. Green's second intermediate Jacobian $J_2^2(S)$ is defined as the universal push-out of all the above diagrams as C is allowed to vary. The Higher Abel-Jacobi map is defined as the natural homomorphism

$$\Psi_2^2 : F^2\,\mathrm{CH}^2(S) \to J_2^2(S).$$

By the above lemma it follows that $J_2^2(S)$ is a quotient of $\mathrm{H}^2(S)_{\mathrm{tr}} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2}$. The question is whether this constructs the required Ext^2.

Problem 1 (Green)**.** Is Ψ_2^2 injective?

2. Non-injectivity of Green's Map

We now compute Green's Higher Abel-Jacobi map for the case of a surface of the form $\mathrm{Sym}^2(C)$, where C is a smooth projective curve. Using this we show that this map is not injective when C is a curve of genus at least two whose Jacobian is a simple abelian variety.

Lemma 2. *Let $Z \in \mathrm{CH}^2(D \times C \times S)$ be a cycle, where D, C are smooth curves and S a smooth surface. Then we have a commutative diagram*

$$\begin{array}{ccccc} F^1\,\mathrm{CH}^1(D) \otimes F^1\,\mathrm{CH}^1(C) & & \xrightarrow{p_{3*}(p_{12}^*(\_)\cdot Z)} & & F^2\,\mathrm{CH}^2(S) \\ \downarrow & & & & \downarrow \\ \mathrm{IJ}^1(H^1(D)) \otimes \mathrm{IJ}^1(H^1(C)) & \xrightarrow{z} & \mathrm{H}^2(S)_{\mathrm{tr}} \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} & \to & J_2^2(S) \end{array}$$

Here the map z is the composite as follows. The cohomology class of Z gives a map $\mathrm{H}^1(D) \otimes \mathrm{H}^1(C) \to \mathrm{H}^2(S)$; we further project to $\mathrm{H}^2(S)_{\mathrm{tr}}$. Now tensor with $(\mathbb{R}/\mathbb{Z})^{\otimes 2}$ and identify the resulting left-hand term with the product of the Intermediate Jacobians.

We note that the vertical arrow on the left is an isomorphism.

Proof. By the functoriality of the Abel-Jacobi map we have a commutative diagram

$$\begin{array}{ccc} F^1\,\mathrm{CH}^1(D) & \xrightarrow{p_{23*}(p_1^*(\_)\cdot Z)} & F^1\,\mathrm{CH}^2(C \times S) \\ \downarrow & & \downarrow \\ \mathrm{IJ}^1(\mathrm{H}^1(D)) & \xrightarrow{1_{(\mathbb{R}/\mathbb{Z})} \otimes p_{23*}(p_1^*(\_)\cup[Z])} & \mathrm{IJ}^2(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)) \end{array}$$

By projection we can replace the bottom right corner with $\mathrm{IJ}^2(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)_{\mathrm{tr}})$. Now we tensor this with the Abel-Jacobi map for C to obtain,

$$\begin{array}{ccc} F^1\,\mathrm{CH}^1(D) \otimes F^1\,\mathrm{CH}^1(C) & \to & F^1\,\mathrm{CH}^1(C) \otimes F^1\,\mathrm{CH}^2(C \times S) \\ \downarrow & & \downarrow \\ \mathrm{IJ}^1(\mathrm{H}^1(D)) \otimes \mathrm{IJ}^1(\mathrm{H}^1(C)) & \to & \mathrm{IJ}^1(\mathrm{H}^1(C)) \otimes \mathrm{IJ}^2(\mathrm{H}^1(C) \otimes \mathrm{H}^2(S)_{\mathrm{tr}}) \end{array}$$

The required commutative diagram now follows from the definition of $J_2^2(S)$. □

We now apply this lemma to the case $C = D$ and $S = \mathrm{Sym}^2(C)$. In this case we take Z to be the graph of the quotient morphism $q : C \times C \to \mathrm{Sym}^2(C)$. We then compute that the cohomological correspondence given by $[Z]$ factors as

$$\mathrm{H}^1(C) \otimes \mathrm{H}^1(C) \to \bigwedge^2 \mathrm{H}^1(C) \to \mathrm{H}^2(\mathrm{Sym}^2(C))_{\mathrm{tr}}$$

By the above lemma we obtain a factoring,

$$\begin{array}{ccc} F^1\,\mathrm{CH}^1(C) \otimes F^1\,\mathrm{CH}^1(C) & \xrightarrow{p_{3*}(p_{12}^*(\_)\cdot Z)} & F^2\,\mathrm{CH}^2(\mathrm{Sym}^2(C)) \\ \downarrow & & \downarrow \\ (H^1(C)\otimes \mathbb{R}/\mathbb{Z})^{\otimes 2} \quad \to & \bigwedge^2 \mathrm{H}^1(C) \otimes (\mathbb{R}/\mathbb{Z})^{\otimes 2} & \to \quad J_2^2(\mathrm{Sym}^2(C)) \end{array}$$

The image of the tensor product of a pair of elements of $\mathrm{IJ}^1(\mathrm{H}^1(C))$ of the form $v \otimes \alpha$ and $v \otimes \beta$ must therefore be 0 in $J_2^2(\mathrm{Sym}^2(C))$.

The description of $F^2\,\mathrm{CH}^2(\mathrm{Sym}^2(C))$ is given by the following lemma that is similar to one in [3],

Lemma 3. *The homomorphism*

$$Z_* : F^1\,\mathrm{CH}^1(C) \otimes F^1\,\mathrm{CH}^1(C) \to F^2\,\mathrm{CH}^2(\mathrm{Sym}^2(C))$$

is surjective.

Proof. The following composite map is multiplication by 2

$$F^2\,\mathrm{CH}^2(\mathrm{Sym}^2(C)) \xrightarrow{q^*} F^2\,\mathrm{CH}^2(C \times C) \xrightarrow{q_*} F^2\,\mathrm{CH}^2(\mathrm{Sym}^2(C))$$

By the divisibility of $F^2\,\mathrm{CH}^2(S)$ for a surface S we see that the lemma follows from the following result. □

Sublemma 1. *Fix a base point p on C. Then the filtration F of $\mathrm{CH}^2(C\times C)$ is explicitly described as follows*

$$\begin{aligned} F^2\,\mathrm{CH}^2(C \times C) &= \mathrm{im}(J(C) \otimes J(C)) \subset \\ F^1\,\mathrm{CH}^2(C \times C) &= F^2\,\mathrm{CH}^2(C \times C) + \mathrm{im}(J(C) \times p) + \mathrm{im}(p \times J(C)) \\ &\subset \mathrm{CH}^2(C \times C) = F^1\,\mathrm{CH}^2(C \times C) + \mathbb{Z}\cdot(p,p) \end{aligned}$$

Proof. Let a, b be points on C; we get points $[a-p]$ and $[b-p]$ of $J(C)$. The image of $[a-p]\otimes[b-p]$ in $\mathrm{CH}^2(C\times C)$ is $(a,b)+(p,p)-(a,p)-(p,b)$. Thus, we have an expression

$$(a,b) = \mathrm{im}([a-p]\otimes[b-p]) + \mathrm{im}([a-p]\times p) + \mathrm{im}(p \times [b-p]) + (p,p)$$

Now, any cycle ξ in $F^1\,\mathrm{CH}^2(C\times C)$ can be written as $\sum_{i=1}^n (a_i,b_i) - n\cdot(p,p)$. The Albanese variety of $C \times C$ is $J(C) \oplus J(C)$ and the image of ξ under the Albanese map is $(\sum_{i=1}^n [a_i - p], \sum_{i=1}^n [b_i - p])$. Thus, if the cycle is in

$F^2\,\mathrm{CH}^2(C \times C)$, then $\sum_{i=1}^n [a_i - p] = 0 = \sum_{i=1}^n [b_i - p]$. Now we combine this with the above expression to obtain

$$\xi = \sum_{i=1}^{n} \mathrm{im}([a_i - p] \otimes [b_i - p])$$

which proves the result. □

Lemma 4. *If C is a curve of genus at least 2 such that its Jacobian variety is a simple abelian variety then Ψ_2^2 has a non-trivial kernel.*

Proof. By Mumford's result there are non-trivial classes in $F^2\,\mathrm{CH}^2(S)$. The Jacobian variety $J(C) = \mathrm{IJ}^1(\mathrm{H}^1(C)) = \mathrm{H}^1(C) \otimes \mathbb{R}/\mathbb{Z}$ is spanned by decomposable elements. Moreover $F^1\,\mathrm{CH}^1(C) \cong J(C)$. Thus there is a pair of elements of $F^1\,\mathrm{CH}^1(C)$ of the form $v \otimes \alpha$, $w \otimes \beta$ such that the image of their tensor product in $F^2\,\mathrm{CH}^2(S)$ is non-zero. By a result of Roitman, for any fixed class f in $F^1\,\mathrm{CH}^1(C)$, the collection

$$K_f = \{e \in J(C) | e \otimes f \mapsto 0 \text{ in } F^2\,\mathrm{CH}^2(S)\}$$

forms a countable union of abelian subvarieties of $J(C)$. Since $w \otimes \beta$ does not lie in $K_{v\otimes\alpha}$, the latter is a proper subgroup of $J(C)$. Since $J(C)$ is assumed to be simple this is forced to be a countable set. In particular, there is an element of the form $v \otimes \gamma$ which is *not* in $K_{v\otimes\alpha}$; so that the product of this with $v \otimes \alpha$ is non-zero in $F^2\,\mathrm{CH}^2(S)$. But we just saw that all such elements are mapped to 0 in $J_2^2(S)$. □

3. Absolute Deligne-Beilinson Cohomology

The fundamental idea underlying the following constructions and definitions is as follows. A variety V over $\mathbb{C}$ can be thought of as a family of varieties over the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$ of the field of rational numbers. Even when the variety is defined over $\mathbb{Q}$ the Chow group of such a variety (when considered over $\mathbb{C}$) may contain cycles that are defined over larger fields. In particular, the usual examples of non-trivial elements in $F^2\,\mathrm{CH}^2(S)$ are defined over fields of transcendence degree 2 (see [18]). Thus, in order to detect such cycles we must use the full force of such a "family"-like structure.

For any variety V over $\mathbb{C}$ we consider the collection of Cartesian diagrams

$$\begin{array}{ccc} V & \rightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,\mathbb{C} & \rightarrow & S \end{array}$$

where S and $\mathcal{V}$ are varieties defined over $\overline{\mathbb{Q}}$, and the lower horizontal arrow factors through the generic point of S. Assume for the moment that V is smooth projective, and that S and $\mathcal{V}$ are smooth and $\mathcal{V} \rightarrow S$ is proper and smooth. Then the relative de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(\mathcal{V}/S)$ carry the Gauss-Manin connection; moreover, after base change to $S \otimes \mathbb{C}$ the associated local system is a variation of Hodge structure. This has been generalised by M. Saito (see [16]) for all V and all choices of S and $\mathcal{V}$ as follows.[2] There is a (mixed) Hodge module $\mathrm{R}^i_{\mathrm{dR}}(\mathcal{V}/S)$ on S in the above context so that its pull-back via $\operatorname{Spec}\mathbb{C} \to S$ is the (mixed) Hodge structure on the cohomology of V. The category $\mathrm{MHM}(S)$ of Hodge modules over S is an abelian category which *has* non-trivial Ext^2's when S has dimension at least 1. Moreover, we have a natural spectral sequence

$$E_1^{a,b} = \mathrm{Ext}^b_{\mathrm{MHM}(S)}(\mathbb{Q}(c), \mathrm{R}^a_{\mathrm{dR}}(\mathcal{V}/S)) \Rightarrow \mathrm{Ext}^{a+b}_{\mathrm{MHM}(\mathcal{V})}(\mathbb{Q}(c), \mathbb{Q})$$

We are interested in the case $a = 2p - k$, $b = k$ and $c = -p$. In this case the latter term can be identified with the Deligne-Beilinson cohomology $\mathrm{H}^{2p}_{\mathrm{Db}}(\mathcal{V}, \mathbb{Q}(p))$ (see [15] and [2]).

Definition 2. Let us define the absolute Deligne-Beilinson cohomology of V as the direct limit

$$\mathrm{H}^n_{\mathrm{ADb}}(V, \mathbb{Q}(c)) = \varinjlim \mathrm{Ext}^n_{\mathrm{MHM}(\mathcal{V})}(\mathbb{Q}(c), \mathbb{Q})$$

where the limit is taken over all diagrams such as the one above.

Since any algebraic cycle on V (and V itself) is defined over some finitely generated field, we have

$$\mathrm{CH}^p(V) = \varinjlim CH^p(\mathcal{V})$$

The cycle class map in Deligne-Beilinson cohomology then gives us a cycle class map

$$\mathrm{cl}^p_{\mathrm{ADb}} : \mathrm{CH}^p(V) \to \mathrm{H}^{2p}_{\mathrm{ADb}}(V, \mathbb{Q}(p))$$

The filtration on the latter group induced by the above spectral sequence induces a filtration on $\mathrm{CH}^p(V)$. We can then ask whether this is the filtration as required by Bloch's conjecture.

It is well known (see [13]) that the cycle class map for Deligne-Beilinson cohomology combines the usual cycle class map to singular cohomology with the Abel-Jacobi map. Thus, the following conjecture implies that $\mathrm{cl}^p_{\mathrm{ADb}}$ is injective.

Conjecture 1 (Bloch-Beilinson)**.** *If V is a variety defined over a number field then $F^2\,\mathrm{CH}^p(V) = 0$.*

We (of course) offer no proof of this conjecture. However, there are examples due to C. Schoen and M. V. Nori (see [17]), discovered independently by M. Green and the third author, which show that one cannot relax the conditions in this conjecture. A paper [6] by M. Green and and the third author contains these and other examples showing that $F^2\,\mathrm{CH}^p(V)$ can be non-zero for V a variety over a field of transcendence degree at least one.

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[2] Since we have chosen an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$ we can think of Hodge modules as being associated with varieties over $\overline{\mathbb{Q}}$ rather than with varieties over $\mathbb{C}$

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