LEFSCHETZ THEOREMS FOR PRINCIPAL BUNDLES

AJNEET DHILLON AND G. V. RAVINDRA

Abstract. We study the problem of lifting a principal bundle from a closed subscheme to an ambient scheme. The general framework in [SGA2] and [H] is used to tackle this problem along with Tannaka duality. As an application, we prove an analogue of a splitting criterion of Kempf to principal bundles on projective hypersurfaces and an analogue of the Noether-Lefschetz theorem for G-bundles.

1. Introduction

Let Y be a closed subscheme of X and let P be a principal G-bundle on Y. In this paper we study the problem of lifting P to X. In the case where $G = \mathrm{GL}_n$, the problem was studied in [SGA2] (see also [H]). The procedure is to first lift the vector bundle to a formal vector bundle on $\mathfrak{X}$, the formal completion of X along Y and then to consider the problem of algebraization of this formal vector bundle.

In the case of a principal G-bundle for G an affine group scheme, the problem of lifting a principal G-bundle $\pi : P \to Y$ to any infinitesimal thickening Z of Y in the ambient subscheme X is well understood (see [I]). The obstruction for such a lift to exist is an element of the cohomology group $\mathrm{H}^2(Y, \mathrm{ad}(P) \otimes I_{Y/Z})$. Here $I_{Y/Z}$ is the ideal sheaf defining Y in Z and $\mathrm{ad}(P)$ is the *adjoint bundle* of P. It follows from results in [EGA1] that the vanishing of these obstructions is necessary and sufficient for P to lift to *geometric principal bundle* $\mathfrak{P} \to \mathfrak{X}$ (see §2 for details).

However the algebraization problem, i.e., that of finding a principal bundle $\widetilde{P} \to X$ such that $\widetilde{P} \otimes \mathcal{O}_{\mathfrak{X}} \cong \mathfrak{P}$ turns out to be more subtle. To achieve this, we use the *Tannakian formalism* of Nori [M] which allows us to view a principal bundle on any (formal) scheme as a functor from the cateogory of G-representations to the category of vector bundles on that (formal) scheme. By Grothendieck's algebraization theorem, each of these vector bundles on $\mathfrak{X}$ admits a lift to an open set. However these open sets are not necessarily the same for each of these vector bundles, and hence lifting the functor associated to $\mathfrak{P}$ to a functor associated to $\widetilde{P}$ is not guaranteed.

When G is reductive we solve the algebraization problem (Theorem 4.4) by first extending the principal G-bundle P to a principal $\mathrm{GL}(V)$-bundle $\mathbf{Frame}(P(V))$ on Y. The associated vector bundle, by Grothendieck's algebraization theorem, lifts to a vector bundle on an open set $U \subset X$ containing Y. This then yields a principal $\mathrm{GL}(V)$-bundle on U. We then show that this admits a reduction of structure group to G to yield us a principal G-bundle $\widetilde{P}$ which we then verify is indeed a lift of the G-bundle $P \to Y$. This last part is achieved by working out a *Tannakian* interpretation of the reduction of structure group inspired by the results of [BG]. The reductive hypothesis is used here in identifying $\mathrm{GL}(V)/G$ with $\mathbf{Spec}(k[\mathrm{GL}(V)]^G)$. The Tannakian description of the reduction of structure group, along with its formal scheme analog (3.11), when combined with results in [EGA1, H]) produces a Lefschetz type theorem, see (4.4).

As an application of our algebraization theorem, Theorem 4.4 we prove an analogue of Kempf's splitting criterion for principal bundles, Theorem 5.8. Our theorem strengthens the result in [B] which was provided a criterion for splitting of principal bundles over a projective space. As a final application, we prove a G-bundle analogue of the Noether-Lefschetz theorem, Theorem 6.1.

One of the main applications of the Lefschetz formalism of Grothendieck in *op. cit.* is an algebraic proof of the following statement, now known as the *Grothendieck-Lefschetz therorem*: if X is a smooth, projective variety of dimension at least 4 over an algebraically closed field of characteristic 0, and $Y \subset X$ is a smooth ample divisor, then the restriction of the Picard groups is isomorphic.

In the context of local commutative algebra, the Grothendieck-Lefschetz theorem says that if R is a Noetherian local complete intersection ring of dimension ≥ 4, then any line bundle on the punctured spectrum $U_R := \operatorname{Spec} R \setminus \{\mathfrak{m}\}$ is trivial. Here $\mathfrak{m}$ is the irrelevant maximal ideal in R. In [Dao], H. Dao proposed the following generalisation of this theorem to higher rank bundles:

> Let M be a reflexive module on R which is locally free on the punctured spectrum U_R and such that $\mathrm{depth}_{\mathrm{R}}(\mathrm{End}_R(M)) \geq 4$. Then M is free.

Notice that when M is a rank 1 module, then $\mathrm{End}_R(M) \cong R$. Since R is Cohen-Macaulay, we have $\mathrm{depth}(R) = \dim R$ and thus we recover the Grothendieck-Lefschetz theorem. Dao's conjecture was proved in [KC]. As an application, a splitting criterion for vector bundles on complete intersections of dimension at least 3 is proved (see *op. cit.* Theorem 4.1).

The depth condition in Dao's conjecture implies the vanishing of the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(R, \mathrm{End}(M))$ for $i = 2, 3$. There are isomorphisms $\mathrm{H}^i_{\mathfrak{m}}(R, \mathrm{End}(M)) \cong \mathrm{H}^{i-1}_*(X, \mathrm{End}(E))$ for all $i \geq 2$. Thus the vanishing of these cohomology groups imply that the vector bundle E is a sum of line bundles. We refer the reader to [KC] for more details.

A slightly stronger version of this result (see Theorem 5, [RT]) was proved when X is a *smooth* complete intersection of dimension at least 3; the corresponding result for hypersurfaces can be strengthened even more (see Theorem 3, in *op. cit.*). In addition, analogues of the Noether-Lefschetz theorem, which do not follow from Dao's conjecture, were also established (Theorems 7 and 8, *op. cit.*). A crucial ingredient (apart from Grothendieck's Lefschetz formalism which plays a key role in both [KC] and [RT]) is a strengthened version of a theorem of Kempf [Kempf] due to Mohan Kumar [MK] which states that any bundle on projective space with vanishing $\bigoplus_{\nu<0} \mathrm{H}^1(\mathbb{P}^n, End(E)(\nu))$ is a sum of line bundles.

1.1. **Organization of the paper.** We briefly describe the organization of this paper.

§2 of the paper recalls the deformation theory of principal bundles and applies these results to lifting to formal schemes. §3 starts with Nori's Tannakian interpretation of principal bundles in ([M]). The result is extended to reduction of structure group. In §4, we start by recalling the Lefschetz conditions. The main results are stated and proved here. In §5, we prove analogues of Kempf's splitting criterion for principal bundles.

Notations and conventions

All principal bundles will be right bundles.

k	Our ground field
G	An affine group scheme over k
H	A closed subgroup of G, often assumed reductive
$\mathrm{Vect}(X)$	The category of finite rank vector bundles on a scheme or formal scheme or algebraic stack
$\mathrm{Vect}(BG)$	The category of finite dimensional representations of G.
$\mathbf{Qcoh}(X)$	The category of quasi-coherent sheaves on X
$\mathbf{Qcoh}(BG)$	The category of representations of G

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2. Deformation theory of principal bundles and lifting from a closed subscheme.

Let X be a scheme of finite type over k and let Y be a closed subscheme defined by a sheaf of ideals $\mathcal{I}$. We will denote by $\mathfrak{X}$ the formal completion of X along Y. We let Y_n be the nth infinitesimal thickening of Y in X, that is the scheme defined by $\mathcal{I}^{n+1}$. In this section we are interested in the problem of lifting a principal G-bundle on Y to $\mathfrak{X}$.

2.1. Principal bundles on formal schemes.

Definition 2.1. Let X/k be a scheme and G/k an algebraic group. A *principal G-bundle on X* is a scheme P/X with a G-action such that the structure map $P \to X$ is G-equivariant for the trivial action on X and the induced map

$$G \times_k X \to X \times_k X$$

is an isomorphism.

Definition 2.2. A *geometric principal G-bundle on $\mathfrak{X}$* is a Noetherian formal scheme $\mathfrak{P}$ together with an adic morphism

$$f : \mathfrak{P} \to \mathfrak{X}$$

and a G action on $\mathfrak{P}$ such that the morphism f is G-equivariant when G acts on $\mathfrak{X}$ trivially. We further require that

$$G \times_k \mathfrak{P} \xrightarrow{\sim} \mathfrak{P} \times_{\mathfrak{X}} \mathfrak{P}$$

where the morphism is induced by the projection and the action.

Recall [EGA1, 10.12], that a morphsim $f : \mathfrak{P} \to \mathfrak{X}$ of formal schemes is said to be *adic* if I is an ideal of definition of $\mathfrak{X}$ then f^*I is an ideal of definition of $\mathfrak{P}$.

In this situation, we set $X_n = \mathfrak{X}/I^n$. An inductive system of X_n-schemes consists of Noetherian schemes $Y_n \to X_n$ such that $Y_m \cong Y_n \times_{X_n} X_m$ whenever $n \geq m$. There are categories of inductive systems of Noetherian X_n-schemes and adic $\mathfrak{X}$-schemes defined in the obvious way.

Theorem 2.3. *There is an equivalence of categories between the category of adic Noetherian formal $\mathfrak{X}$-schemes and the category of inductive systems of Noetherian X_n-schemes given by the obvious functor.*

Proof. This is essentially [EGA1, 10.12.3], one just needs to check the Noetherian hypothesis is preserved in the equivalence. In this direction, we remark that the topology on $\mathfrak{X}$ and X_n is the same. □

2.2. **Deformation theory.** Consider a principal G-bundle $P \to Y$, where Y is smooth over k. As we have seen (or defined) in the previous subsection, the problem of extending P to $\mathfrak{X}$ consists of a sequence of deformation theory problems. More precisely, given $P_n \to Y_n$, lifting P, can we find an extension $P_{n+1} \to Y_{n+1}$ of P_n to Y_{n+1} so that the following diagram is Cartesian

$$\begin{array}{ccc} P_n & \longrightarrow & P_{n+1} \\ \downarrow & & \downarrow \\ Y_n & \longrightarrow & Y_{n+1} \end{array}$$

and P_{n+1} is a principal G-bundle.

Recall that the adjoint bundle of P is the vector bundle

$$\mathrm{ad}P := P \times_G \mathrm{Lie}(G)$$

The following theorem is well-known:

Theorem 2.4. *Let Y/k be smooth and suppose that we have a lift P_{n-1} of P_1 to Y_n. Then there is an obstruction in*

$$\mathrm{H}^2(Y, \mathrm{ad}(P_0) \otimes \mathcal{I}^n/\mathcal{I}^{n+1})$$

whose vanishing is sufficient for the existence of a lift P_n to Y_{n+1}.

Proof. This is [I, pg 209, theorem 2.4.4] combined with [I, 2.4.2.13]. □

Corollary 2.5. *In the above situation, if*

$$\mathrm{H}^2(Y, \mathrm{ad}(P_0) \otimes I^n/I^{n+1}) = 0$$

for all n then we can lift P to principal bundle over $\mathfrak{X}$.

Example 2.6. The obstructions sometimes do vanish. For example for the trivial G-bundle on $\mathbb{P}^r \subseteq \mathbb{P}^{r+1}$ when $r \geq 3$.

3. Tannaka Duality

3.1. **The Tannakian interpretation of principal bundles.** In this section we recall one of the results of [M].

Let X/k be a scheme. We denote by $\mathrm{Vect}(X)$ the $\otimes$-category of finite rank vector bundles on X.

We consider functors $F : \mathrm{Vect}(BG) \to \mathrm{Vect}(X)$ that satisfy the following axioms:

F1 The functor is a k-additive exact monoidal functor.
F2 The functor preserves rank and is faithful.

Example 3.1. If $P \to X$ is a principal G-bundle then the functor

$$\begin{array}{rcl} \mathrm{Vect}(BG) & \longrightarrow & \mathrm{Vect}(X) \\ V & \longmapsto & P \times_G V \end{array}$$

satifies these conditions. Recall that $\mathrm{Vect}(BG)$ is the category of G-representations. This functor will play an important role in what is to come and we will denote it by F_P. It is useful to reinterpret F_P in terms of locally free sheaves on X. As G is affine, the principal bundle G corresponds to a sheaf of algebras $\mathcal{A}$ on X with an action of G. Then $(\mathcal{A} \otimes_k V)^G$ is a locally free sheaf corresponding to $F_P(V)$.

If T is a scheme or more generally an algebraic stack, we denote by $\mathbf{Qcoh}(T)$ the category of quasi-coherent sheaves on T. For G/k an algebraic group, the category $\mathbf{Qcoh}(BG)$ just amounts to the category of co-modules over the Hopf algebra $k[G]$.

Let F be a functor satisfying the conditions above. Then F admits a unique extension to a direct limit preserving functor $\mathbf{Qcoh}(BG) \to \mathbf{Qcoh}(X)$ such that $F(V)$ is faithfully flat for all non-zero $V \in \mathbf{Qcoh}(BG)$, see [M, 2.1].

If $G = \mathrm{Spec}\, A$ then A has a G-action. Then $F(A)$ is a sheaf of algebras on X and one can show that $P(F) := \mathrm{Spec}(F(A))$ is a principal G-bundle, see [M, §2].

Theorem 3.2. *There is a bijection between isomorphism classes of functors satisfying the conditions F1-2 and isomorphism classes of principal G-bundles.*

Proof. See [M]. In the cited paper an extra condition that F preserves the $\otimes$ identity is needed. However, this is part of being monoidal. □

3.2. **Reductions of structure group.** Let G be an algebraic group and H a closed subgroup. We begin with a couple of well-known results whose proofs are included here for the sake of completeness.

Lemma 3.3. *If $P \to X$ is a principal G-bundle then to give a reduction of structure group of P to H is equivalent to giving a section σ of $P/H := P \times_G G/H$.*

Proof. Given a section $\sigma : X \to P/H$, the reduction Q is given by forming the Cartesian square:

$$\begin{array}{ccc} Q := \sigma^*(P) & \longrightarrow & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & P/H. \end{array}$$

Conversely, suppose we have a principal H-bundle Q which is a reduction of P. This is the same as giving an H-equivariant isomorphism $P \cong Q \times_H G$. Now consider the H-equivariant inclusion $\iota : Q \hookrightarrow P$, given by $q \mapsto (q, 1_G)$. Going modulo the action of H, ι descends to an inclusion

$$\sigma : Q/H = X \hookrightarrow P/H.$$

□

We can further unwind the data of a section $X \to P/H$.

Lemma 3.4. *There is a one-to-one correspondence between morphisms $f : P \to G/H$ satisfying $f(p \cdot g) = g^{-1}f(p)$ and sections $\sigma : X \to P/H$.*

Proof. Given a morphism $f : P \to G/H$ satisfying $f(p \cdot g) = g^{-1}f(p)$, we define a section

$$\sigma : X \to P/H, \quad \sigma(x) := (p, f(p),$$

where $\pi(p) = x$ for the morphism $\pi : P \to X$.

For any $g \in G$, $\pi(p \cdot g) = \pi(p)$, and also

$$(p \cdot g, f(p \cdot g)) = (p \cdot g, g^{-1}f(p)) = (p, f(p)) \cdot g = (p, f(p)).$$

This shows that $\sigma(x)$ is well-defined.

To see the converse, suppose that we are given a section $\sigma : X \to P/H$. The data of a section is the same as giving a G-bundle $\widetilde{P} \to X$ and an equivariant morphism $\widetilde{\sigma} : \widetilde{P} \to P \times G/H$ so that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{P} & \longrightarrow & P \times G/H \\ \downarrow & & \downarrow \\ X & \longrightarrow & (P \times G/H)/G. \end{array}$$

The first component of $\widetilde{\sigma}$ identifies $\widetilde{P}$ with P and the second produces the desired morphism. □

3.3. **Tannakian interpretation of sections.** In the case where H is reductive, we can give a Tannakian interpretation of the section. To do this, we introduce the *functor of H-invariants*

$$(-)^H_X : \mathrm{Vect}(BG) \longrightarrow \mathrm{Vect}(X),$$

defined by

$$V \longmapsto (V)^H \otimes_k \mathcal{O}_X$$

This functor is lax monoidal, that is there are morphisms

$$(V)^H_X \otimes (W)^H_X \to (V \otimes W)^H_X$$

that form a natural transformation.

Recall that given a principal G-bundle P we have a functor

$$F_P : \mathrm{Vect}(BG) \to \mathrm{Vect}(X).$$

defined by

$$F_P(V) := P \times_G V.$$

Theorem (3.2) says that this functor determines P. Combining these constructions we are able to give a Tanakian description of reductions of structure group of G-bundles to a reductive subgroup.

Theorem 3.5. *In the above setting, assume further that H is reductive.*

To give a section of P/H is the same as giving a monoidal natural transformation

$$\eta : (-)^H_X \longrightarrow F_P.$$

Remark 3.6. Being monoidal means that the following diagrams commute:

$$\begin{array}{ccc} (V)^H_X \otimes (W)^H_X & \xrightarrow{\eta_V \otimes \eta_W} & F_P(V) \otimes F_P(W) \\ \downarrow & & \downarrow \\ (V \otimes W)^H_X & \xrightarrow{\eta_{V \otimes W}} & F_P(V \otimes W). \end{array}$$

Proof. To prove this we need to recall a few elements of the proof in [M]. To recover P from F_P one extends F_P to a colimit preserving functor

$$F_P : \mathbf{Qcoh}(BG) \to \mathbf{Qcoh}(X).$$

The extension is constructed as follows. Given $V \in \mathbf{Qcoh}(BG)$ we write it as

$$V = \varinjlim V_i$$

where the colimit is over all finite dimensional subrepresentations of V. Then set

$$F_P(V) = \varinjlim F_P(V_i).$$

Nori observes that $F_P(k[G])$ is a sheaf of algebras on X and $P = \mathbf{Spec}(F_P(k[G]))$.

To prove the theorem, one observes that $(-)^H$ also extends to

$$(-)^H_X : \mathbf{Qcoh}(BG) \to \mathbf{Qcoh}(X)$$

in the obvious way.

Now suppose that we are given a monoidal natural transformation

$$\eta : (-)^H_X \longrightarrow F_P.$$

We can extend it to a natural transformation of extended functors. Evaluating at $k[G]$ gives a morphism

$$k[G]^H \to F_P(k[G]).$$

Taking **Spec** yields the section by the previous remark.

Now suppose that we have a section. As we observed in (3.3), the data of a section is the same as giving an H-principal bundle $Q \to X$ that is a reduction of structure group of P to H. Then if V is a representation of G, we have

$$Q \times_H V^H \to Q \times_H V \cong P \times_G V$$

and it is easy to see that this yields a natural transformation. □

3.4. **Principal bundles on formal schemes.** In this section we let $\mathfrak{X}$ be a Noetherian formal scheme over k. We let G be an affine group scheme of finite type over k.

Definition 3.7. A *Tannakian principal G-bundle* on $\mathfrak{X}$ is a functor

$$F : \mathrm{Vect}(BG) \to \mathrm{Vect}(\mathfrak{X})$$

satisfying the conditions F1-2.

We have previously defined the notion of a geometric principal G-bundle on $\mathfrak{X}$, see (2.2).

Theorem 3.8. *There is a bijection between Tannakian principal G-bundles on $\mathfrak{X}$ and geometric principal G-bundles on $\mathfrak{X}$.*

Proof. Recall that by reduction mod I^n a vector bundle on $\mathfrak{X}$ is equivalent to an inductive system of vector bundles on X_n. As Theorem 3.2 applies to the schemes X_n we see that, using Theorem 2.3, Tannakian principal G-bundles are in bijection with inductive systems of principal G-bundles on X_n. So to prove the theorem, it suffices to show that there is a bijection between isomorphism classes of inductive systems of principal G-bundles and geometric principal G-bundles on $\mathfrak{X}$.

Each geometric principal G-bundle produces an inductive system of principal X_n-bundles by reduction mod I^n. Conversely, suppose that we are given an inductive system $P_n \to X_n$ of G-bundles. We set $\mathfrak{P} = \varinjlim P_n$, c.f [EGA1, 10.12.3.1]. Then by the universal property, we see that $\mathfrak{P}$ has a G-action for which the morphism $\mathfrak{P} \to \mathfrak{X}$ is eqivariant. We need to show that there is an isomorphism

$$G \times \mathfrak{P} \xrightarrow{\sim} \mathfrak{P} \times_{\mathfrak{X}} \mathfrak{P}.$$

It suffices to show that morphism above is an isomorphism mod I^n for each n. But it is, as P_n is a G-bundle. □

Definition 3.9. Let $\mathfrak{P}$ be a principal G-bundle on $\mathfrak{X}$. If $H \subseteq G$ is a closed subgroup we define

$$\mathfrak{P}/H := \lim P_n \times_G G/H.$$

An *adic section of* $\mathfrak{P}/H \to \mathfrak{X}$ is a morphism $\sigma : \mathfrak{X} \to \mathfrak{P}/H$ of the form

$$\sigma = \lim \sigma_n$$

where $\sigma_n : X_n \to P_n/H$ are sections so that for $n \geq m$ the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\sigma_n} & P_n/H \\ \downarrow & & \downarrow \\ X_m & \xrightarrow{\sigma_m} & P_m/H \end{array}$$

is Cartesian.

Example 3.10. Suppose that H is a closed subgroup of G. Let $\mathfrak{P} \to \mathfrak{X}$ be a principal H-bundle. We can form a principal G-bundle $\mathfrak{P} \times_H G$ by extending the structure group. It corresponds to the X_n-inductive system $P_n \times_H G$. Then $\mathfrak{P} \times_H G \times_G G/H$ has an adic section.

Let H be a subgroup of G. We have a lax monoidal functor

$$(-)^H_{\mathfrak{X}} : \ \mathrm{Vect}(BG) \longrightarrow \ \mathrm{Vect}(\mathfrak{X})$$

given by $(W)^H := W^H \otimes \mathcal{O}_{\mathfrak{X}}$.

Theorem 3.11. *Let H be a reductive subgroup of G. There is a bijection between the set of adic sections and monoidal natural transformations*

$$(-)^H_{\mathfrak{X}} \ \longrightarrow \ F_{\mathfrak{P}}.$$

Proof. Combine (3.5) and (2.3). □

4. The Lefschetz formalism for principal bundles

We start by recalling a few facts from [SGA2] and [H].

Let X be a scheme and let $Y \subseteq X$ be a closed subscheme. Denote by $\mathfrak{X}$ the formal completion of X along Y. Given an open subscheme $U \subseteq X$, containing Y, there is a restriction functor

$$\widehat{\ } : \mathrm{Vect}(U) \to \mathrm{Vect}(\mathfrak{X}).$$

We say that the *Lefschetz condition* holds for (X, Y), if for every open set $U \supseteq Y$ and every locally free sheaf F on U, we can find an open V with $Y \subseteq V \subseteq U$ so that

$$\mathrm{H}^0(V, F|_V) \tilde{\to} \mathrm{H}^0(\mathfrak{X}, \hat{F}).$$

We say that the *effective Lefschetz condition* holds for (X, Y), if the Lefschetz condition holds and every locally free sheaf $\hat{F}$ on $\mathfrak{X}$ lifts to some open neighbourhood of Y, that is there exists $U \supseteq Y$ and a locally free sheaf F on U so that $F|_{\mathfrak{X}} \cong \hat{F}$.

Remark 4.1. Note that if X is projective and Y is an ample divisor in X then any neighbourhood of Y in X is the complement of a finite set of points. Further, the effective Lefschetz condition for (X, Y) will often hold, see Theorem 4.3 below.

We denote by $\mathrm{Vect}_L(Y)$ the category of *germs of locally free sheaves near* Y. The objects of this category are equivalence classes of pairs (U, F) where U is an open neighbourhood of Y inside X and F is a locally free sheaf on U. Two pairs are equivalent, if they are isomorphic after refining to a common open neighbourhood. Morphisms are defined in the obvious way. There is a functor

$$\widehat{\ } : \mathrm{Vect}_L(Y) \to \mathrm{Vect}(\mathfrak{X}).$$

Proposition 4.2. *Suppose that the effective Lefschetz condition holds for (Y, X) then*

$$\widehat{\ } : \mathrm{Vect}_L(Y) \to \mathrm{Vect}(\mathfrak{X})$$

is a $\otimes$-equivalence of categories.

Proof. It is clear that the given functor is a $\otimes$-functor. The Lefschetz condition ensures that it is fully faithfaul and the effective Lefschetz condition ensures that it is essentially surjective. □

The main source of examples come from the following theorem.

Theorem 4.3. *Suppose that X is a non-singular projective variety. Let $Y \subseteq X$ be a complete intersection subscheme. If $\dim Y \geq 2$, then the effective Lefschetz condition holds for (X, Y).*

Proof. This is [H, Ch IV, Theorem 1.5]. □

Theorem 4.4. *Let G be a reductive linear algebraic group and suppose that the effective Lefschetz condition holds for the pair (X, Y). If*

$$F_P : \mathrm{Vect}(BG) \to \mathrm{Vect}(\mathfrak{X})$$

is a principal G-bundle on $\mathfrak{X}$, then P lifts to a neighbourhood U of Y.

Proof. As G is linear algebraic, it has a faithful represention V. We write $\mathbf{Frame}(P(V))$ for the functor associated to the principal $\mathrm{GL}(V)$-bundle $F_P(V)$, i.e.,

$$\mathbf{Frame}(P(V)) : \mathrm{Vect}(B\mathrm{GL}(V)) \to \mathrm{Vect}(\mathfrak{X}).$$

To simplify notation, we refer to $\mathbf{Frame}(P(V))$ as both the $\mathrm{GL}(V)$-principal bundle associated to $P(V)$ as well as the functor it defines in the sense of [M].

By Lemma 3.3, there is an adic section

$$\sigma : \mathfrak{X} \to \mathbf{Frame}(P(V))/G$$

corresponding to the reduction of structure group to G. We view $\mathbf{Frame}(P(V))$ as a monoidal functor and the section as a natural transformation as in Theorem 3.5. Denote the natural transformation corresponding to σ by

$$\widehat{\eta} : (-)^G \to \mathbf{Frame}(P(V)).$$

To prove the theorem we will lift the pair $(\mathbf{Frame}(P(V)), \widehat{\eta})$ to a neighbourhood of U of Y in X. The result follows then from Lemma (refr:reduction and Theorem 3.5.

By the effective Lefschetz condition, the formal vector bundle $F_P(V)$, and hence the principal $\mathrm{GL}(V)$-bundle $\mathbf{Frame}(P(V))$ on $\mathfrak{X}$ lifts to a vector bundle on a neighbourhood U of Y in X. We denote the lift of $\mathbf{Frame}(P(V))$ by $\widetilde{\mathbf{Frame}(P}(V))$. We need to now lift $\widehat{\eta}$. The data of $\widehat{\eta}$ is a collection of morphisms

$$\widehat{\eta}_W : W^G \otimes \mathcal{O}_{\mathfrak{X}} \to \mathbf{Frame}(P(V))(W)$$

for each representation W of $\mathrm{GL}(V)$. By the Lefschetz condition, these morphisms lift uniquely to morphisms

$$\eta_W : W^G \otimes \mathcal{O}_U \to \widetilde{\mathbf{Frame}(P}(V)).$$

In view of the uniqueness of the lifts, we obtain a natural transformation

$$\eta : (-)^G \to \widetilde{\mathbf{Frame}(P}(V)).$$

Once again, by uniqueness it is a monoidal natural transformation and so we are done by Theorem 3.5. □

Remark 4.5. The proof shows that if G has a faithful representation V and $P \times_G \mathrm{GL}(V)$ lifts to an open set U then so does P.

5. A splitting criterion

We assume throughout this section that G is a reductive algebraic group.

In this section we prove an analogue of Kempf's splitting criterion for principal G-bundles on hypersurfaces in projective space. The version that we prove here (see Theorem 5.8) may be viewed as a *Grothendieck-Lefschetz theorem for higher rank bundles.* As mentioned in §1, the statement for this theorem was conjectured by Dao (see [Dao]), and was first established by Česnavičius in [KC]. Our version here is a slightly stronger statement; however unlike the theorem in [KC], we require an additional hypothesis that the hypersurface be *smooth*.

Theorem 5.1 (Kempf). *A vector bundle E on $\mathbb{P}^n$ for $n \geq 2$ splits into a sum of line bundles if and only if*

(1) $\mathrm{H}^1(\mathbb{P}^n, \mathrm{End}(E)(\nu)) = 0$ *for all* $\nu < 0$*, and*
(2) E extends to a vector bundle on $\mathbb{P}^{n+1}$.

In [MK], it was shown that (1) $\implies$ (2) in the above theorem. This was independently proved in [B] as well where Kempf's criterion was generalized to prove a splitting theorem for any principal G-bundle, with G reductive. One of the crucial points in this context is that a bundle E on $\mathbb{P}^n$ admits, for any point $\mathbf{x} \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$, via the projection map

$$\pi : \mathbb{P}^{n+1} \setminus \{\mathbf{x}\} \to \mathbb{P}^n,$$

an extension $\pi^*(E)$ to the open set $U := \mathbb{P}^{n+1} \setminus \{\mathbf{x}\}$ containing $\mathbb{P}^n$. The implication (1) $\implies$ (2) is then established in [MK, B] by showing that the vanishing in condition (1) implies that for any two distinct points $\mathbf{x}$, $\mathbf{y} \in \mathbb{P}^{n+1} \setminus \mathbb{P}^n$, the pull back bundles patch together to give an extension of E as a bundle to all of $\mathbb{P}^{n+1}$. Unfortunately, when working with hypersurfaces of degree at least 2, one does not have such an extension (of the bundle E on the hypersurface X) to an open set (U containing X) for free; the vanishing of the cohomology groups $\mathrm{H}^2(X, \mathrm{End}(E)(\nu))$ for $\nu < 0$ ensures that we do have such an extension.

We will make use of the following results.

Lemma 5.2. *Let F and G be vector bundles on $\mathbb{P}^n$. Let $U \subseteq \mathbb{P}^n$ be an open subset with codimension $\mathbb{P}^n \setminus U$ at least 2. Then the restriction map*

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(F|_U, G|_U)$$

is an isomorphism.

Proof. Note that F is the reflexive hull of $i_* i^* F$ where $i : U \hookrightarrow \mathbb{P}^n$ is the inclusion. As a similar description holds for E, the result follows. □

Lemma 5.3. *Let $n \geq 3$ and let F be a coherent sheaf on $\mathbb{P}^n$. Suppose that we have a short exact sequence*

$$0 \to F(-1) \to F \to \bigoplus_{i=1}^{r} \mathcal{O}_W(a_i) \to 0$$

where $W \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ is a hyperplane and a_i are integers. Then $\mathrm{Ext}^1(F, L) = 0$ *for* $L \in \mathrm{Pic}(\mathbb{P}^n)$*.*

Proof. By Serre duality we need to show $H^{n-1}(\mathbb{P}^n, F(m)) = 0$ for all integers m. We know that this holds for $m \gg 0$ so we use descending induction on m. The result follows from the fact that $H^{n-2}(\mathbb{P}^{n-1}, \mathcal{O}(a)) = 0$ for each integer a. □

Lemma 5.4. *Let $n \geq 3$ and let F be a reflexive sheaf on $\mathbb{P}^n$. Suppose that there is a closed subset $Z \subseteq \mathbb{P}^n$ of codimension at least 2, so that $F_{\mathbb{P}^n \setminus Z}$ is a vector bundle. Suppose that we have a short exact sequence*

$$0 \to F(-1) \to F \xrightarrow{\phi} \bigoplus_{i=1}^{r} \mathcal{O}_W(a_i) \to 0$$

where $W \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ is a hyperplane and a_i are finitely many integers. If $\phi|_W$ is an isomorphism then $F \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i)$.

Proof. There is a short exact sequence

$$0 \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(a_i - 1) \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(a_i) \to \bigoplus_{i=1}^{r} \mathcal{O}_W(a_i) \to 0.$$

By applying the functor $\mathrm{Hom}(F, -)$ to this sequence and applying the prior lemma we see that ϕ lifts to a homomorphism

$$\Phi : F \to \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n}(a_i).$$

Now, $\wedge^r \Phi$ is an isomorphism on some neighbourhood of W. Hence it is an isomorphism away from finitely many points. However, there is some open subset U with $\mathbb{P}^n \setminus U$ having codimension at least 2, on which the two reflexive sheaves F and $\bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^n}(a_i)$ are isomorphic. But then both these sheaves are reflexive extensions of the same vector bundle, and hence must be isomorphic. □

Lemma 5.5. *Let $U \subseteq \mathbb{P}^n$ be an open subset so that the closed set $\mathbb{P}^n \setminus U$ has codimension at least 2 in $\mathbb{P}^n$. Let P be a principal G-bundle on U. If for each $V \in \mathrm{Vect}(BG)$ the vector bundle $P \times_G V$ extends to a vector bundle on $\mathbb{P}^n$ then P exends to a principal bundle $\tilde{P}$ on $\mathbb{P}^n$. Note that as extensions of vector bundles on U to $\mathbb{P}^n$ are unique we must have that $\tilde{P} \times_G V$ is the unique extension of the vector bundle $P \times_G V$.*

Proof. Fix a faithful representation V of G and consider the $GL(V)$-bundle $Q := P \times_G \mathrm{GL}(V)$. As $P \times_G V$ has an extension to $\mathbb{P}^n$ so does Q. We will call this extension $\tilde{Q}$.

The $\mathrm{GL}(V)$-bundle Q has a reduction of structure group to G that produces P. To prove the result it suffices to show that this reduction extends to $\mathbb{P}^n$. The reduction corresponds to a natural transformation as in Theorem 3.5. The hypothesis of the lemma insure that this natural transformation extends using Lemma 5.2. □

Lemma 5.6. *Let Y be a smooth projective variety of dimension at least 3 with ample line bundle $\mathcal{O}_Y(1)$. Let F be a reflexive sheaf on Y. Further assume that there are finitely many closed points $y_1, \ldots, y_n \in Y$ so that $F_{Y \setminus \{y_1, \ldots, y_n\}}$ is locally free. Then we have*

$$\mathrm{H}^0(Y, F(\nu)) = \mathrm{H}^1(Y, F(\nu)) = 0 \ \forall \ \nu \ll 0.$$

Proof. See [RT, Lemma 2]. □

Lemma 5.7. *Let Y be a smooth projective variety of dimension at least 3 with ample line bundle $\mathcal{O}_Y(1)$. Let F be a reflexive sheaf on Y. Further assume that there are finitely many closed points $y_1, \ldots, y_n \in Y$ so that $F_{Y \setminus \{y_1, \ldots, y_n\}}$ is locally free. Let $Z \subseteq Y$ be a smooth hypersurface defined by the vanishing of a section of $\mathcal{O}_Y(a)$ for $a > 0$. Then the natural map*

$$\mathrm{Ext}^i(F(\nu), \omega_Y) \to \mathrm{Ext}^i(F(\nu - a), \omega_Y)$$

is an isomorphism for $\nu < 0$.

Proof. This is [RT, Corollary 2]. □

Theorem 5.8. *Let $Y \subseteq \mathbb{P}^n$ be an smooth hypersurface with $\dim Y \geq 3$. Then a principal G-bundle P admits a reduction of structure group to a one parameter subgroup if and only if*

$$\mathrm{H}^1(Y, \mathrm{ad}P(\nu)) = \mathrm{H}^2(Y, \mathrm{ad}P(\nu)) = 0,$$

for all $\nu < 0$.

Proof. The forward implication is a straightforward application of the cohomology of line bundles on $\mathbb{P}^n$.

For the reverse implication, we see that P lifts to $\mathfrak{X}$, the formal completion of $\mathbb{P}^n$ along Y using Theorem 2.4. Using Theorems 4.3 and 4.4 we can lift P to a neighbourhood U of Y in $\mathbb{P}^n$. Call the lift $\widetilde{P}$. As Y is a hypersurface, any positive dimensional subvariety of $\mathbb{P}^n$ intersects it, and so we see that $U = \mathbb{P}^n \setminus \{y_1, \ldots y_m\}$ for finitely many closed points $y_i \in \mathbb{P}^n$. The vector bundle $\mathrm{ad}\widetilde{P}$ extends to a reflexive sheaf F on $\mathbb{P}^n$ and we have exact sequences

$$0 \to F(\nu - d) \to F(\nu) \to \mathrm{ad}P(v) \to 0,$$

where $\nu \in \mathbb{Z}$ and d is the degree of Y in $\mathbb{P}^n$. Using the hypothesis $\mathrm{H}^1(Y, \mathrm{ad}P(\nu)) = 0$ for all $\nu < 0$, we see that $\mathrm{H}^1(\mathbb{P}^n, F(\nu)) = 0$ for $\nu < 0$, using Lemma 5.6. We also have isomorphisms

$$\mathrm{H}^2(\mathbb{P}^n, F(\nu - d)) \cong \mathrm{H}^2(\mathbb{P}^n, F(\nu)) \ \ \forall\, \nu < 0.$$

As U is the complement of finitely many points, so we may find a hyperplane $W = \mathbb{P}^{n-1}$ contained within U. We have an exact sequence

$$0 \to F(-1) \to F \to \mathrm{ad}\widetilde{P}|_W \to 0.$$

Using the vanishing $\mathrm{H}^1(\mathbb{P}^n, F(\nu)) = 0$, established above, together with Lemma 5.7, we see that

$$\mathrm{H}^1(W, \mathrm{ad}\widetilde{P}(\nu)|_W) = 0 \ \ \forall\, \nu < 0.$$

By [B] we see that $\mathrm{ad}\widetilde{P}|_W$ splits into a sum of line bundles. Arguing as in the proof of Theorem 4 in [RT], we see that this splitting lifts to a splitting of F into a sum of line bundles on $\mathbb{P}^n$. Consequently, $\mathrm{ad}P$ splits into a sum of line bundles.

By [BCT, Proposition 1] there is a 1-parameter subgroup $\mathbb{G}_\mathrm{m} \to G$ so that $\widetilde{P}|_W$ admits a reduction of structure group to $\mathbb{G}_\mathrm{m}$. Call the $\mathbb{G}_\mathrm{m}$-bundle Q. Let V be a representation of G. The vector bundle $\widetilde{P} \times_G V$ extends from U to $\mathbb{P}^n$ by Lemma 5.4 as

$$\widetilde{P} \times_G V|_W = Q \times_{\mathbb{G}_\mathrm{m}} V.$$

Furthermore, this extension is a direct sum of line bundles. By Lemma 5.5, the principal G-bundle $\widetilde{P}$ extends to $\mathbb{P}^n$. As we have already observed, ad$P$ splits into a sum of line bundles, and this decomposition extends to $\mathbb{P}^n$. Therefore, the extended principal bundle on $\mathbb{P}^n$, and consequently its restriction P on the hypersurface, also admit a reduction of the structure group to $\mathbb{G}_{\mathrm{m}}$ by a second application of [BCT, Proposition 1]. □

6. The Noether-Lefschetz theorem

We now state the analogue of the Noether-Lefschetz theorem to G-bundles.

Theorem 6.1. *Let Y be a general (in particular smooth) hypersurface of degree $d \geq 4$ in $\mathbb{P}^3$ and P be a principal G-bundle. Suppose that P satisfies the following conditions:*

(1) $\mathrm{H}^1(Y, \mathrm{ad}P(\nu)) = 0$ *for all* $\nu < 0$.
(2) the multiplication map

$$\mathrm{H}^0(Y, \mathrm{ad}P \otimes K_Y) \otimes \mathrm{H}^0(Y, O_Y(a)) \to \mathrm{H}^0(Y, \mathrm{ad}P \otimes K_Y(a))$$

is surjective for all $a \geq 0$.

Then P admits a reduction of structure group to a one parameter subgroup.

Proof. It is enough to show that condition (2) implies that ad$P$ extends to a reflexive sheaf on $\mathbb{P}^3$. Then by arguing as in the proof of Theorem 5.8, we will be done.

Let $\mathcal{Y} \to T$ be the universal family of smooth degree d hypersurfaces in $\mathbb{P}^3$. The hypothesis that P is a principal bundle on a general hypersurface means by definition the following: there is a smooth scheme $S \to T$ which is étale over its image and a G-bundle $\mathcal{P}$ on $\mathcal{Y}$ such that for each $s \in S$, $P_s := \mathcal{P} \otimes \mathcal{O}_{Y_s}$ is a principal G-bundle over the fibre Y_s. Furthermore, $\mathrm{ad}\mathcal{P} \otimes \mathcal{O}_{Y_s} \cong \mathrm{ad}P_s$.

Let $o \in S$ be any point parameterising a smooth hypersurface Y. If $\mathfrak{m}$ is the maximal ideal of the point o in the local ring $\mathcal{O}_{S,o}$, then one has the following identications for the dual of the tangent space

$$\Omega^1_{S,0} := \mathfrak{m}/\mathfrak{m}^2 \cong \mathrm{H}^0(Y, \mathcal{O}_Y(d)).$$

In our situation, we then have that under the Kodaira-Spencer map

$$\begin{array}{ccc} \mathrm{H}^2(Y, \mathrm{ad}P(-d)) & \to & \mathrm{Hom}(T_{S,o}, \mathrm{H}^2(Y, \mathrm{ad}P)) \\ \xi & \mapsto & (\partial/\partial(x) \mapsto \partial/\partial(x)(\xi)) \end{array}$$

the obstruction class $\eta \mapsto 0$.

The Kodaira-Spencer map can be rewritten as

$$\mathrm{H}^2(Y, \mathrm{ad}P(-d)) \to \mathrm{H}^2(Y, \mathrm{ad}P) \otimes V^*,$$

or using Serre duality as

$$\mathrm{H}^0(Y, \mathrm{ad}P \otimes K_Y) \otimes \mathrm{H}^0(Y, \mathcal{O}_Y(d)) \to \mathrm{H}^0(Y, \mathrm{ad}P \otimes K_Y(d)).$$

However, by our hypothesis, this map is surjective, and hence its dual map is injective. Since $\eta \mapsto 0$, this implies that $\eta = 0$. In particular, this means that ad$P$ extends to a bundle on $Y_1 \subset \mathbb{P}^3$, the first order thickening of Y in $\mathbb{P}^3$.

Suppose that we have been able to extend ad$P$ to Y_{m-1}. Then we consider the higher order Kodaira-Spencer map

$$\mathrm{H}^2(Y, \mathrm{ad}P(-md)) \to \mathrm{Hom}(\mathrm{Sym}^m T_{S,o}, \mathrm{H}^2(Y, \mathrm{ad}P)).$$

Let $\eta_m \in \mathrm{H}^2(Y, \mathrm{ad}P(-md))$ be obstruction for P_{m-1} to lift to a principal bundle P_m on Y_m. The same argument as above implies that the element $\eta_m \mapsto 0$. Consequently, we see that $\eta_m = 0$ for all $m \geq 0$ and so we have an extension of ad$P$ to the thickenings Y_m for all $m \geq 0$. □

Remark 6.2. One can extend the Lefschetz theorems to complete intersections as well. We refer the reader to [RT] for the precise statements.

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Department of Mathematics, Western University, Canada
Email address: adhill3@uwo.ca

Department of Mathematics, University of Missouri-St Louis
Email address: girivarur@umsl.edu