ISOGENIES OF CERTAIN K3 SURFACES OF RANK 18

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Abstract. We construct geometric isogenies between three types of two-parameter families of K3 surfaces of Picard rank 18. One is the family of Kummer surfaces associated with Jacobians of genus-two curves admitting an elliptic involution, another is the family of Kummer surfaces associated with the product of two non-isogeneous elliptic curves, and the third is the twisted Legendre pencil. The isogenies imply the existence of algebraic correspondences between these K3 surfaces and prove that the associated four-dimensional Galois representations are isomorphic. We also apply our result to several subfamilies of Picard rank 19. The result generalizes work of Van Geemen and Top in [55].

In [1], Ahlgren, Ono and Penniston computed the L-series and zeta function, over the finite field $\mathbb{F}_p$, for certain families of K3 surfaces of Picard rank 19. The K3 families in question are associated with a specific one-parameter elliptic curve family $\mathcal{E}_t$ and include the Legendre pencil $\mathcal{X}_t$, as well as the Kummer surfaces $\mathrm{Kum}(\mathcal{E}_t \times \mathcal{E}_t)$. The Ahlgren-Ono-Penniston results imply that two three-dimensional ℓ-adic Galois representations, associated with $\mathcal{X}_t$ and $\mathrm{Kum}(\mathcal{E}_t \times \mathcal{E}_t)$, respectively, are isomorphic, where ℓ is any prime number different from p. In turn, the above isomorphism determines a Galois invariant class in a certain étale cohomology group of $\mathcal{X}_t \times \mathrm{Kum}(\mathcal{E}_t \times \mathcal{E}_t)$. The Tate conjecture predicts then the existence of an algebraic cycle on $\mathcal{X}_t \times \mathrm{Kum}(\mathcal{E}_t \times \mathcal{E}_t)$, with coefficients in $\mathbb{Q}_\ell$, that realizes this Galois invariant class. The explicit algebraic correspondence determining the predicted cycle was subsequently constructed by Van Geemen and Top [55]. This correspondence is obtained as the graph of a dominant rational map from $\mathcal{X}_t$ to $\mathrm{Kum}(\mathcal{E}_t \times \mathcal{E}_t)$, defined over a finite extension of $\mathbb{Q}(t)$.

The present article indicates a generalization of the above work. We extend the above results to certain two-parameter families of K3 surfaces of generic Picard rank 18. The K3 families in question include: Kummer surfaces associated with Jacobians of genus-two curves admitting an elliptic involution, Kummer surfaces associated with the product of two elliptic curves, and the two-parameter twisted Legendre pencil.

To begin with, let us briefly introduce some notation. Let V be a smooth projective variety over $\mathbb{Q}$, with separable closure $\overline{\mathbb{Q}}$ and let $\mathrm{G}_{\mathbb{Q}} = \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. Fix a prime number ℓ and consider the ℓ-adic cohomology groups, i.e., the cohomology groups of the base extension $V_{\overline{\mathbb{Q}}}$ of V to $\overline{\mathbb{Q}}$ with coefficients in the ℓ-adic integers $\mathbb{Z}_\ell$, and corresponding scalars then extended to the ℓ-adic numbers

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$\mathbb{Q}_\ell$. The étale cohomology theory determines then ℓ-adic cohomology groups for the algebraic variety V, which carry many of the same properties that the usual singular cohomology groups have. In addition, however, the étale cohomology groups are representations of $\mathrm{G}_\mathbb{Q}$. They also satisfy a form of Poincaré duality on non-singular projective varieties, and a Künneth formula holds. In particular, when V is a non-singular algebraic curve of genus g, $H^1_{\text{ét}}$ is a free $\mathbb{Z}_\ell$-module of rank $2g$, dual to the Tate module of the Jacobian variety of V. In turn, $H^1_{\text{ét}}$ is isomorphic to the singular cohomology (of the complex algebraic curve V) with $\mathbb{Z}_\ell$-coefficients. Getting back to the general V case, for $r > 0$, a codimension-r subvariety of V, defined over $\mathbb{Q}$, determines an element of the cohomology group $H^{2r}_{\text{ét}}(V_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell(r))$ fixed by $\mathrm{G}_\mathbb{Q}$. Here, $\mathbb{Q}_\ell(r)$ denotes the rth Tate twist. In this context, the Tate conjecture states that the subspace of Galois invariant classes is in fact the $\mathbb{Q}_\ell$-vector space of algebraic cycles on V of codimension r. For more details on this aspect, we refer the reader to the excellent survey in [54].

In this article, we consider several special families of K3 surfaces. First, the two-parameter twisted Legendre pencil $\mathcal{X}$ is, by definition, given by the family of K3 surfaces that are minimal resolutions of double covers of $\mathbb{P}^2 = \mathbb{P}(z_1, z_2, z_3)$ branched over a special union of six lines (and hence over a sextic curve). Specifically, consider the family of double sextics, given by

$$\mathcal{X}: \quad y^2 = z_1 z_2 (z_1 - z_2)(z_1 - z_3)(z_3 - A z_2)(z_3 - B z_2)\,, \tag{0.1}$$

where the parameters A, B satisfy $A \neq B$. The branch locus of a general member of this family is the union of six lines, one of which is coincident, at two distinct points, with two other pairs of lines. We assume then:

$$A = \left(\frac{1+\lambda_2}{1-\lambda_2}\right)^2, \qquad B = \left(\frac{1+\lambda_3}{1-\lambda_3}\right)^2 \tag{0.2}$$

with $\lambda_2, \lambda_3 \in \mathbb{Q}$ with $\lambda_2, \lambda_3 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $\lambda_2 \neq \lambda_3^{\pm 1}$. In the generic case the Picard rank of $\mathcal{X}$ is 18. Hence, one has an isomorphism of $\mathrm{G}_\mathbb{Q}$-representations:

$$H^2_{\text{ét}}\Big(\mathcal{X}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\Big) \;\cong\; \mathrm{T}^{(1)}_\ell \oplus \mathbb{Q}_\ell(-1)^{\oplus 18}\,, \tag{0.3}$$

where $\mathrm{T}^{(1)}_\ell$ is a certain four-dimensional ℓ-adic representation.

We also consider Kummer surfaces $\mathrm{Kum}(\mathbf{A})$, of generic Picard rank 18, associated with certain principally polarized abelian surfaces $\mathbf{A}$. If we denote the minus identity involution denoted by $-\mathbb{I}$, then $\mathrm{Kum}(\mathbf{A})$ is, by definition, the minimal resolution of the quotients $\mathbf{A}/\langle -\mathbb{I}\rangle$. We shall realize the quotients $\mathbf{A}/\langle -\mathbb{I}\rangle$ as double quadric or double sextic surfaces, as well as Jacobian elliptic surfaces. We shall refer to the minimal resolutions of the corresponding quadratic-twist surfaces as twisted Kummer surfaces $\mathrm{Kum}(\mathbf{A})^{(\varepsilon)}$ with twist factor ε.

One choice of abelian surface we shall consider is the product $\mathbf{A} = \mathcal{E}_1 \times \mathcal{E}_2$, where the elliptic curves $\mathcal{E}_l$, for $l = 1, 2$, are not mutually isogenous. Specifically, we consider the (families of) elliptic curves given by the equation

$$\mathcal{E}_l: \quad y_l^2 z_l = x_l\big(x_l - z_l\big)\big(x_l - \Lambda_l z_l\big)\,, \tag{0.4}$$

such that

$$\Lambda_1\Lambda_2 = \frac{(\lambda_2+\lambda_3)^2 - 4\lambda_2\lambda_3}{(1-\lambda_2)^2(1-\lambda_3)^2}\,, \qquad \Lambda_1+\Lambda_2 = -\frac{2(\lambda_2+\lambda_3)}{(1-\lambda_2)(1-\lambda_3)}\,. \tag{0.5}$$

The Picard rank of $\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)$ is generically 18. Hence, one has an isomorphism of $\mathrm{G}_{\mathbb{Q}}$-representations, namely

$$H^2_{\text{ét}}\Big(\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) \;\cong\; \mathrm{T}_\ell^{(2)} \oplus \mathbb{Q}_\ell(-1)^{\oplus 18}\,. \tag{0.6}$$

Here $\mathrm{T}_\ell^{(2)}$ is a four-dimensional ℓ-adic representation. At the level of $\mathrm{G}_{\mathbb{Q}}$-representations, one then has the isomorphism:

$$\mathrm{T}_\ell^{(2)} \cong H^1_{\text{ét}}\Big(\mathcal{E}_1,\mathbb{Q}_\ell\Big)\times H^1_{\text{ét}}\Big(\mathcal{E}_2,\mathbb{Q}_\ell\Big)\,. \tag{0.7}$$

Finally, we consider Kummer surfaces $\mathrm{Kum}(\mathbf{A})$ with abelian surface $\mathbf{A}$ realized as the Jacobian $\mathrm{Jac}\,(\mathcal{C}_0)$ of a smooth genus-two curve $\mathcal{C}_0$ admitting an elliptic involution. The curves $\mathcal{C}_0$ are given explicitly by

$$\mathcal{C}_0:\quad Y^2 = XZ\big(X-Z\big)\big(X-\lambda_2\lambda_3 Z\big)\big(X-\lambda_2 Z\big)\big(X-\lambda_3 Z\big)\,. \tag{0.8}$$

Due to the elliptic involution existing on the genus-two curve $\mathcal{C}_0$, the transcendental lattice of $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$ generically has rank four. As $\mathrm{G}_{\mathbb{Q}}$-representations, one has an isomorphism:

$$H^2_{\text{ét}}\Big(\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) \;\cong\; \mathrm{T}_\ell^{(3)} \oplus \mathbb{Q}_\ell(-1)^{\oplus 18}\,, \tag{0.9}$$

where $\mathrm{T}_\ell^{(3)}$ is a second four-dimensional ℓ-adic representation and ε is a certain quadratic-twist factor.

The main result of this article can be stated as follows:

Theorem 0.1. *Assume that $\lambda_2,\lambda_3\in\mathbb{Q}$ satisfy $\lambda_2,\lambda_3\notin\{0,1\}$, $\lambda_2\neq\lambda_3^{\pm1}$, and are generic otherwise. Further assume that Λ_1,Λ_2 satisfy*

$$\Lambda_1\Lambda_2 = \frac{(\lambda_2+\lambda_3)^2 - 4\lambda_2\lambda_3}{(1-\lambda_2)^2(1-\lambda_3)^2}\,, \qquad \Lambda_1+\Lambda_2 = -\frac{2(\lambda_2+\lambda_3)}{(1-\lambda_2)(1-\lambda_3)}\,, \tag{0.10}$$

and $\varepsilon = \lambda_2\lambda_3$. Then, there are explicit algebraic correspondences

$$\begin{aligned}
\Gamma^{(1,2)} &\subset \mathrm{Kum}\big(\mathcal{E}_1\times\mathcal{E}_2\big)\times\mathcal{X}\,,\\
\Gamma^{(2,3)} &\subset \mathcal{X}\times\mathrm{Kum}\big(\,\mathrm{Jac}\,\mathcal{C}_0\big)^{(\varepsilon)}\,,\\
\Gamma^{(3,1)} &\subset \mathrm{Kum}\big(\,\mathrm{Jac}\,\mathcal{C}_0\big)^{(\varepsilon)}\times\mathrm{Kum}\big(\mathcal{E}_1\times\mathcal{E}_2\big)\,,
\end{aligned} \tag{0.11}$$

defined over $\mathbb{Q}$, which induce ℓ-adic $\mathrm{G}_{\mathbb{Q}}$-representation isomorphisms:

$$\Big[\Gamma^{(i,j)}\Big]:\quad \mathrm{T}_\ell^{(i)} \;\xrightarrow{\;\cong\;}\; \mathrm{T}_\ell^{(j)}\,, \tag{0.12}$$

for $i,j\in\{1,2,3\}$ and $i\neq j$.

We note that the result above contains three special sub-families, associated with the non-generic situation when the elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic, up to a quadratic twist. The $\mathrm{G}_{\mathbb{Q}}$-representations involved are then three-dimensional. The details on this case are included in Theorem 3.29. One of these specializations is exactly the case considered in [55]. To our knowledge, the other two cases have not appeared in the literature. In addition, in Theorem 3.31 we prove analogous results for two other subfamilies of Picard rank 19 and 18 - one where the elliptic curves are two-isogenous to each other and one where one of the elliptic curves is constant with complex multiplication and j-invariant 12^3.

The article structure is as follows. In Section 1, we derive normal forms for Jacobians of genus-two curves admitting an elliptic involution, their elliptic quotient-curves and certain $(2,2)$-isogenous abelian surfaces. In Section 2, we construct the Kummer surfaces associated with the Jacobian varieties of genus-two curves admitting an elliptic involution and the product abelian surfaces of two non-isogeneous elliptic curves. These Kummer surfaces are explicitly described as quartic hypersurfaces, double quadric surfaces, double sextic surfaces, and Jacobian elliptic surfaces. As a side observation, we show that Legendre's glueing construction, i.e., the construction of a genus-two curve with involution from its elliptic-curve quotients over a small field of definition, has an exact analogue for quartic Kummer surfaces. This is discussed in Theorem 2.31. Using the results from Section 1, we are then able to construct explicit isogenies between the various Kummer surfaces involved; see Theorem 2.38. In Section 3 we prove that the two-parameter twisted Legendre pencil fits into a Kummer sandwich of dominant rational maps, directly relating it to the Kummer surfaces from Section 2. This geometrical aspect is presented in Theorem 3.18. The rational maps involved are defined only over a finite field extension of the field of definition for the given varieties. The varieties involved have rational models, but one has to choose the right quadratic-twists in order to obtain non-trivial rational correspondences between them. We achieve this goal by arranging for the pullbacks between natural generators of the corresponding $H^{2,0}$-parts in cohomology to be rational. Theorem 3.28 is the main result of this article and proves that the aforementioned four-dimensional ℓ-adic representations – as obtained from the corresponding transcendental lattices of families of K3 surfaces of Picard rank 18 – are isomorphic. Finally, Theorems 3.29 and 3.31 discuss subfamilies of generic Picard rank 19, one of which recovers the main result from [55].

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1. Abelian surfaces admitting an elliptic involution

Let $\mathcal{C}$ be an irreducible, smooth, projective curve of genus two, defined over the complex numbers $\mathbb{C}$. Let $\mathcal{M}$, be the coarse moduli space of smooth curves of genus two. We denote by $[\mathcal{C}]$ the isomorphism class of $\mathcal{C}$, i.e., the corresponding point in $\mathcal{M}$. For a smooth genus-two curve $\mathcal{C}$ given as a sextic $Y^2 = f_6(X,Z)$ in the weighted complex projective space $\mathbb{WP}_{(1,3,1)} = \mathbb{P}(X,Y,Z)$, we send three roots $(\lambda_4,\lambda_5,\lambda_6)$ to $(0,1,\infty)$ to obtain an isomorphic curve in Rosenhain normal form, given by

$$\mathcal{C}: \quad Y^2 = X\,Z\,\big(X-Z\big)\big(X-\lambda_1 Z\big)\big(X-\lambda_2 Z\big)\big(X-\lambda_3 Z\big)\,. \tag{1.1}$$

We denote the hyperelliptic involution on $\mathcal{C}$ by $\imath_{\mathcal{C}}$. The Weierstrass points of $\mathcal{C}$ are the fixed points of $\imath_{\mathcal{C}}$, and they are $P_i : [X:Y:Z] = [\lambda_i : 0 : 1]$ for $i = 1,\ldots,5$, and $P_6 : [X:Y:Z] = [1:0:0]$. The tuple $(\lambda_1,\lambda_2,\lambda_3)$ where the roots λ_i are all distinct and different from $\{0,1,\infty\}$ determines a point in the moduli space of genus-two curves with six marked Weierstrass points, denoted by $\mathcal{M}(2)$. The Jacobian variety $\mathrm{Jac}\,(\mathcal{C})$ is the moduli space of degree-zero line bundles on $\mathcal{C}$. It is the connected component of the identity in the Picard group of $\mathcal{C}$, hence an abelian surface. We refer to a genus-two curve as *generic* if it is smooth and its Jacobian $\mathrm{Jac}\,(\mathcal{C})$ has no extra automorphisms.

The Siegel three-fold is the quasi-projective variety of dimension three, obtained from the Siegel upper half-plane $\mathbb{H}_2$ of degree two[1] divided by the action of the modular transformations $\Gamma_2 := \mathrm{Sp}_4(\mathbb{Z})$, i.e.,

$$\mathcal{A}_2 = \mathbb{H}_2/\Gamma_2\,. \tag{1.2}$$

Each $\tau = \left(\begin{smallmatrix}\tau_{11} & \tau_{12}\\ \tau_{12} & \tau_{22}\end{smallmatrix}\right) \in \mathbb{H}_2$ determines a complex abelian surface $\mathbf{A} = \mathbb{C}^2/\Lambda$ obtained from the lattice $\Lambda = \langle \mathbb{Z}^2 \oplus \tau\,\mathbb{Z}^2\rangle$ with the period matrix $(\mathbb{I}_2,\tau) \in \mathrm{Mat}(2,4;\mathbb{C})$. We consider two abelian surfaces $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ isomorphic if and only if there is an element $M \in \Gamma_2$ such that $\widetilde{\tau} = M(\tau)$. The *canonical principal polarization* of $\mathbf{A}$ is given by the positive definite hermitian forms $\mathbf{h}$ on $\mathbb{C}^2$ such that $\alpha = \mathrm{Im}\,\mathbf{h}(\Lambda,\Lambda) \subset \mathbb{Z}$ where α is the Riemann form $\alpha(x_1+x_2\tau, y_1+y_2\tau) = x_1^t\cdot y_2 - y_1^t\cdot x_2$ on $\mathbb{Z}^2\oplus\tau\,\mathbb{Z}^2$. Such a hermitian form determines a line bundle $\mathcal{L}\to\mathbf{A}$ in the Néron-Severi lattice $\mathrm{NS}(\mathbf{A})$. A general fact from linear algebra asserts that one can always choose a basis of Λ such that α is given by the matrix $\left(\begin{smallmatrix}0 & D\\ -D & 0\end{smallmatrix}\right)$ with $D = \left(\begin{smallmatrix}d_1 & 0\\ 0 & d_2\end{smallmatrix}\right)$ where $d_1,d_2 \in \mathbb{N}$, $d_1,d_2 \geq 0$, and d_1 divides d_2. For a principal polarization one has $(d_1,d_2) = (1,1)$. Since transformations in Γ_2 preserve the Riemann form α, it follows that the Siegel three-fold $\mathcal{A}_2$ is also the set of isomorphism classes of principally polarized abelian surfaces, i.e., abelian surfaces with a polarization of type $(1,1)$. We also define the subgroup $\Gamma_2(2) = \{M \in \Gamma_2 |\, M \equiv \mathbb{I} \mod 2\}$ such that $\Gamma_2/\Gamma_2(2) \cong S_6$ where S_6 is the permutation group of six elements representing the permutations of the Rosenhain roots $(\lambda_1,\lambda_2,\lambda_3,0,1,\infty)$. Then, $\mathcal{A}_2(2)$ is the three-dimensional moduli space of principally polarized abelian surfaces with marked level-two structure.

[1] By definition $\mathbb{H}_2$ is the set of two-by-two symmetric matrices over $\mathbb{C}$ whose imaginary part is positive definite

For a principally polarized abelian variety $\mathbf{A}$ the line bundle $\mathcal{L}$ defining its principal polarization is ample and satisfies $h^0(\mathcal{L}) = 1$. There exists an effective divisor Θ such that $\mathcal{L} = \mathcal{O}_{\mathbf{A}}(\Theta)$, uniquely defined only up to translations. The divisor $\Theta \in \mathrm{NS}(\mathbf{A})$ is called a *theta divisor* associated with the polarization. It is known that the abelian surface $\mathbf{A}$ is not the product of two elliptic curves if and only if Θ is an irreducible divisor. Torelli's theorem states that the map sending a curve $\mathcal{C}$ to its Jacobian $\mathrm{Jac}(\mathcal{C})$ is injective and defines a birational map $\mathcal{M} \dashrightarrow \mathcal{A}_2$ which restricts to a rational map $\mathcal{M}(2) \dashrightarrow \mathcal{A}_2(2)$ between the coarse moduli spaces of smooth curves of genus two with marked Weierstrass points and principally polarized abelian surfaces with marked level-two structures, respectively. Conversely, if the period matrix τ of a principally polarized abelian variety $\mathbf{A}$ is not equivalent to a point with $\tau_{12} = 0$ relative to Γ_2, then a Θ-divisor on $\mathbf{A}$ is given by a non-singular curve $\mathcal{C}$ of genus-two, i.e., $\Theta = [\mathcal{C}]$ such that $\mathbf{A} = \mathrm{Jac}(\mathcal{C})$. Thus, a principally polarized abelian surface is either the Jacobian of a smooth curve of genus two with a theta divisor or the product of two complex elliptic curves, with the product polarization.

1.1. **Isogenies of Jacobian surfaces.** Translations of the Jacobian $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$ by order-two points in $\mathbf{A}[2]$ are isomorphisms of the Jacobian and map the set of two-torsion points to itself. Moreover, a *Göpel group* is a two-dimensional subspace $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ of $\mathbf{A}[2]$ such that $\mathbf{A}' = \mathbf{A}/G$ is again a principally polarized abelian surface [45, Sec. 23]. The corresponding isogeny $\Psi : \mathbf{A} \to \mathbf{A}'$ between principally polarized abelian surfaces has as its kernel $G \leqslant \mathbf{A}[2]$ and is called a $(2,2)$-*isogeny*.

For the Jacobian of a genus-two curve, every nontrivial two-torsion point is a difference of Weierstrass points $P_i \in \mathcal{C}$ for $1 \le i \le 6$. In fact, the sixteen order-two points of $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$ are obtained using the embedding of the curve into the connected component of the identity in the Picard group, i.e., $\mathcal{C} \hookrightarrow \mathrm{Jac}\,(\mathcal{C}) \cong \mathrm{Pic}^0(\mathcal{C})$. We obtain the 15 elements $P_{ij} = [P_i + P_j - 2\,P_6] \in \mathbf{A}[2]$ with $1 \le i < j \le 6$ and set $P_0 = P_{66} = [0]$. For $\{i,j,k,l,m,n\} = \{1,\dots,6\}$, the group law on $\mathbf{A}[2]$ is given by the relations

$$P_0 + P_{ij} = P_{ij}\,, \quad P_{ij} + P_{ij} = P_0\,, \quad P_{ij} + P_{kl} = P_{mn}, \quad P_{ij} + P_{jk} = P_{ik}\,. \tag{1.3}$$

The space $\mathbf{A}[2]$ of two-torsion points admits a symplectic bilinear form, called the *Weil pairing* such that the two-dimensional, maximal isotropic subspace of $\mathbf{A}[2]$ with respect to the Weil pairing are the Göpel groups. It is easy to check that there are exactly 15 inequivalent Göpel groups, and they are of the form

$$\Big\{P_0, P_{ij}, P_{kl}, P_{mn}\Big\} \tag{1.4}$$

such that $\{i,j,k,l,m,n\} = \{1,\dots,6\}$.

For a generic genus-two curve $\mathcal{C}$, one may ask whether the $(2,2)$-isogenous abelian surface $\mathbf{A}' = \mathbf{A}/G$ satisfies $\mathbf{A}' = \mathrm{Jac}\,(\mathcal{C}')$ for some smooth curve $\mathcal{C}'$ of genus two. The geometric moduli relationship between the two curves of genus two was found by Richelot [49]; see also [8]. If we choose for $\mathcal{C}$ a sextic equation $Y^2 = f_6(X,Z)$, then from any factorization $f_6 = A \cdot B \cdot C$ into three degree-two polynomials A, B, C one obtains a new genus-two curve $\mathcal{C}'$, given by

$$\mathcal{C}' : \qquad \Delta_{ABC} \cdot Y^2 = [A,B]\,[A,C]\,[B,C]\,, \tag{1.5}$$

where we have set $[A,B]\cdot Z = B\,\partial_X A - A\,\partial_X B$ with ∂_X denoting the derivative with respect to X and Δ_{ABC} is the determinant of (A,B,C) with respect to the basis X^2, XZ, Z^2.

1.2. **Humbert surfaces.** Sets of abelian surfaces with the same endomorphism ring form sub-varieties within $\mathcal{A}_2$. The endomorphism ring of principally polarized abelian surface tensored with $\mathbb{Q}$ is either a quartic CM field, an indefinite quaternion algebra, a real quadratic field or in the generic case $\mathbb{Q}$. Irreducible components of the corresponding subsets in $\mathcal{A}_2$ have dimensions $0,1,2$ and are known as CM points, Shimura curves, and Humbert surfaces, respectively. The Humbert surface $\mathcal{H}_\Delta$ with invariant Δ is the space of principally polarized abelian surfaces admitting a symmetric endomorphism with discriminant Δ. It turns out that Δ always is a positive integer satisfying $\Delta \equiv 0,1 \mod 4$ and uniquely determined $\mathcal{H}_\Delta$. In fact, $\mathcal{H}_\Delta$ is the image inside $\mathcal{A}_2$ under the projection of the rational divisor associated with the equation

$$a\,\tau_{11} + b\,\tau_{12} + c\,\tau_{22} + d\,(\tau_{12}^2 - \tau_{11}\,\tau_{22}) + e = 0\,, \tag{1.6}$$

with integers a,b,c,d,e satisfying $\Delta = b^2 - 4\,a\,c - 4\,d\,e$ and $\tau = \left(\begin{smallmatrix}\tau_{11} & \tau_{12}\\ \tau_{12} & \tau_{22}\end{smallmatrix}\right) \in \mathbb{H}_2$. The following was proven by Birkenhake and Lange in [6]:

Theorem 1.1. *For $\delta \in \mathbb{N}$ the Humbert surface $\mathcal{H}_{\delta^2}$ is the locus of principally polarized abelian surfaces $(\mathbf{A},\mathcal{L}) \in \mathcal{A}_2$ admitting an isogeny of degree δ^2, given by*

$$\Phi:\quad \Big(\mathcal{E}_1\times\mathcal{E}_2,\ \mathcal{O}_{\mathcal{E}_1}(\delta)\boxtimes\mathcal{O}_{\mathcal{E}_2}(\delta)\Big)\ \longrightarrow\ \Big(\mathbf{A},\mathcal{L}\Big), \tag{1.7}$$

where $\mathcal{O}_{\mathcal{E}_l}(\delta)$ is a line bundle of degree δ on an elliptic curve $\mathcal{E}_l$ for $l = 1,2$.

In particular, inside of $\mathcal{A}_2$ sit the Humbert surfaces $\mathcal{H}_1$ and $\mathcal{H}_4$ that are defined as the images under the projection of the rational divisor associated to $\tau_{12} = 0$ and $\tau_{11} - \tau_{22} = 0$, respectively. In fact, the singular locus of $\mathcal{A}_2$ has $\mathcal{H}_1$ and $\mathcal{H}_4$ as its two connected components. As analytic spaces, $\mathcal{H}_1$ and $\mathcal{H}_4$ are each isomorphic to the Hilbert modular surface

$$\Big((\mathrm{SL}_2(\mathbb{Z})\times\mathrm{SL}_2(\mathbb{Z}))\rtimes\mathbb{Z}_2\Big)\backslash\Big(\mathbb{H}\times\mathbb{H}\Big). \tag{1.8}$$

For a detailed introduction to Siegel modular forms relative to Γ_2, Humbert surfaces, and the Satake compactification of the Siegel modular threefold we refer to Freitag's book [18]. Igusa proved [30, 31] that the ring of modular forms is generated by the Siegel modular forms ψ_4, ψ_6, χ_{10}, χ_{12} and by one more cusp form χ_{35} of odd weight

35 whose square is the following polynomial [30, p. 849] in the even generators

$$
\begin{aligned}
\chi_{35}^2 = \frac{1}{2^{12}\,3^9}\,\chi_{10}\Big(&2^{24}\,3^{15}\,\chi_{12}^5 - 2^{13}\,3^9\,\psi_4^3\,\chi_{12}^4 - 2^{13}\,3^9\,\psi_6^2\,\chi_{12}^4 + 3^3\,\psi_4^6\,\chi_{12}^3 \\
&- 2\cdot 3^3\,\psi_4^3\,\psi_6^2\,\chi_{12}^3 - 2^{14}\,3^8\,\psi_4^2\,\psi_6\,\chi_{10}\,\chi_{12}^3 - 2^{23}\,3^{12}\,5^2\,\psi_4\,\chi_{10}^2\,\chi_{12}^3 + 3^3\,\psi_6^4\,\chi_{12}^3 \\
&+ 2^{11}\,3^6\,37\,\psi_4^4\,\chi_{10}^2\,\chi_{12}^2 + 2^{11}\,3^6\,5\cdot 7\,\psi_4\,\psi_6^2\,\chi_{10}^2\,\chi_{12}^2 - 2^{23}\,3^9\,5^3\,\psi_6\,\chi_{10}^3\,\chi_{12}^2 \\
&- 3^2\,\psi_4^7\,\chi_{10}^2\,\chi_{12} + 2\cdot 3^2\,\psi_4^4\,\psi_6^2\,\chi_{10}^2\,\chi_{12} + 2^{11}\,3^5\,5\cdot 19\,\psi_4^3\,\psi_6\,\chi_{10}^3\,\chi_{12} \\
&+ 2^{20}\,3^8\,5^3\,11\,\psi_4^2\,\chi_{10}^4\,\chi_{12} - 3^2\,\psi_4\,\psi_6^4\,\chi_{10}^2\,\chi_{12} + 2^{11}\,3^5\,5^2\,\psi_6^3\,\chi_{10}^3\,\chi_{12} - 2\,\psi_4^6\,\psi_6\,\chi_{10}^3 \\
&- 2^{12}\,3^4\,\psi_4^5\,\chi_{10}^4 + 2^2\,\psi_4^3\,\psi_6^3\,\chi_{10}^3 + 2^{12}\,3^4\,5^2\,\psi_4^2\,\psi_6^2\,\chi_{10}^4 + 2^{21}\,3^7\,5^4\,\psi_4\,\psi_6\,\chi_{10}^5 \\
&- 2\,\psi_6^5\,\chi_{10}^3 + 2^{32}\,3^9\,5^5\,\chi_{10}^6\Big)\,.
\end{aligned}
\tag{1.9}
$$

Hence, the expression $Q := 2^{12}\,3^9\,\chi_{35}^2/\chi_{10}$ is a polynomial of degree 60 in the even generators. Igusa also proved that each Siegel modular form (with trivial character) of odd weight is divisible by the form χ_{35}. The following fact is well-known [20]:

Proposition 1.2. *The vanishing divisor of the cusp form χ_{10} is the Humbert surface $\mathcal{H}_1$, i.e., a period point τ is equivalent to a point with $\tau_{12} = 0$ relative to Γ_2 if and only if $\chi_{10}(\tau) = 0$. The vanishing divisor of Q is the Humbert surface $\mathcal{H}_4$, i.e., a period point τ is equivalent to a point with $\tau_{11} = \tau_{22}$ relative to Γ_2 if and only if $Q = 0$.*

It is known that one has $\chi_{10}(\tau) = 0$ if and only if the principally polarized abelian surface $\mathbf{A}$ is a product of two elliptic curves $\mathbf{A} = \mathcal{E}_{\tau_{11}} \times \mathcal{E}_{\tau_{22}}$ with the transcendental lattice $\mathrm{T}_{\mathbf{A}} = H \oplus H$; see [36]. Here, H denotes the lattice $\mathbb{Z}^2$ with quadratic form $q(\vec{v}) = 2v_1v_2$. Moreover, for $Q(\tau) = 0$ the transcendental lattice of the corresponding abelian surface $\mathbf{A}$ is given by $\mathrm{T}_{\mathbf{A}} = H \oplus \langle 2\rangle \oplus \langle -2\rangle$. Here, $\langle m\rangle$ denotes the rank-one lattice $\mathbb{Z}v$ with $q(v) = m$. Similarly, for $\tau \in \mathcal{H}_{16}$, the transcendental lattice of $\mathbf{A}$ is given by $\mathrm{T}_{\mathbf{A}} = H \oplus \langle 4\rangle \oplus \langle -4\rangle$.

1.3. **Components of the Humbert surface $\mathcal{H}_4$.** The ring of invariants of binary sextics is generated by the so-called Igusa-Clebsch invariants (I_2, I_4, I_6, I_{10}) which were studied in [43, p. 319] and also in [29, p. 17]. Igusa [30, p. 848] proved that the relations between the Igusa invariants of a binary sextic $Y^2 = f_6(X, Z)$ defining a smooth genus-two curve $\mathcal{C}$ and the even Siegel modular forms for the associated principally polarized abelian surface $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$ with period matrix τ are as follows:

$$
\begin{aligned}
I_2(f_6) &= -2^3\cdot 3\,\frac{\chi_{12}(\tau)}{\chi_{10}(\tau)}\,, \\
I_4(f_6) &= \;\; 2^2\,\psi_4(\tau)\,, \\
I_6(f_6) &= -\frac{2^3}{3}\,\psi_6(\tau) - 2^5\,\frac{\psi_4(\tau)\,\chi_{12}(\tau)}{\chi_{10}(\tau)}\,, \\
I_{10}(f_6) &= -2^{14}\,\chi_{10}(\tau) \neq 0\,.
\end{aligned}
\tag{1.10}
$$

Here, the Igusa invariant I_{10} is the discriminant of the sextic $f_6(X, Z)$ and we are using the same normalization as in [36, 40].

1.3.1. *Sextics with extra involution.* Bolza [7] described the possible automorphism groups of genus-two curves defined by sextics. In particular, he proved that a sextic curve $Y^2 = f_6(X,Z)$ defining a genus-two curve $\mathcal{C}_0$ admits an elliptic involution if and only if $Q(\tau) = 0$. Moreover, he proved that one can always represent this extra involution as $[X:Y:Z] \mapsto [-X:Y:Z]$. It follows that the smooth sextic curve $\mathcal{C}_0$ with extra involution can be brought into the normal form given by

$$\mathcal{C}_0: \quad Y^2 = X^6 + s_1 X^4 Z^2 + s_2 X^2 Z^4 + Z^6 . \tag{1.11}$$

One uses Equations (1.10) and Equation (1.9) to check that $Q = 2^{12}\,3^9\,\chi_{35}^2/\chi_{10}$ always vanishes for the genus-two curve $\mathcal{C}_0$.

Given the elliptic involution $[X:Y:Z] \mapsto [-X:Y:Z]$ on $\mathcal{C}_0$, its composition with the hyperelliptic involution defines a second elliptic involution. Thus, the elliptic involutions on $\mathcal{C}_0$ come naturally in pairs. The two involutions define two elliptic subfields of degree two for the function field of $\mathcal{C}_0$. We introduce the elliptic curves $\mathcal{E}_l$ in $\mathbb{P}^2 = \mathbb{P}(x_l, y_l, z_l)$ for $l = 1, 2$, given by

$$\mathcal{E}_1: \; y_1^2 z_1 = x_1^3 + s_2 x_1^2 z_1 + s_1 x_1 z_1^2 + z_1^3 , \qquad \mathcal{E}_2: \; y_2^2 z_2 = x_2^3 + s_1 x_2^2 z_2 + s_2 x_2 z_2^2 + z_2^3 , \tag{1.12}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ have the j-invariants $j_1 = j(\mathcal{E}_1)$ and $j_2 = j(\mathcal{E}_2)$ with

$$j_1 = \frac{2^8\left(3s_1 - s_2^2\right)^3}{4(s_1^3 + s_2^3) - (s_1 s_2)^2 - 18 s_1 s_2 + 27} , \qquad j_2 = \frac{2^8\left(3s_2 - s_1^2\right)^3}{4(s_1^3 + s_2^3) - (s_1 s_2)^2 - 18 s_1 s_2 + 27} , \tag{1.13}$$

respectively. Here, we use the standard normalization of the j-invariant where the square torus with the complex structure i satisfies $j = 1728 = 12^3$. The degree-two quotient maps $p_l : \mathcal{C}_0 \to \mathcal{E}_l$ associated with the involutions are given by

$$\pi_{\mathcal{E}_1}: \quad \mathcal{C}_0 \longrightarrow \mathcal{E}_1 , \qquad \Big[X:Y:Z\Big] \mapsto \Big[x_1:y_1:z_1\Big] = \Big[XZ^2 : Y : X^3\Big], \tag{1.14}$$

and

$$\pi_{\mathcal{E}_2}: \quad \mathcal{C}_0 \longrightarrow \mathcal{E}_2 , \qquad \Big[X:Y:Z\Big] \mapsto \Big[x_2:y_2:z_2\Big] = \Big[X^2Z : Y : Z^3\Big], \tag{1.15}$$

respectively, for $XZ \neq 0$.

1.3.2. *Pringsheim decomposition.* We can also start with a smooth genus-two curve $\mathcal{C}$ in Rosenhain form given by Equation (1.1). We use the aforementioned relations between the Igusa invariants and the Siegel modular forms to expand the generators of the ring of modular forms in terms of the Rosenhain roots. We obtain

$$\begin{aligned}
-\frac{2^{22}\chi_{35}^2(\tau)}{\chi_{10}^4(\tau)} &= \left(\lambda_1 - \lambda_2\lambda_3\right)^2\left(\lambda_2 - \lambda_1\lambda_3\right)^2\left(\lambda_3 - \lambda_1\lambda_2\right)^2 \\
&\times \left(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_2\lambda_3\right)^2\left(-\lambda_1 + \lambda_2 - \lambda_3 + \lambda_1\lambda_3\right)^2\left(-\lambda_1 - \lambda_2 + \lambda_3 + \lambda_1\lambda_2\right)^2 \\
&\times \left(\lambda_1\lambda_2 + \lambda_1\lambda_3 - \lambda_2\lambda_3 - \lambda_1\right)^2\left(\lambda_1\lambda_2 + \lambda_2\lambda_3 - \lambda_1\lambda_3 - \lambda_2\right)^2\left(\lambda_1\lambda_3 + \lambda_2\lambda_3 - \lambda_1\lambda_2 - \lambda_3\right)^2 \\
&\times \begin{cases} \left(\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_1 + \lambda_3\right)^2\left(\lambda_1\lambda_3 - \lambda_2\lambda_3 - \lambda_1 + \lambda_3\right)^2\left(\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_1 + \lambda_2\right)^2 \\ \left(\lambda_1\lambda_2 - \lambda_1\lambda_3 + \lambda_1 - \lambda_2\right)^2\left(\lambda_1\lambda_3 - \lambda_2\lambda_3 + \lambda_2 - \lambda_3\right)^2\left(\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_2 + \lambda_3\right)^2 . \end{cases}
\end{aligned} \tag{1.16}$$

The vanishing divisor of Equation (1.16) defines fifteen components in $\mathbb{H}_2/\Gamma_2(2)$ of discriminant $\Delta = 4$. Notice that each line in Equation (1.16) is arranged to be invariant under permutations of the three roots $\lambda_1, \lambda_2, \lambda_3$. Pringsheim proved the following statement[2] in [48]:

Proposition 1.3 (Pringsheim)**.** *There are exactly 15 components in $\mathbb{H}_2/\Gamma_2(2)$ covering $\mathcal{H}_4$, given in Table 1. Each of the component is equivalent to $\tau_{11} = \tau_{22}$ relative to the modular group Γ_2. Moreover, there is a transposition of the six roots $(\lambda_1, \lambda_2, \lambda_3, 0, 1, \infty)$ in $\Gamma_2/\Gamma_2(2) \cong S_6$ that permutes each pair of components.*

Remark 1.4. *In Table 1 for each entry the equation of a rational divisor in $\mathbb{H}_2$ is given whose projection to $\mathbb{H}_2/\Gamma_2(2)$ realizes a component in Proposition 1.3. The corresponding vanishing divisor in terms of the Rosenhain roots is given as well. The group names (I, II, III, IV) were introduced by Pringsheim, and they are colored to match the corresponding terms in Equation (1.16).*

I	
$2\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0$	$\lambda_1 = \lambda_2\lambda_3$
II	
$\tau_{11} + 2\tau_{12} = 0$	$\lambda_1 - \lambda_2 = \lambda_3(1 - \lambda_2)$
$\tau_{11} + 2\tau_{12} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_1(1 - \lambda_3) = \lambda_2(\lambda_1 - \lambda_3)$
III	
$2\tau_{12} - \tau_{22} = 0$	$\lambda_3 = \lambda_1\lambda_2$
$2\tau_{12} - \tau_{22} + (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_2 = \lambda_1\lambda_3$
$\tau_{11} - \tau_{22} = 0$	$\lambda_1 - \lambda_2 = \lambda_2(\lambda_1 - \lambda_3)$
$\tau_{11} - \tau_{22} + (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_1 - \lambda_3 = \lambda_3(\lambda_1 - \lambda_2)$
IV	
$2\tau_{12} = 1$	$\lambda_2 - \lambda_3 = -\lambda_1(1 - \lambda_2)$
$2\tau_{12} + \tau_{22} = 1$	$\lambda_3(1 - \lambda_2) = -\lambda_1(\lambda_2 - \lambda_3)$
$\tau_{11} + 2\tau_{12} = 1$	$\lambda_2 - \lambda_3 = \lambda_1(1 - \lambda_3)$
$\tau_{11} + 2\tau_{12} + \tau_{22} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1(\lambda_2 - \lambda_3) = \lambda_2(1 - \lambda_3)$
$4\tau_{12} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_2 - \lambda_3 = \lambda_2(\lambda_1 - \lambda_3)$
$4\tau_{12} + \tau_{22} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1 - \lambda_3 = \lambda_1(\lambda_2 - \lambda_3)$
$\tau_{11} + 4\tau_{12} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_2 - \lambda_3 = -\lambda_3(\lambda_1 - \lambda_2)$
$\tau_{11} + 4\tau_{12} + \tau_{22} + 2(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1 - \lambda_2 = -\lambda_1(\lambda_2 - \lambda_3)$

TABLE 1. Pringsheim components of $\mathcal{H}_4$

1.3.3. *Normal forms with level-two structure.* We will use component (I) in Proposition 1.3 to construct a smooth genus-two curve in Rosenhain normal form admitting an elliptic involution. We consider the smooth genus-two curve $\mathcal{C}_0$, given by

$$\mathcal{C}_0: \quad Y^2 = XZ\left(X - Z\right)\left(X - \lambda_2\lambda_3 Z\right)\left(X - \lambda_2 Z\right)\left(X - \lambda_3 Z\right), \tag{1.17}$$

[2]We corrected two minor typos in the statement of the main theorem.

with a discriminant given by

$$\lambda_2^6\lambda_3^6(\lambda_2-1)^4(\lambda_3-1)^4(\lambda_2\lambda_3-1)^4(\lambda_2-\lambda_3)^2\,. \tag{1.18}$$

Let us denote by $\Lambda_l \in \mathbb{P}^1\backslash\{0,1,\infty\}$ the modular parameter for the elliptic curves $\mathcal{E}_l$ in $\mathbb{P}^2 = \mathbb{P}(x_l, y_l, z_l)$ for $l = 1,2$ defined by the *Legendre normal form*

$$\mathcal{E}_l: \quad y_l^2 z_l = x_l\Big(x_l - z_l\Big)\Big(x_l - \Lambda_l z_l\Big)\,, \tag{1.19}$$

with the hyperelliptic involution given by $\imath_{\mathcal{E}_l} : [x_l : y_l : z_l] \mapsto [x_l : -y_l : z_l]$. We have the following:

Lemma 1.5. *The function field of the smooth genus-two curve $\mathcal{C}_0$ contains the subfields given by the function fields of the elliptic curves $\mathcal{E}_l$ for $l = 1,2$ if*

$$\Lambda_1\Lambda_2 = \frac{(\lambda_2+\lambda_3)^2 - 4\lambda_2\lambda_3}{(1-\lambda_2)^2(1-\lambda_3)^2}\,, \qquad \Lambda_1 + \Lambda_2 = -\frac{2(\lambda_2+\lambda_3)}{(1-\lambda_2)(1-\lambda_3)}\,. \tag{1.20}$$

Proof. The genus-two curve in Bolza normal form in Equation (1.11) is isomorphic to Rosenhain curve in Equation (1.17) if we set

$$\lambda_2 = \frac{(\mu^2\nu+1)(\mu-\nu)}{(\mu^2\nu-1)(\mu+\nu)}\,, \qquad \lambda_3 = \frac{(\mu^2\nu-1)(\mu-\nu)}{(\mu^2\nu+1)(\mu+\nu)}\,, \tag{1.21}$$

and

$$s_1 = \mu^2 + \nu^2 + \frac{1}{\mu^2\nu^2}\,, \qquad s_2 = \frac{1}{\mu^2} + \frac{1}{\nu^2} + \mu^2\nu^2\,, \tag{1.22}$$

for $\mu, \nu \in \mathbb{C}^\times$ such that $(\mu^2-\nu^2)(\mu^4\nu^2-1) \neq 0$. This statement is proved by computing the Igusa-Clebsch invariants, using the normalization in [36, 40]. Denoting the Igusa-Clebsch invariants of the genus-two curve in Equation (1.17) and Equation (1.11) by $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}_{(2,4,6,10)}$ and $[I_2' : I_4' : I_6' : I_{10}']$, respectively, one checks that

$$[I_2 : I_4 : I_6 : I_{10}] = [s^2 I_2' \ : \ s^4 I_4' \ : \ s^6 I_6' \ : \ s^{10} I_{10}'] = [I_2' : I_4' : I_6' : I_{10}'] \tag{1.23}$$

with $s \in \mathbb{C}^\times$. The remainder of the proof follows from computing the j-invariants for the curves in Equation (1.19) and comparing them with Equations (1.13). The solution of Equations (1.20), up to interchanging $\Lambda_1 \leftrightarrow \Lambda_2$, is then given by

$$\Lambda_1 = \frac{-\mu^2+\nu^2}{\mu^4\nu^4-\mu^2}\,, \qquad \Lambda_2 = \frac{-\mu^4\nu^2+\mu^2\nu^4}{\mu^2\nu^4-1}\,, \tag{1.24}$$

and $j(\mathcal{E}_1) = j_1$ and $j(\mathcal{E}_2) = j_2$ in Equations (1.13). □

We have the following:

Proposition 1.6. *Assume that $\lambda_2, \lambda_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2 \neq \lambda_3^{\pm 1}$, and the moduli of the curves $\mathcal{C}_0$ and $\mathcal{E}_l$ in Equations (1.17) and (1.19) satisfy Equations (1.20). Quotient maps $\pi_{\mathcal{E}_l} : \mathcal{C}_0 \to \mathcal{E}_l$ for $l = 1,2$ are given by*

$$\pi_{\mathcal{E}_l}: \quad \mathcal{C}_0 \ \longrightarrow \ \mathcal{E}_l\,, \qquad \Big[X : Y : Z\Big] \mapsto \Big[x_l : y_l : z_l\Big] \ \textit{for } l = 1,2, \tag{1.25}$$

with

$$\Big[x_l : y_l : z_l\Big] \ = \ \Big[r\big(X-\lambda_2 Z\big)\big(X-\lambda_3 Z\big)XZ : \ \big(X-(-1)^l qZ\big)Y : \ r^3X^2Z^2\Big], \tag{1.26}$$

for $XZ \neq 0$*, and* $[x_l : y_l : z_l] = [1 : 0 : 0]$ *otherwise. Here,* q, r *are square roots of* $q^2 = \lambda_2\lambda_3$ *and* $r^2 = (1-\lambda_2)(1-\lambda_3)$*, respectively. The elliptic involutions* $\jmath_l$ *on* $\mathcal{C}_0$*, given by*

$$\jmath_l : \quad \Big[X : Y : Z\Big] \mapsto \Big[\lambda_2\lambda_3 Z : \ (-1)^{l+1}\lambda_2\lambda_3 q Y : \ X\Big], \tag{1.27}$$

satisfy $\pi_{\mathcal{E}_l} \circ \jmath_l = \pi_{\mathcal{E}_l}$ *for* $l = 1, 2$.

Proof. The genus-two curve $\mathcal{C}_0$ in Equation (1.17) is assumed to be smooth. Thus, one must have $\lambda_2, \lambda_3 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ and $\lambda_2 \neq \lambda_3^{\pm 1}$. The remainder of the proof follows by explicit computation. We note that a choice of square root for r is changed by composing $\pi_{\mathcal{E}_l}$ with the elliptic involution on $\mathcal{E}_l$. The choice of square root for q is changed by interchanging $\pi_{\mathcal{E}_1} \leftrightarrow \pi_{\mathcal{E}_2}$. □

In the context of hyperelliptic period integrals, the construction of the rational double covers is known as the (classical) *Jacobi reduction* of a genus-two curve with elliptic involution [42]. We note that we are using component (I) in Proposition 1.3 to construct smooth normal forms for the genus-two curve in Rosenhain normal form and its elliptic subfields representing a specific divisor of $\mathcal{H}_4 \subset \mathcal{A}_2/\Gamma_2(2)$:

Remark 1.7. *Using a covering space of component (I) in Proposition 1.3, given by the set of tuples* (k_1, k_2) *with* $\lambda_l = k_l^2$ *for* $l = 2, 3$ *and* $\lambda_1 = (k_2 k_3)^2$*, we obtain for the modular parameters of the elliptic-curve quotients* $\mathcal{E}_l$ *in Lemma 1.5 the following algebraic solutions of Equation (1.20):*

$$\Lambda_1 = -\frac{(k_2 - k_3)^2}{(1-k_2^2)(1-k_3^2)}\,, \qquad \Lambda_2 = -\frac{(k_2 + k_3)^2}{(1-k_2^2)(1-k_3^2)}\,. \tag{1.28}$$

A change of square roots $(k_2, k_3) \mapsto (\pm k_2, \pm k_3)$ *leaves Equations (1.28) invariant, whereas the change* $(k_2, k_3) \mapsto (\pm k_2, \mp k_3)$ *interchanges* Λ_1 *and* Λ_2*. This means that the elliptic curves* $\mathcal{E}_l$ *with level-two structure can be constructed over* $\mathbb{Q}(k_2, k_3)$ *as a finite field extension of* $\mathbb{Q}(\Lambda_1, \Lambda_2)$.

1.3.4. *Legendre's gluing construction.* The Weierstrass points P_i of the curve $\mathcal{C}_0$ in Equation (1.17), given by

$$\begin{aligned} P_1 : [\lambda_2\lambda_3 : 0 : 1]\,, \quad P_2 : [\lambda_2 : 0 : 1]\,, \quad P_3 : [\lambda_3 : 0 : 1]\,, \\ P_4 : [0 : 0 : 1]\,, \quad P_5 : [1 : 0 : 1]\,, \quad P_6 : [1 : 0 : 0]\,, \end{aligned} \tag{1.29}$$

are the fixed points of the hyperelliptic involution $\imath_{\mathcal{C}_0}$. The two elliptic involutions $\jmath_l$ in Equation (1.27) for $l = 1, 2$ (each) pairwise interchange the Weierstrass points, i.e.,

$$\jmath_l : \quad P_2 \leftrightarrow P_3\,, \quad P_1 \leftrightarrow P_5\,, \quad P_4 \leftrightarrow P_6\,. \tag{1.30}$$

The points $Q_{l,\pm}$, given by

$$\begin{aligned} Q_{1,\pm} : \quad & [-q : \pm i q^{\frac{3}{2}}(k_2 k_3 + 1)(k_2 + k_3) : 1]\,, \\ Q_{2,\pm} : \quad & [\ \ q : \pm\ q^{\frac{3}{2}}(k_2 k_3 - 1)(k_2 - k_3) : 1]\,, \end{aligned} \tag{1.31}$$

are the ramification points of $\pi_{\mathcal{E}_l}$ in Equation (1.25) for $l = 1$ and $l = 2$, respectively. The involution $\jmath_1$ fixes the points $Q_{1,+}$ and $Q_{1,-}$, and interchanges the points

$Q_{2,+} \leftrightarrow Q_{2,-}$. An analogous statements holds for the involution $\jmath_2$. The hyperelliptic involution of the curve $\mathcal{C}_0$ interchanges the elements of the two pairs simultaneously, i.e., $\imath_{\mathcal{C}_0} : Q_{l,+} \leftrightarrow Q_{l,-}$ for $l = 1, 2$.

The pairs $\{P_2, P_3\}$, $\{P_1, P_5\}$, $\{Q_{2,+}, Q_{2,-}\}$, and $\{P_5, P_6\}$ are mapped by $\pi_{\mathcal{E}_1}$ to the two-torsion points $p_0 : [x_1 : y_1 : z_1] = [0 : 0 : 1]$, $p_1 : [1 : 0 : 1]$, $p_{\Lambda_1} : [\Lambda_1 : 0 : 1]$, and the identity $p_\infty : [1 : 0 : 0]$ in $\mathcal{E}_1[2]$. Similarly, the pairs $\{P_2, P_3\}$, $\{P_1, P_5\}$, $\{Q_{1,+}, Q_{1,-}\}$, and $\{P_5, P_6\}$ are mapped by $\pi_{\mathcal{E}_2}$ to the two-torsion points $p_0 : [x_2 : y_2 : z_2] = [0 : 0 : 1]$, $p_1 : [1 : 0 : 1]$, $p_{\Lambda_2} : [\Lambda_2 : 0 : 1]$, and the identity $p_\infty : [1 : 0 : 0]$ in $\mathcal{E}_2[2]$.

We associate with the branch locus of $\pi_{\mathcal{E}_1}$ and $\pi_{\mathcal{E}_2}$ the effective divisor of degree two $B_1 = [\pi_{\mathcal{E}_1}(Q_{1,+}) + \pi_{\mathcal{E}_1}(Q_{1,-})]$ on $\mathcal{E}_1$ and $B_2 = [\pi_{\mathcal{E}_2}(Q_{2,+}) + \pi_{\mathcal{E}_2}(Q_{2,-})]$ on $\mathcal{E}_2$, respectively. One checks that for $\pi_{\mathcal{E}_1}(Q_{1,\pm})$ one has $[x_1 : z_1] = [\Lambda_2 : 1]$, and for $\pi_{\mathcal{E}_2}(Q_{2,\pm})$ one has $[x_2 : z_2] = [\Lambda_1 : 1]$. Because of $\mathrm{Pic}^0(\mathcal{E}_l) \cong \mathcal{E}_l$ the line bundle $\mathcal{O}_{\mathcal{E}_l}(B_l)$ associated with the branch locus B_l is equivalent to the line bundle $\mathcal{O}_{\mathcal{E}_l}(2p_\alpha)$ if and only if

$$\pi_{\mathcal{E}_l}\big(Q_{l,+}\big) \oplus \pi_{\mathcal{E}_l}\big(Q_{l,-}\big) = 2p_\alpha = p_\alpha \oplus p_\alpha \,, \tag{1.32}$$

where $\oplus$ refers to the addition with respect to the elliptic curve group law on $\mathcal{E}_l$. Using the properties $\pi_{\mathcal{E}_l}$ in Equation (1.25) one finds that

$$\pi_{\mathcal{E}_l}\big(Q_{l,+}\big) \oplus \pi_{\mathcal{E}_l}\big(Q_{l,-}\big) = 0 \,, \tag{1.33}$$

and $p_\alpha \in \mathcal{E}_l[2]$ is a two-torsion point. Thus, there are four line bundle $\mathcal{L}_l \to \mathcal{E}_l$ such that $\mathcal{L}_l^{\otimes 2} \cong \mathcal{O}_{\mathcal{E}_l}(B_l)$, namely $\mathcal{L}_l \cong \mathcal{O}_{\mathcal{E}_l}(p_\alpha)$. It follows $h^0(\mathcal{E}_l, \mathcal{L}_l) = 1$ by the Riemann-Roch theorem. Thus, the preimage of the vanishing locus of a section of $\mathcal{L}_l \to \mathcal{E}_l$ determines a unique double cover $p_l : \mathcal{C}_0 \to \mathcal{E}_l$. The composition on $\mathcal{C}_0$ of the elliptic involution with the hyperelliptic involution defines a second elliptic involution, and a second elliptic-curve quotient is obtained. Thus, the data $(\mathcal{E}_l, \mathcal{L}_l, B_l)$ for either $l = 1$ or $l = 2$ determines the curve $\mathcal{C}_0$ uniquely. As we have seen, this data is equivalent to an (unordered) pair $\{\Lambda_1, \Lambda_2\}$ of modular parameters $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ with $\Lambda_1 \neq \Lambda_2$. Legendre constructed an explicit model for $\mathcal{C}_0$ over $\mathbb{Q}(\Lambda_1, \Lambda_2)$; this is often referred to as *Legendre's gluing method*. Following Serre's explanation [50, Sec. 27] of Legendre's gluing method we obtain:

Proposition 1.8. *Assume that $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ satisfy $\Lambda_1 \neq \Lambda_2$, and the moduli of $\mathcal{C}_0$ and $\mathcal{E}_l$ in Equations (1.17) and (1.19) satisfy Equations (1.20). The smooth genus-two curve, given by*

$$\widetilde{\mathcal{C}}_0 : \quad Y^2 = \Big(X^2 - Z^2\Big)\Big(X^2 - \frac{\Lambda_1}{\Lambda_2} Z^2\Big)\Big(X^2 - \frac{1 - \Lambda_1}{1 - \Lambda_2} Z^2\Big) \,, \tag{1.34}$$

is isomorphic to $\mathcal{C}_0$ over a finite field extension of $\mathbb{Q}(\lambda_2, \lambda_3)$. The curve is also isomorphic to each of the curves obtained by replacing $\{\Lambda_1, \Lambda_2\}$ in Equation (1.34) by one of the following pairs:

$$\Big\{\frac{1}{1 - \Lambda_1}, \frac{1}{1 - \Lambda_2}\Big\}, \Big\{\frac{\Lambda_1 - 1}{\Lambda_1}, \frac{\Lambda_2 - 1}{\Lambda_2}\Big\}, \Big\{\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2}\Big\}, \Big\{\frac{\Lambda_1}{\Lambda_1 - 1}, \frac{\Lambda_2}{\Lambda_2 - 1}\Big\}, \Big\{1 - \Lambda_1, 1 - \Lambda_2\Big\}. \tag{1.35}$$

Proof. Using Remark 1.7 and denoting the Igusa-Clebsch invariants of any pairs genus-two curves in the statement of the proposition by $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}_{(2,4,6,10)}$

and $[I_2' : I_4' : I_6' : I_{10}']$, respectively, one checks that

$$[I_2 : I_4 : I_6 : I_{10}] = [s^2 I_2' \ : \ s^4 I_4' \ : \ s^6 I_6' \ : \ s^{10} I_{10}'] = [I_2' : I_4' : I_6' : I_{10}'] \tag{1.36}$$

with $s \in \mathbb{Q}(\lambda_2, \lambda_3, \sqrt{\lambda_2\lambda_3})$. □

The six pairs, given by $\{\Lambda_1, \Lambda_2\}$ and Equation (1.35), correspond to the orbit of the data $(\mathcal{E}_l, \mathcal{L}_l, B_l)$ under $\Gamma/\Gamma(2)$, i.e., the orbit under simultaneous action of the *anharmonic group* on an elliptic curve in Legendre form. The anharmonic group is generated by the two transformations $\Lambda \mapsto 1-\Lambda, \Lambda/(\Lambda-1)$. In contrast, the action of the anharmonic group on only one of the modular parameters generates non-isomorphic genus-two curves. We have the following:

Corollary 1.9. *Assume that* $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ *satisfy*

$$\Lambda_1 \ \notin \ \left\{\Lambda_2, \frac{1}{1-\Lambda_2}, \frac{\Lambda_2 - 1}{\Lambda_2}, \frac{1}{\Lambda_2}, \frac{\Lambda_2}{\Lambda_2 - 1}, 1-\Lambda_2\right\}. \tag{1.37}$$

Using $\{\Lambda_1, \Lambda_2\}$ *and the pairs*

$$\left\{\Lambda_1, \frac{1}{1-\Lambda_2}\right\}, \left\{\Lambda_1, \frac{\Lambda_2-1}{\Lambda_2}\right\}, \left\{\Lambda_1, \frac{1}{\Lambda_2}\right\}, \left\{\Lambda_1, \frac{\Lambda_2}{\Lambda_2-1}\right\}, \left\{\Lambda_1, 1-\Lambda_2\right\}, \tag{1.38}$$

one obtains smooth genus-two curves admitting an elliptic involution whose elliptic-curve quotients are given by Equations (1.19). Generically, the six curves are non-isomorphic and satisfy $\mathrm{Jac}\,(\widetilde{\mathcal{C}_0}) \in \mathcal{H}_4$.

Proof. The proof follows from an explicit computation similar to the one in the proof of the previous proposition. □

1.3.5. *Relation to Göpel groups.* Since the Picard group $\mathrm{Pic}^0(\mathcal{C}_0) \cong \mathrm{Jac}\,(\mathcal{C}_0)$ consists of elements of the from $[P + Q - 2P_6]$ for $P, Q \in \mathcal{C}_0$, an isogeny Ψ of abelian surfaces is defined by setting

$$\Psi: \quad \mathrm{Jac}\,(\mathcal{C}_0) \longrightarrow \mathcal{E}_1 \times \mathcal{E}_2\,, \quad [P+Q-2P_6] \mapsto \Big(\pi_{\mathcal{E}_1}(P) \oplus \pi_{\mathcal{E}_1}(Q),\ \pi_{\mathcal{E}_2}(P) \oplus \pi_{\mathcal{E}_2}(Q)\Big). \tag{1.39}$$

It follows from the construction of the line bundles $\mathcal{L}_l \to \mathcal{E}_l$ in Section 1.3.4 that the line bundle $\mathcal{L} \to \mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$ associated with the principal polarization of $\mathbf{A}$ satisfies $\mathcal{L} \cong \Psi^*(\mathcal{L}_1 \boxtimes \mathcal{L}_2)$. The elliptic involutions $\jmath_l$ on $\mathcal{C}_0$ extend to involutions on the Jacobian $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$ that coincide on $\mathbf{A}[2]$. We denote this induced involution on $\mathbf{A}[2]$ by $\jmath : \mathbf{A}[2] \to \mathbf{A}[2]$.

Given the marking $(P_1, \ldots, P_6) = (\lambda_1 = \lambda_2\lambda_3, \lambda_2, \lambda_3, 0, 1, \infty)$ of the Weierstrass points for the curve $\mathcal{C}_0$ as before, we have the following:

Proposition 1.10. *In the situation above, the kernel of* $\Psi : \mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0) \to \mathcal{E}_1 \times \mathcal{E}_2$ *is given by the Göpel group*

$$\ker \Psi \ = \ \Big\{P_0, P_{15}, P_{23}, P_{46}\Big\} \ \cong \ (\mathbb{Z}/2\mathbb{Z})^2 \ \subset \ \mathbf{A}[2]\,. \tag{1.40}$$

In particular, Ψ *is a* $(2,2)$*-isogeny and* $\mathbf{A}' = \mathbf{A}/\ker\Psi \cong \mathcal{E}_1 \times \mathcal{E}_2$ *with* $\mathrm{T}_{\mathbf{A}'} = H \oplus H$.

Proof. The induced actions of the two elliptic involutions $\jmath_l$ on $\mathrm{Jac}\,(\mathcal{C}_0)$ fix the divisors P_{15}, P_{23}, P_{46}, and $[Q_{l,+} + Q_{l,-} - 2P_6]$. It follows from the explicit formulas for p_l in Proposition that $(P, Q) \in \ker \Psi$ if and only if $P, Q \in \mathbf{A}[2]$ and $\jmath_l(P) = Q$ for $l = 1, 2$. □

The remaining Göpel groups fall within two cases: the first case consists of six Göpel groups such that the involution $\jmath$ fixes one non-trivial element and interchanges the other two. These groups are given by

$$\begin{aligned} &\left\{P_0, P_{12}, P_{35}, P_{46}\right\}, \quad \left\{P_0, P_{13}, P_{25}, P_{46}\right\}, \quad \left\{P_0, P_{14}, P_{23}, P_{56}\right\}, \\ &\left\{P_0, P_{16}, P_{23}, P_{45}\right\}, \quad \left\{P_0, P_{15}, P_{24}, P_{36}\right\}, \quad \left\{P_0, P_{15}, P_{26}, P_{34}\right\}. \end{aligned} \tag{1.41}$$

By construction, each of them contains one non-trivial element from the special Göpel group $\{P_0, P_{15}, P_{23}, P_{46}\}$ that yields the quotient $\mathcal{E}_1 \times \mathcal{E}_2$. We have the following:

Proposition 1.11. *For each Göpel group G in Equation (1.41) the abelian surface $\mathbf{A}' = \mathbf{A}/G$ satisfies $\mathbf{A}' \cong \mathrm{Jac}\,(\mathcal{C}_0')$ for some smooth genus-two curve $\mathcal{C}_0'$ admitting an elliptic involution. In particular, we have $\mathrm{Jac}\,(\mathcal{C}_0') \in \mathcal{H}_4$ and $\mathrm{T}_{\mathbf{A}'} = H \oplus \langle 2\rangle \oplus \langle -2\rangle$.*

Proof. The proof follows by constructing the Richelot curve in Equation (1.5) and checking that $Q = 0$ in Proposition 1.2 for each Göpel subgroup G_n with $n = 1, \ldots, 6$, in Equation (1.41). □

Remark 1.12. *The six genus-two curve in Proposition 1.11 correspond to the six curves in Corollary 1.9 under $(2, 2)$-isogeny.*

For the elliptic curve $\mathcal{E}_l$ in Equation (1.19), there are three order-two points that can be used to construct a two-isogenous elliptic curve. Using the order-two point $[x_l : y_l : z_l] = [0 : \Lambda_l : 1]$, one easily obtains an elliptic curves $\mathcal{E}_l'$ in $\mathbb{P}^2 = \mathbb{P}(X_l, Y_l, Z_l)$ for $l = 1, 2$, given by

$$\mathcal{E}_l' : \quad Y_l^2 Z_l = X_l\Big(X_l^2 + 2\big(1 - 2\Lambda_l\big)X_l Z_l + Z_l^2\Big), \tag{1.42}$$

together with an isogenies $\chi_{\mathcal{E}_l} : \mathcal{E}_l \to \mathcal{E}_l'$, and its dual two-isogenies $\chi_{\mathcal{E}_l}'$. Assuming Equations (1.20), the two-isogenous elliptic curves can be brought into Legendre normal form, but only over the finite field extension $\mathbb{Q}(k_1, k_2)$. In fact, the elliptic curve $\mathcal{E}_l'$ is isomorphic over $\mathbb{Q}(k_1, k_2)$ to the quadratic-twist curve

$$\mathcal{E}_l' : \quad Y_l^2 Z_l = \delta' X_l \left(X_1 - Z_1\right)\left(X_1 - \Lambda_1' Z_1\right), \tag{1.43}$$

where we have set

$$\Lambda_1' = \frac{(k_2 - 1)^2 (k_3 + 1)^2}{(k_2 + 1)^2 (k_3 - 1)^2}, \qquad \Lambda_2' = \frac{(k_2 + 1)^2 (k_3 + 1)^2}{(k_2 - 1)^2 (k_3 - 1)^2}, \qquad \delta' = -(1 - k_2^2)(1 - k_3^2). \tag{1.44}$$

A change of square roots $(k_2, k_3) \mapsto (\pm k_2, \pm k_3)$ leaves Equations (1.44) invariant or acts as the modular transformation $\Lambda_l' \mapsto 1/\Lambda_l'$, whereas the change $(k_2, k_3) \mapsto (\mp k_2, \pm k_3)$ in Equations (1.44) interchanges Λ_1' and Λ_2' (up to a modular transformation). Applying Proposition 1.8 one obtains the common double cover

$$\widetilde{\mathcal{C}}_0' : \quad Y^2 = \left(X^2 - Z^2\right)\left(X^2 - \frac{\Lambda_1'}{\Lambda_2'} Z^2\right)\left(X^2 - \frac{1 - \Lambda_1'}{1 - \Lambda_2'} Z^2\right). \tag{1.45}$$

We have the following:

Proposition 1.13. *Assume that $\lambda_2, \lambda_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2 \neq \lambda_3^{\pm 1}$ and $\lambda_2 \neq -1$, and $\mathcal{C}_0$ is given by Equation (1.17) with $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$. The genus-two curve*

$$\mathcal{C}_0' : \quad Y^2 = XZ\big(X-Z\big)\big(X-\mu_2\mu_3 Z\big)\big(X^2-(\mu_2+\mu_3)XZ+\mu_2\mu_3 Z^2\big)\,, \tag{1.46}$$

is smooth, admits an elliptic involution, and satisfies $\mathbf{A}' \cong \mathbf{A}/G$ for $\mathbf{A}' = \mathrm{Jac}\,(\mathcal{C}_0')$ and $G = \{P_0, P_{12}, P_{35}, P_{46}\}$. Here, μ_2, μ_3 are given by

$$\begin{aligned} \mu_2 &= \frac{\lambda_2\lambda_3+\lambda_2-\lambda_3-1+2d}{\lambda_2\lambda_3+\lambda_2-\lambda_3-1-2d}\,, \\ \mu_3 &= \frac{\big(\lambda_2\lambda_3+\lambda_2-\lambda_3-1+2d\big)\big((1+\lambda_2)^2k_3+(1-\lambda_2)d-2(1+\lambda_3)\lambda_2\big)}{\big(\lambda_2\lambda_3+\lambda_2-\lambda_3-1-2d\big)\big((1+\lambda_2)^2k_3-(1-\lambda_2)d-2(1+\lambda_3)\lambda_2\big)}\,, \end{aligned} \tag{1.47}$$

with $k_3^2 = \lambda_3$ and $d^2 = (\lambda_2-\lambda_3)(\lambda_2\lambda_3-1)$. In particular, the curve $\mathcal{C}_0'$ is isomorphic to $\widetilde{\mathcal{C}}_0'$ in Equation (1.45) over a finite field extension of $\mathbb{Q}(\lambda_2,\lambda_3)$.

Remark 1.14. *In Equations (1.47) the sign of k_3 and d can be chosen independently. A change $(k_3, d) \mapsto (\mp k_3, \mp d)$ acts on the Rosenhain roots as $(\mu_2,\mu_3) \mapsto (1/\mu_2, 1/\mu_3)$. The sign change $(k_3, d) \mapsto (\pm k_3, \mp d)$ acts on the Rosenhain roots as $(\mu_2,\mu_3) \mapsto (\mu_2, \mu_2^2/\mu_3)$ or $(\mu_2,\mu_3) \mapsto (1/\mu_2, \mu_3/\mu_2^2)$. Isomorphic curves are also obtained by interchanging $\lambda_2 \leftrightarrow \lambda_3$, $k_2 \leftrightarrow k_3$, and changing $d \mapsto id$ in Equations (1.47) (while assuming $\lambda_3 \neq -1$ instead of $\lambda_2 \neq -1$).*

Proof. The result is obtained by applying the results of [15,16] in the case $\lambda_1 = \lambda_2\lambda_3$. In fact, [15, Prop. 2.1] yields the curve

$$Y^2 = \Big(X-2(1+\lambda_2^2)\lambda_3 Z\Big)\Big(c_2X^2+c_1XZ+c_0Z^2\Big)\Big(\big(X+(1+\lambda_2^2)\lambda_3 Z\big)^2-36\lambda_2^2\lambda_3^2Z^2\Big)Z\,, \tag{1.48}$$

with

$$\begin{aligned} c_2 &= 1+\lambda_3-2k_3\,, \\ c_1 &= 2\lambda_3(\lambda_2^2-6\lambda_2+1)(\lambda_3+1)+8(1+\lambda_2^2)\lambda_3k_3\,, \\ c_0 &= \lambda_3^2(\lambda_2^4+24\lambda_2^3-34\lambda_2^2+24\lambda_2+1)(\lambda_3+1)+2\lambda_3^2(\lambda_2^2-5)(5\lambda_2^2-1)k_3\,. \end{aligned}$$

If in Equation (1.48) one substitutes

$$X = 3(1+\lambda_2)^2\lambda_3X'-(1+6\lambda_2+\lambda_2^2)\lambda_3Z'\,, \qquad Y = 3^{\frac{5}{2}}(1+\lambda_2)^5(1-k_3)Y'\,, \quad Z = Z'\,,$$

one obtains the curve

$$(Y')^2 = X'Z'\big(X'-Z'\big)\big(X'-\lambda_2'\lambda_3'Z'\big)\big((X')^2-(\lambda_2'+\lambda_3')X'Z'+\lambda_2'\lambda_3'(Z')^2\big)\,, \tag{1.49}$$

where λ_2', λ_3' satisfy

$$\lambda_2'\lambda_3' = \frac{4\lambda_2}{(1+\lambda_2)^2}\,, \qquad \lambda_2'+\lambda_3' = 2\left(1-\frac{(1-\lambda_2)^2(1+\lambda_3)}{(1+\lambda_2)^2(1+\lambda_3-2k_3)}\right)\,, \tag{1.50}$$

with $\lambda_3 = k_3^2$. Introducing new moduli μ_2, μ_3, given by

$$\big(\lambda_2', \lambda_3'\big) = \left(\frac{1-\mu_2}{1-\mu_3}\,,\ \frac{(1-\mu_2)\mu_3}{(1-\mu_3)\mu_2}\right) \ \Leftrightarrow\ (\mu_2,\mu_3) = \left(\frac{1-\lambda_2'}{1-\lambda_3'}\,,\ \frac{(1-\lambda_2')\lambda_3'}{(1-\lambda_3')\lambda_2'}\right)\,, \tag{1.51}$$

one obtains an isomorphic curve over $\mathbb{Q}(\mu_2,\mu_3)$ given by Equation (1.46). Because of $\lambda_2 \neq \lambda_3^{\pm 1}$ we have $d \neq 0$. Since we also assumed $\lambda_2 \neq -1$, we have $\mu_2,\mu_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ and $\mu_2 \neq \mu_3^{\pm 1}$ and the curve is smooth.

Denoting the Igusa-Clebsch invariants of the genus-two curves in Equation (1.45) and Equation (1.49) by $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}_{(2,4,6,10)}$ and $[I_2' : I_4' : I_6' : I_{10}']$, respectively, one checks that

$$[I_2 : I_4 : I_6 : I_{10}] = [s^2 I_2' \ : \ s^4 I_4' \ : \ s^6 I_6' \ : \ s^{10} I_{10}'] = [I_2' : I_4' : I_6' : I_{10}'] \tag{1.52}$$

with $s \in \mathbb{Q}(k_2,k_3)$. □

The last case of Göpel groups to consider consists of four pairs that are pairwise interchanged by the action of the involution $\jmath$. These groups are given by

$$\begin{array}{ccccc} \left\{P_0,P_{12},P_{34},P_{56}\right\}, & \left\{P_0,P_{12},P_{36},P_{45}\right\}, & \left\{P_0,P_{13},P_{24},P_{56}\right\}, & \left\{P_0,P_{13},P_{26},P_{45}\right\}, \\ \jmath: \quad \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \left\{P_0,P_{14},P_{26},P_{35}\right\}, & \left\{P_0,P_{16},P_{24},P_{35}\right\}, & \left\{P_0,P_{14},P_{25},P_{36}\right\}, & \left\{P_0,P_{16},P_{25},P_{34}\right\}. \end{array} \tag{1.53}$$

We have the following:

Proposition 1.15. *For each pair of Göpel subgroups $G_\pm$ in Equation (1.53) which are pairwise interchanged by the action of the involution $\jmath$, the abelian surfaces $\mathbf{A}'_\pm = \mathbf{A}/G_\pm$ satisfy $\mathbf{A}'_\pm \cong \mathrm{Jac}\,(\mathcal{C}'_{0,\pm})$ for some smooth genus-two curves $\mathcal{C}'_{0,\pm}$ with $\mathcal{C}'_{0,+} \cong \mathcal{C}'_{0,-}$ and $\mathrm{Jac}\,(\mathcal{C}'_{0,\pm}) \in \mathcal{H}_{16}$. In particular, we have $\mathrm{T}_{\mathbf{A}'_\pm} = H \oplus \langle 4\rangle \oplus \langle -4\rangle$.*

Proof. The proof follows by constructing the Richelot curves $\mathcal{C}'_\pm$ in Equation (1.5) for each pair of Göpel groups in Equation (1.53). Denoting the Igusa-Clebsch invariants of the genus-two curves $\mathcal{C}'_{0,\pm}$ by $[I_2^\pm : I_4^\pm : I_6^\pm : I_{10}^\pm] \in \mathbb{WP}_{(2,4,6,10)}$, one checks that

$$[I_2^+ : I_4^+ : I_6^+ : I_{10}^+] = [I_2^- : I_4^- : I_6^- : I_{10}^-]\,. \tag{1.54}$$

One then checks by a direct computation that $Q \neq 0$ for each $\mathcal{C}'_{0,\pm}$ whence it follows $\mathrm{Jac}\,(\mathcal{C}'_{0,\pm}) \notin \mathcal{H}_4$. Using the $(2,2)$-isogeny Φ from Theorem 1.1, we obtain the following compositions of isogenies

$$\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\Phi} \mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0) \xrightarrow{\Psi} \mathbf{A}'_{n,\pm} = \mathrm{Jac}\,(\mathcal{C}'_{0,\pm}) = \mathbf{A}/G_\pm\,. \tag{1.55}$$

Since Ψ is obtained from the projection onto the quotient by a Göpel group, it is a $(2,2)$-isogeny as well. Thus, $\Psi \circ \Phi$ is a $(4,4)$-isogeny. Using Theorem 1.1 once more, it follows that $\mathrm{Jac}\,(\mathcal{C}'_{0,\pm}) \in \mathcal{H}_{16}$. □

We have the following:

Corollary 1.16. *For a genus-two curve $\mathcal{C}_0$ admitting an elliptic involution, there is exactly one Göpel group $G \leqslant \mathbf{A}[2]$ of $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$ such that $\mathbf{A}/G \cong \mathcal{E}_1 \times \mathcal{E}_2$ for two elliptic curves $\mathcal{E}_1,\mathcal{E}_2$. In particular, G is the unique Göpel group of $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$ which is fixed under the action of the involution $\jmath$ on $\mathbf{A}[2]$.*

Remark 1.17. *The special Göpel group from Corollary 1.16 has one non-trivial element in common with each Göpel group in Proposition 1.11 whose associated* $(2,2)$*-isogenous Richelot curve has a Jacobian in* $\mathcal{H}_4$*. In contrast, the intersection with each Göpel group in Proposition 1.15 is trivial.*

2. Kummer surfaces and isogenies

On a principally polarized abelian surface $(\mathbf{A}, \mathcal{L})$ with the minus identity involution denoted by $-\mathbb{I}$, one can always choose a theta divisor $\mathcal{L} = \mathcal{O}_{\mathbf{A}}(\Theta)$ to satisfy $(-\mathbb{I})^*\Theta = \Theta$, that is, to be a *symmetric theta divisor*. The abelian surface $\mathbf{A}$ then maps to the complete linear system $|2\Theta|$, and the rational map $\varphi_{\mathcal{L}^2} : \mathbf{A} \to \mathbb{P}^3$ associated with the line bundle $\mathcal{L}^2$ factors via an embedding through the projection $\mathbf{A} \to \mathbf{A}/\langle -\mathbb{I}\rangle$; see [5]. In this way, one identifies $\mathbf{A}/\langle -\mathbb{I}\rangle$ with its image $\varphi_{\mathcal{L}^2}(\mathbf{A})$ in $\mathbb{P}^3$, which is a singular quartic projective surface $\mathcal{K}_{\mathbf{A}}$ with sixteen ordinary double points, called a *nodal surface*. The map from $\mathbf{A}$ onto its image $\mathcal{K}_{\mathbf{A}} \subset \mathbb{P}^3$ is two-to-one, except on sixteen points of order two where it is injective. Its minimum resolution is the Kummer surface $\mathrm{Kum}(\mathbf{A})$ associated with the principally polarized abelian surface $(\mathbf{A}, \mathcal{L})$. Due to the 16_6 configuration on $\mathcal{K}_{\mathbf{A}}$, there are six hyperplanes containing the double point $\varphi_{\mathcal{L}^2}(0)$ and touching $\mathcal{K}_{\mathbf{A}}$ along a double conic. The linear projection with center $\varphi_{\mathcal{L}^2}(0)$ maps these six hyperplanes onto the six lines $\mathsf{T}_1, \dots, \mathsf{T}_6$ in a projective plane $\mathbb{P}^2$. The configuration $(\mathbb{P}^2; \mathsf{T}_1, \dots, \mathsf{T}_6)$ is called the *Kummer plane* associated with $(\mathbf{A}, \mathcal{L})$. The rich geometry of six-line configurations in the Kummer plane, as well as their strong connection with theta functions have been the subject of multiple studies [11, 17, 19, 27, 36, 38, 51] over the last century and a half.

In this section we will construct the Kummer surfaces associated with the product of two elliptic curves and the Jacobian of a genus-two curve as a double quadric, a double sextic, and a quartic projective surface. We will also construct several Jacobian elliptic fibrations on these surfaces. As a reminder, an elliptic surface is a (relatively) minimal complex surface $\mathcal{X}$, together with a *Jacobian elliptic fibration*, that is, a holomorphic map $\pi_{\mathcal{X}} : \mathcal{X} \to \mathbb{P}^1$ to $\mathbb{P}^1$ such that the general fiber is a smooth curve of genus one together with a distinguished section $\Sigma : \mathbb{P}^1 \to \mathcal{X}$ that marks a smooth point in each fiber. To each Jacobian elliptic fibration $\pi_{\mathcal{X}} : \mathcal{X} \to \mathbb{P}^1$ there is an associated Weierstrass model obtained by contracting all components of reducible fibers not meeting Σ. By a slight abuse of notation we will denote the elliptic surface and its associated Weierstrass model by the same symbol. The complete list of possible singular fibers has been given by Kodaira [33]. It encompasses two infinite families $(I_n, I_n^*, n \geq 0)$ and six exceptional cases $(II, III, IV, II^*, III^*, IV^*)$. The Weierstrass model of a smooth K3 surface can always be written in the form

$$y^2 z = 4x^3 - g_2(v)\, x z^2 - g_3(v) z^3 \,, \tag{2.1}$$

where v is a suitable coordinate on the base curve $\mathbb{P}^1$, and g_2 and g_3 are polynomials of degree 8 and 12, respectively. The section is given by the point at infinity in each smooth fiber. We denote the Mordell-Weil group of sections on the Jacobian elliptic surface $\pi_{\mathcal{X}} : \mathcal{X} \to \mathbb{P}^1$ by $\mathrm{MW}(\mathcal{X}, \pi_{\mathcal{X}})$. If a Jacobian elliptic fibration admits

in addition a two-torsion section $T \in \mathrm{MW}(\mathcal{X}, \pi_{\mathcal{X}})$, we use a change of coordinates to write Equation (2.1) in the form

$$\mathcal{X}: \quad y^2 z = x^3 + p_1(v)\, x^2 z + p_2(v)\, x z^2 \,, \tag{2.2}$$

where p_1 and p_2 are polynomials of degree 4 and 8, respectively. A natural holomorphic two-form on the K3 surface, obtained as the minimal resolution of the Weierstrass model, is the pullback of the holomorphic two-form $dv \wedge dx/y$ in the (affine coordinate) chart $z = 1$ in Equation (2.2).

2.1. **Kummer surfaces associated with two elliptic curves.** The Kummer surface $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$ is the minimal resolutions of quotient surface of the product abelian surface $\mathcal{E}_1 \times \mathcal{E}_2$ by the inversion automorphism where the elliptic curves $\mathcal{E}_l$ for $l = 1, 2$ are not mutually isogenous. Oguiso proved that these surfaces admit eleven distinct Jacobian elliptic fibrations [47]. Kuwata and Shioda furthered Oguiso's work in [35] and computed their Weierstrass models, reducible fibers, and Mordell-Weil lattices.

Let us define a double quadric surface closely related to $\mathcal{E}_1 \times \mathcal{E}_2$ for the elliptic curves given by Equation (1.19). In general, a *double quadric surface* is obtained as the double cover branched along a locus of bi-degree $(4, 4)$ in $\mathbb{P}^1 \times \mathbb{P}^1$; see [4]. It is well known that the minimal resolution of any double quadric is a K3 surface. In our situation, we identify $\mathbb{P}^1 = \mathbb{P}(x_l, z_l)$ for $l = 1, 2$ and define the special double quadric surface $\mathcal{Z}$ using the equation

$$\mathcal{Z}: \quad y_{1,2}^2 = x_1 z_1 \big(x_1 - z_1\big)\big(x_1 - \Lambda_1 z_1\big)\, x_2 z_2 \big(x_2 - z_2\big)\big(x_2 - \Lambda_2 z_2\big) \,. \tag{2.3}$$

There is a natural projection map $\pi : \mathcal{E}_1 \times \mathcal{E}_2 \dashrightarrow \mathcal{Z}$ given by $y_{1,2} = z_1 z_2 y_1 y_2$, invariant under the action of $\imath_{\mathcal{E}_1} \times \imath_{\mathcal{E}_2}$. It follows that the minimal resolution of $\mathcal{Z}$ is the Kummer surface $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$.

A holomorphic two-form on $\mathcal{Z}$ is given by $\omega_{\mathcal{Z}} = dx_1 \wedge dx_2 / y_{1,2}$ in the coordinate chart $z_1 = z_2 = 1$ in Equation (2.3). The regular two-from on $\mathcal{E}_1 \times \mathcal{E}_2$, given by the outer tensor product of the elliptic-curve holomorphic one-forms, i.e.,

$$\frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2} := \frac{dx_1}{y_1} \boxtimes \frac{dx_2}{y_2} \,, \tag{2.4}$$

descends to the quotient $(\mathcal{E}_1 \times \mathcal{E}_2)/\langle -\mathbb{I} \rangle$ and coincides with $\omega_{\mathcal{Z}}$. If ϵ is the minimal resolution, we obtain a holomorphic two-form $\omega_{\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)}$ on $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$ by pullback, i.e.,

$$\omega_{\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)} = \epsilon^* \omega_{\mathcal{Z}} \,, \quad \text{with } \epsilon\colon \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2) = \widehat{\mathcal{Z}} \dashrightarrow \mathcal{Z} \,. \tag{2.5}$$

We also introduce the quadratic twist of $\mathcal{Z}$, given by

$$\mathcal{Z}^{(\varepsilon)}: \quad \hat{y}_{1,2}^2 = \varepsilon x_1 z_1 \big(x_1 - z_1\big)\big(x_1 - \Lambda_1 z_1\big)\, x_2 z_2 \big(x_2 - z_2\big)\big(x_2 - \Lambda_2 z_2\big) \,, \tag{2.6}$$

such that for general $\varepsilon \in \mathbb{Q}$ the surfaces $\mathcal{Z}$ and $\mathcal{Z}^{(\varepsilon)}$ are isomorphic only over $\mathbb{Q}(\sqrt{\varepsilon})$. We refer to the minimal resolution of $\mathcal{Z}^{(\varepsilon)}$ as twisted Kummer surface and denote it by $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)^{(\varepsilon)}$. Holomorphic two-forms on the twisted double quadric surface and the minimal resolution are then obtained analogously to Equation (2.5). We make the following:

Remark 2.1. *Projecting to either* $\mathbb{P}^1$ *in Equation (2.3) defines a Jacobian elliptic fibration on the Kummer surface* $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$ *denoted* $\mathcal{J}_4$ *in* [35]*, i.e., a fibration with the singular fibers* $4I_0^*$ *and a Mordell-Weil group of sections* $(\mathbb{Z}/2\mathbb{Z})^2$*.*

Remark 2.2. *If the elliptic curves* $\mathcal{E}_l$ *for* $l = 1, 2$ *are not mutually isogenous, then the Kummer surface* $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$ *has Picard rank 18. It follows from* [44] *that the transcendental lattice is given by*

$$\mathrm{T}_{\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)} \cong \mathrm{T}_{\mathcal{E}_1\times\mathcal{E}_2}(2) = H(2) \oplus H(2)\,. \tag{2.7}$$

Moreover, we will consider $\mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2')$, i.e., the minimal resolution of the quotient surface of the product abelian surface $\mathcal{E}_1' \times \mathcal{E}_2'$ by the inversion automorphism – associated with the two-isogenous elliptic curves $\mathcal{E}_l'$ in Equation (1.42). The associated double quadric surface is given by

$$\mathcal{Z}' : \quad Y_{1,2}^2 = X_1 Z_1 \Big(X_1^2 + 2(1-2\Lambda_1) X_1 Z_1 + Z_1^2 \Big) X_2 Z_2 \Big(X_2^2 + 2(1-2\Lambda_2) X_2 Z_2 + Z_2^2 \Big)\,. \tag{2.8}$$

On $\mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2')$ a holomorphic two-form $\omega_{\mathrm{Kum}(\mathcal{E}_1'\times\mathcal{E}_2')}$ is obtained as the pullback of the two-form $\omega_{\mathcal{Z}'} = dX_1 \wedge dX_2 / Y_{1,2}$ in the chart $Z_1 = Z_2 = 1$ on $\mathcal{Z}'$ in Equation (2.8). The Cartesian product of the dual two-isogenies $\chi'_{\mathcal{E}_1} \times \chi'_{\mathcal{E}_2}$ of elliptic curves induces a rational cover $\mathcal{Z}' \to \mathcal{Z}$. We have the following:

Lemma 2.3. *The map* $\chi : \mathcal{Z}' \dashrightarrow \mathcal{Z}^{(2^4)}$*, given by*

$$\chi : \quad \Big(X_1, Z_1, X_2, Z_2,\ Y_{1,2} \Big) \ \mapsto\ \Big(x_1, z_1, x_2, z_2,\ \hat{y}_{1,2} \Big) \tag{2.9}$$

with

$$\begin{aligned} x_1 &= 4\Lambda_1 X_1 Z_1\,, \qquad & z_1 &= (X_1 + Z_1)^2\,, \\ x_2 &= 4\Lambda_2 X_2 Z_2\,, \qquad & z_2 &= (X_2 + Z_2)^2\,, \\ \hat{y}_{1,2} &= 16\,\Lambda_1\Lambda_2 (X_1^2 - Z_1^2)(X_2^2 - Z_2^2) Y_{1,2}\,, \end{aligned} \tag{2.10}$$

is a rational covering map defined over $\mathbb{Q}(\Lambda_1, \Lambda_2)$*. For the holomorphic two-forms* $\omega_{\mathcal{Z}^{(2^4)}} = dx_1 \wedge dx_2 / \hat{y}_{1,2}$ *in the chart* $z_1 = z_2 = 1$ *on* $\mathcal{Z}^{(2^4)}$ *and* $\omega_{\mathcal{Z}'} = dX_1 \wedge dX_2 / Y_{1,2}$ *in the chart* $Z_1 = Z_2 = 1$ *on* $\mathcal{Z}'$*, it follows* $\omega_{\mathcal{Z}'} = \chi^* \omega_{\mathcal{Z}^{(2^4)}}$*.*

Proof. The proof follows by an explicit computation using Equations (2.10). □

2.1.1. *Certain elliptic fibrations.* On the double quadric surface $\mathcal{Z}$, a non-isotrivial elliptic fibration, with fibers over $\mathbb{P}^1 = \mathbb{P}(u_1, u_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(X, Y, Z)$, is given by the Weierstrass model

$$\begin{aligned} Y^2 Z = X \Big(X &+ \frac{1}{2} u_1 u_2 \big(u_1 - u_2\big)\big((\rho_1 - \rho_2) u_1 - (\rho_1 + \rho_2) u_2\big) Z \Big) \\ &\times \Big(X + \frac{1}{2} u_1 u_2 \big((\rho_1 - \rho_2) u_1^2 - 2(\rho_1 + 1) u_1 u_2 + (\rho_1 + \rho_2) u_2^2\big) Z \Big)\,, \end{aligned} \tag{2.11}$$

where we have set

$$\rho_1 = \Lambda_1 + \Lambda_2 - 2\Lambda_1\Lambda_2 - 1\,, \qquad \rho_2 = 1 - \Lambda_1 - \Lambda_2\,. \tag{2.12}$$

The discriminant function of the elliptic fibration is given by

$$2^{-4}u_1^8u_2^8(u_1-u_2)^2\big((\rho_1-\rho)u_1-(\rho_1+\rho_2)u_2\big)^2\big((\rho_1-\rho)u_1^2-2(\rho_1+1)u_1u_2+(\rho_1+\rho)u_2^2\big)^2.$$

We have the immediate:

Lemma 2.4. *Equation (2.11) defines a Jacobian elliptic fibration with the singular fibers $2I_2^*+4I_2$ and the Mordell-Weil group of sections $(\mathbb{Z}/2\mathbb{Z})^2$.*

Equation (2.11) defines the Jacobian elliptic fibration on $\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)$ denoted $\mathcal{J}_6$ in [35]. We have the following:

Proposition 2.5. *For $\Lambda_1,\Lambda_2\in\mathbb{P}^1\backslash\{0,1,\infty\}$ and parameters ρ_1,ρ_2 given by*

$$\rho_1=\Lambda_1+\Lambda_2-2\Lambda_1\Lambda_2-1\,,\qquad \rho_2=1-\Lambda_1-\Lambda_2\,,$$

the surfaces in Equation (2.3) and (2.11) are birational equivalent over $\mathbb{Q}(\Lambda_1,\Lambda_2)$.

Proof. In the chart $Z=u_2=1$ in Equation (2.11) and $z_1=z_2=1$ in Equation (2.3), a birational map is given by

$$u_1=\frac{x_1x_2}{(x_1-1)(x_2-1)}\,,\qquad Y=\frac{x_1^2x_2(x_1+x_2-1)(x_1x_2-\Lambda_2(x_1+x_2)+\Lambda_2)y_{1,2}}{(x_1-1)^4(x_2-1)^5}\,,$$
$$X=\frac{x_1x_2(x_1+x_2-1)(x_1-\Lambda_1)(x_1x_2-\Lambda_2(x_1+x_2)+\Lambda_2)}{(x_1-1)^3(x_2-1)^3}\,. \tag{2.13}$$

□

The holomorphic two-forms are related as follows:

Corollary 2.6. *The birational equivalence of Proposition 2.5 identifies the holomorphic two-form $du_1\wedge dX/Y$ in the chart $Z=u_2=1$ in Equation (2.11) with $\omega_{\mathcal{Z}}=dx_1\wedge dx_2/y_{1,2}$ in the chart $z_1=z_2=1$ in Equation (2.3).*

Proof. The proof follows by an explicit computation using the transformation provided in the proof of Proposition 2.5. □

On the double quadric surface $\mathcal{Z}'$, another elliptic fibration, with fibers over $\mathbb{P}^1=\mathbb{P}(v_1,v_2)$ embedded into $\mathbb{P}^2=\mathbb{P}(X,Y,Z)$, is given by the Weierstrass model

$$Y^2Z=X^3-2v_2\big(2v_1-v_2\big)\Big(v_1^2+2\rho_1v_1v_2+\rho_2^2v_2^2\Big)X^2Z+v_2^4\Big(v_1^2+2\rho_1v_1v_2+\rho_2^2v_2^2\Big)^2XZ^2\,. \tag{2.14}$$

The discriminant function of the fibration is $16v_1v_2^{10}(v_1-v_2)(v_1^2+2\rho_1v_1v_2+\rho_2^2v_2^2)^6$. We have the immediate:

Lemma 2.7. *Equation (2.14) defines a Jacobian elliptic fibration with the singular fibers $I_4^*+2I_1+2I_0^*$ and the Mordell-Weil group of sections $\mathbb{Z}/2\mathbb{Z}$.*

Equation (2.14) defines the Jacobian elliptic fibration on $\mathrm{Kum}(\mathcal{E}_1'\times\mathcal{E}_2')$ denoted $\mathcal{J}_7$ in [35]. However, to construct a birational equivalence between the Jacobian elliptic fibration and $\mathcal{Z}'$, one has to use a finite field extension $\mathbb{Q}(\kappa_1,\kappa_2)$ with $\Lambda_1=1/(1-\kappa_1^2)$ and $\Lambda_2=1/(1-\kappa_2^2)$. We have the following:

Proposition 2.8. *For $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ let parameters ρ_1, ρ_2 be given by*

$$\rho_1 = \Lambda_1 + \Lambda_2 - 2\Lambda_1\Lambda_2 - 1\,, \qquad \rho_2 = 1 - \Lambda_1 - \Lambda_2\,.$$

The surfaces in Equation (2.8) and (2.14) are birational equivalent over $\mathbb{Q}(\kappa_1, \kappa_2)$ with $\Lambda_l = 1/(1-\kappa_l^2)$ for $l = 1, 2$.

Proof. In the charts $v_2 = 1, Z = 1$ in Equation (3.35) and $Z_1 = Z_2 = 1$ in Equation (2.8), a birational equivalence is given by

$$\begin{aligned} v_1 &= \frac{(\kappa_1^2-1)(\kappa_1\kappa_2^2-1)X_1X_2 - \kappa_1(\kappa_1+1)(\kappa_2^2-1)X_1 + \kappa_1(\kappa_1-1)(\kappa_2^2-1)X_2^2 + (\kappa_1^2-1)(\kappa_1\kappa_2^2+1)X_2}{(1-\kappa_1^2)(1-\kappa_2^2)X_2((\kappa_1+1)X_1+\kappa_1-1)}\,, \\ X &= \frac{((\kappa_1+1)(\kappa_2-1)X_1 - (\kappa_1-1)(\kappa_2+1)X_2)((\kappa_1+1)(\kappa_2+1)X_1 - (\kappa_1-1)(\kappa_2-1)X_2)}{(1-\kappa_1^2)^2(1-\kappa_2^2)^2} \\ &\qquad \times \frac{((\kappa_2+1)X_2+(\kappa_2-1))((\kappa_2-1)X_2+(\kappa_2+1))\kappa_1^2X_1}{((\kappa_1+1)X_1+\kappa_1-1)^2X_2^3}\,, \\ Y &= -\frac{((\kappa_1+1)(\kappa_2-1)X_1 - (\kappa_1-1)(\kappa_2+1)X_2)^2((\kappa_1+1)(\kappa_2+1)X_1 - (\kappa_1-1)(\kappa_2-1)X_2)^2}{(1-\kappa_1^2)^3(1-\kappa_2^2)^3} \\ &\qquad \times \frac{((\kappa_2+1)X_2+(\kappa_2-1))((\kappa_2-1)X_2+(\kappa_2+1))\kappa_1^3Y_{1,2}}{((\kappa_1+1)X_1+\kappa_1-1)^4X_2^5}\,. \end{aligned} \tag{2.15}$$

□

The holomorphic two-forms are related as follows:

Corollary 2.9. *The birational equivalence of Proposition 2.8 identifies the holomorphic two-form $dv_1 \wedge dX/Y$ in the chart $Z = v_2 = 1$ in Equation (2.14) with $\omega_{\mathcal{Z}'} = dX_1 \wedge dX_2/Y_{1,2}$ in the chart $Z_1 = Z_2 = 1$ in Equation (2.8).*

Proof. The proof follows by an explicit computation using the transformation provided in the proof of Proposition 2.8. □

2.2. **Jacobian Kummer surfaces.** The Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ is the minimal resolution of the quotient surface of the Jacobian $\mathrm{Jac}\,(\mathcal{C})$ by the inversion automorphism where $\mathcal{C}$ is a generic, smooth genus-two curve. Kumar classified all Jacobian elliptic fibrations on the Jacobian Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ in [34].

Since the Jacobian $\mathrm{Jac}\,(\mathcal{C})$ is birational to the symmetric product $\mathrm{Sym}^2(\mathcal{C})$, let us consider the function field of $\mathrm{Sym}^2(\mathcal{C}) = (\mathcal{C}\times\mathcal{C})/\langle\sigma\rangle$ where σ interchanges two copies of a generic genus-two curve $\mathcal{C}$. In particular, for a smooth genus-two curve $\mathcal{C}$ in Rosenhain form (1.1), the function field of the variety $\mathrm{Sym}^2(\mathcal{C})/\langle\imath_{\mathcal{C}}\times\imath_{\mathcal{C}}\rangle$ – where $\imath_{\mathcal{C}}\times\imath_{\mathcal{C}}$ is the involution on $\mathrm{Sym}^2(\mathcal{C})$ induced by the hyperelliptic involution $\imath_{\mathcal{C}}$ on $\mathcal{C}$ – is generated by $z_1 = Z^{(1)}Z^{(2)}$, $z_2 = X^{(1)}Z^{(2)} + X^{(2)}Z^{(1)}$, $z_3 = X^{(1)}X^{(2)}$, and $\tilde{z}_4 = Y^{(1)}Y^{(2)}$, subject to the relation

$$\mathcal{W}: \quad \tilde{z}_4^2 = z_1z_3\big(z_1 - z_2 + z_3\big)\prod_{i=1}^{3}\big(\lambda_i^2\,z_1 - \lambda_i\,z_2 + z_3\big)\,. \tag{2.16}$$

Equation (2.16) is a special case of a *double sextic surface*, i.e., the double cover of $\mathbb{P}^2 = \mathbb{P}(z_1, z_2, z_3)$ branched on the union of six lines (and hence over a sextic curve). The minimal resolution of a double sextic surface is always a K3 surface; see [4].

Equation (2.16) is known as *Shioda sextic* associated with $\mathrm{Jac}(\mathcal{C})$; see [53]. It follows that the minimal resolution of $\mathcal{W}$ is the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$.

In Equation (2.16) the double cover is branched along the six lines T_i with $1 \le i \le 6$, given by

$$\lambda_i^2\, z_1 - \lambda_i\, z_2 + z_3 = 0 \text{ with } 1 \le i \le 3, \quad z_1 = 0, \quad z_1 - z_2 + z_3 = 0, \quad z_3 = 0\,. \tag{2.17}$$

The six lines are tangent to the common conic $K_2 = z_2^2 - 4\, z_1 z_3 = 0$. Conversely, it is easy to see that any six lines tangent to a common conic can always be brought into the form of Equations (2.17) using a projective transformation. Humbert proved that the Kummer plane $(\mathbb{P}^2; \mathsf{T}_1, \dots, \mathsf{T}_6)$ inherits essential information of the principally polarized abelian surface $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$ itself; see [28]. A picture is provided in Figure 1.

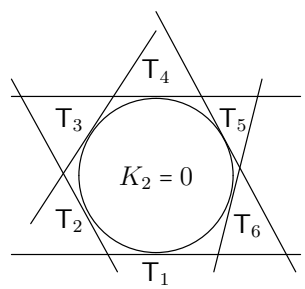

FIGURE 1. Double cover branched along reducible sextic

Remark 2.10. *In* [6] *the geometry of the Kummer plane was determined over several Humbert surfaces* $\mathcal{H}_\Delta$*. In particular, the following statement were proven* [6, Cor. 7.2] *and* [6, Cor. 7.3]*:*

$\Delta = 4$: *if* $(\mathbf{A}, \mathcal{L}) \in \mathcal{H}_4$ *if and only if (numbering the six lines on its Kummer plane* $(\mathbb{P}^2; \mathsf{T}_1, \dots, \mathsf{T}_6)$ *suitably) the three points* $\mathsf{T}_1 \cap \mathsf{T}_2$*,* $\mathsf{T}_3 \cap \mathsf{T}_4$*,* $\mathsf{T}_5 \cap \mathsf{T}_6$ *are collinear.*

$\Delta = 16$: *if* $(\mathbf{A}, \mathcal{L}) \in \mathcal{H}_{16}$ *then its Kummer plane* $(\mathbb{P}^2; \mathsf{T}_1, \dots, \mathsf{T}_6)$ *admits a cubic passing smoothly through three of the 15 points* $\mathsf{T}_m \cap \mathsf{T}_n$ *and touching the singular lines* T_n *in the remaining intersection points with even multiplicity. Conversely, if* $(\mathbb{P}^2; \mathsf{T}_1, \dots, \mathsf{T}_6)$ *admits such a curve, then* $(\mathbf{A}, \mathcal{L}) \in \mathcal{H}_\Delta$ *with* $\Delta \in \{4, 8, 12, 16, 20\}$*.*

For the transcendental lattice it follows $\mathrm{T}_{\mathrm{Kum}(\mathbf{A})} \cong \mathrm{T}_{\mathbf{A}}(2)$ [44, Thm. 10]*. In particular we have*

$$\begin{aligned} \mathrm{T}_{\mathrm{Kum}(\mathbf{A})} &= H(2) \oplus \langle 4 \rangle \oplus \langle -4 \rangle \text{ for } (\mathbf{A}, \mathcal{L}) \in \mathcal{H}_4\,, \\ \mathrm{T}_{\mathrm{Kum}(\mathbf{A})} &= H(2) \oplus \langle 8 \rangle \oplus \langle -8 \rangle \text{ for } (\mathbf{A}, \mathcal{L}) \in \mathcal{H}_{16}\,. \end{aligned} \tag{2.18}$$

Generally, for $\mathbf{A} \in \mathcal{H}_{\delta^2}$ *it follows from* [44] *that* $\mathrm{Kum}(\mathbf{A})$ *has Picard rank 18, and the transcendental lattice is given by*

$$\mathrm{T}_{\mathrm{Kum}(\mathbf{A})} \cong \mathrm{T}_{\mathbf{A}}(2) = H(2) \oplus \langle 2\delta \rangle \oplus \langle -2\delta \rangle\,. \tag{2.19}$$

2.2.1. *Closely related normal forms.* The Shioda sextic is closely related to two quartic equations, known as the *Baker quartic* and *Cassels-Flynn quartic*. The Baker quartic is obtained directly from the Shioda sextic $\mathcal{W}$ in Equation (2.16). Using parameters

$$\begin{aligned} L_4 &= 1+\lambda_1+\lambda_2+\lambda_3\,, & L_3 &= \lambda_1+\lambda_2+\lambda_3+\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3\,,\\ L_1 &= \lambda_1\lambda_2\lambda_3\,, & L_2 &= \lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2\lambda_3\,, \end{aligned} \tag{2.20}$$

and a variable transformation given by

$$\begin{gathered} \mathbf{W} = K_2\, z_1\,, \qquad \mathbf{X} = K_2\, z_2\,, \qquad \mathbf{Y} = K_2\, z_3\,,\\ \mathbf{Z} = 2\tilde{z}_4 - \Big(L_1 z_1^2 z_2 - 2L_2 z_1^2 z_3 + L_3 z_1 z_2 z_3 - 2L_4 z_1 z_3^2 + z_2 z_3^2\Big)\,, \end{gathered} \tag{2.21}$$

Equation (2.16) is equivalent to the quartic surface $\mathcal{U}$ in $\mathbb{P}^3 = \mathbb{P}(\mathbf{W},\mathbf{X},\mathbf{Y},\mathbf{Z})$ given by the vanishing locus of the *Baker determinant*, i.e.,

$$\mathcal{U}: \quad \begin{vmatrix} 0 & L_1\mathbf{W} & -\mathbf{Z} & \mathbf{Y}\\ L_1\mathbf{W} & 2L_2\mathbf{W}+2\mathbf{Z} & L_3\mathbf{W}-\mathbf{Y} & \mathbf{X}\\ -\mathbf{Z} & L_3\mathbf{W}-\mathbf{Y} & 2L_4\mathbf{W}-2\mathbf{X} & \mathbf{W}\\ \mathbf{Y} & \mathbf{X} & \mathbf{W} & 0 \end{vmatrix} = 0\,. \tag{2.22}$$

The Baker determinant was first derived in [3].

In addition to K_2 introduced above, we define homogeneous polynomials $K_l = K_l(z_1,z_2,z_3)$ of degree $4-l$ for $l=0,1$ with

$$\begin{aligned} &K_2 = z_2^2 - 4z_1z_3\,, \qquad K_1 = -2z_2z_3^2 - 2L_1z_1^2z_2 + 4L_2z_1^2z_3 - 2L_3z_1z_2z_3 + 4L_4z_1z_3^2\,,\\ &K_0 = L_1^2z_1^4 - 2L_1L_3z_1^3z_3 + (2L_1 - 4L_2L_4 + L_3^2)z_1^2z_3^2 + 4L_1L_4z_1^2z_2z_3\\ &\qquad + 3(-2L_1z_2^2z_3 + 2L_2z_2z_3^2 - L_3z_3^3)z_1 + z_3^4\,. \end{aligned} \tag{2.23}$$

A variable transformation in Equation (2.16), given by

$$\tilde{z}_4 = \frac{1}{4}\left(K_2z_4 + 2K_1\right)\,, \tag{2.24}$$

or, equivalently, the variable transformation in Equation (2.22), given by

$$\mathbf{W} = K_2\, z_1\,, \qquad \mathbf{X} = K_2\, z_2\,, \qquad \mathbf{Y} = K_2\, z_3\,, \qquad \mathbf{Z} = K_2z_4 + K_1\,, \tag{2.25}$$

yields the quartic projective surface $\mathcal{K}_{\mathbf{A}}$ in $\mathbb{P}^3 = \mathbb{P}(z_1,z_2,z_3,z_4)$ given by

$$\mathcal{K}_{\mathbf{A}}: \quad K_2(z_1,z_2,z_3)\, z_4^2 \;+\; K_1(z_1,z_2,z_3)\, z_4 \;+\; K_0(z_1,z_2,z_3) = 0\,. \tag{2.26}$$

The quartic appeared in Cassels and Flynn [10, Sec. 3] and is called the *Cassels-Flynn quartic*. We have the following:

Proposition 2.11. *Assume that $\lambda_1,\lambda_2,\lambda_3$ are the Rosenhain roots of a smooth genus-two curve $\mathcal{C}$. The surfaces in Equations (2.16), (2.22), and (2.26) are birational equivalent over $\mathbb{Q}(\lambda_1,\lambda_2,\lambda_3)$. The minimal resolution is isomorphic to the Kummer surface* $\mathrm{Kum}(\mathbf{A})$ *associated with the Jacobian* $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$.

Proof. It is known that under birational equivalence the involution minus $-\mathbb{I}$ on $\mathrm{Jac}\,(\mathcal{C})$ restricts to the product involution $\imath_{\mathcal{C}} \times \imath_{\mathcal{C}}$ on $\mathrm{Sym}^2(\mathcal{C})$; see [14]. Hence, the Shioda sextic in Equation (2.16) is birational to $\mathrm{Jac}\,(\mathcal{C})/\langle -\mathbb{I}\rangle$ associated with the Jacobian $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$ of a genus-two curve $\mathcal{C}$ in Rosenhain normal form (1.1). One

nodes on $\mathcal{K}_{\mathbf{A}}$	$[z_1 : z_2 : z_3 : z_4]$
p_0	$[0:0:0:1]$
p_{16}	$[0:1:\lambda_1:\lambda_1^2]$
p_{26}	$[0:1:\lambda_2:\lambda_2^2]$
p_{36}	$[0:1:\lambda_3:\lambda_3^2]$
p_{46}	$[0:1:0:0]$
p_{56}	$[0:1:1:1]$
p_{14}	$[1:\lambda_1:0:\lambda_2\lambda_3]$
p_{24}	$[1:\lambda_2:0:\lambda_1\lambda_3]$
p_{34}	$[1:\lambda_3:0:\lambda_1\lambda_2]$
p_{45}	$[1:1:0:\lambda_1\lambda_2\lambda_3]$
p_{15}	$[1:\lambda_1+1:\lambda_1:\lambda_1(\lambda_2+\lambda_3)]$
p_{25}	$[1:\lambda_2+1:\lambda_2:\lambda_2(\lambda_1+\lambda_3)]$
p_{35}	$[1:\lambda_3+1:\lambda_3:\lambda_3(\lambda_1+\lambda_2)]$
p_{13}	$[1:\lambda_1+\lambda_3:\lambda_1\lambda_3:(\lambda_2+1)\,\lambda_1\lambda_3]$
p_{23}	$[1:\lambda_2+\lambda_3:\lambda_2\lambda_3:(\lambda_1+1)\,\lambda_1\lambda_2]$
p_{12}	$[1:\lambda_1+\lambda_2:\lambda_1\lambda_2:(\lambda_3+1)\,\lambda_1\lambda_2]$

TABLE 2. Nodes on the generic Jacobian Kummer surface $\mathcal{K}_{\mathbf{A}}$

substitutes Equations (2.21) into Equation (2.22) and checks that the strict transform is given by Equation (2.3). Moreover, the variable transformation in Equation (2.22) can be inverted to obtain a rational inverse map. □

Remark 2.12. *The sixteen nodes p_{ij} of the Kummer surface $\mathcal{K}_{\mathbf{A}}$ in Equation (2.26) are given in Table 2. They are the images of the order-two points $P_{ij} \in \mathbf{A}[2]$.*

A holomorphic two-form on $\mathcal{W}$ is given by the two-form $\omega_{\mathcal{W}} = dz_2 \wedge dz_3/\tilde{z}_4$ in the coordinate chart $z_1 = 1$ in Equation (2.16). A regular two-form on $\mathcal{C}\times\mathcal{C}$, given as the antisymmetric linear combination of the outer tensor products of the two holomorphic one-forms on each copy of $\mathcal{C}$, i.e.,

$$\frac{dX^{(1)}}{Y^{(1)}} \boxtimes \frac{X^{(2)}dX^{(2)}}{Y^{(2)}} - \frac{X^{(1)}dX^{(1)}}{Y^{(1)}} \boxtimes \frac{dX^{(2)}}{Y^{(2)}}\,, \tag{2.27}$$

descends to $\mathrm{Sym}^2(\mathcal{C})/\langle \imath_{\mathcal{C}} \times \imath_{\mathcal{C}}\rangle$ and coincides with $\omega_{\mathcal{W}}$. If ϵ is the minimal resolution, we obtain a holomorphic two-form $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})}$ on $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ by pullback, i.e.,

$$\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})} = \epsilon^*\omega_{\mathcal{W}}\,, \quad \text{with } \epsilon\colon \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}) = \widehat{\mathcal{W}} \dashrightarrow \mathcal{W}\,. \tag{2.28}$$

We also introduce the twisted Shioda sextic $\mathcal{W}^{(\varepsilon)}$, given by

$$\mathcal{W}^{(\varepsilon)}: \quad \hat{z}_4^2 = \varepsilon z_1 z_3\big(z_1 - z_2 + z_3\big)\prod_{i=1}^{3}\big(\lambda_i^2\, z_1 - \lambda_i\, z_2 + z_3\big)\,, \tag{2.29}$$

such that for general $\varepsilon \in \mathbb{Q}$ the surfaces $\mathcal{W}^{(\varepsilon)}$ and $\mathcal{W}$ are isomorphic only over $\mathbb{Q}(\sqrt{\varepsilon})$. Further, we introduce another genus-two curve, namely

$$\widetilde{\mathcal{C}}: \quad Y^2 = XZ\big(X-\lambda_0 Z\big)\big(X-\lambda_0\lambda_1 Z\big)\big(X-\lambda_0\lambda_2 Z\big)\big(X-\lambda_0\lambda_3 Z\big)\,, \tag{2.30}$$

and its associated (twisted) Shioda sextic given by

$$\widetilde{\mathcal{W}}^{(\varepsilon)}: \quad \hat{y}_4^2 = \varepsilon y_1 y_3 \big(\lambda_0^2 y_1 - \lambda_0 y_2 + y_3\big) \prod_{i=1}^{3} \big((\lambda_0\lambda_i)^2 y_1 - \lambda_0\lambda_i y_2 + y_3\big) . \tag{2.31}$$

We refer to the minimal resolutions of $\mathcal{W}^{(\varepsilon)}$ and $\widetilde{\mathcal{W}}^{(\varepsilon)}$ as twisted Kummer surfaces and denote them as $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})^{(\varepsilon)}$ and $\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}})^{(\varepsilon)}$, respectively. Holomorphic two-forms on the various twisted Shioda sextic surfaces and their minimal resolutions are then obtained analogously to Equation (2.28). We have the following:

Proposition 2.13. *Assume $\lambda_1, \lambda_2, \lambda_3$ are the Rosenhain roots of a smooth genus-two curve $\mathcal{C}$ and $\lambda_0 \neq 0$. The surfaces $\mathcal{W}^{(\varepsilon)}$ and $\widetilde{\mathcal{W}}^{(\varepsilon/\lambda_0^4)}$ are birational equivalent over $\mathbb{Q}(\lambda_0)$ such that the holomorphic two-form $dy_2 \wedge dy_3/\hat{y}_4$ in the chart $y_1 = 1$ in Equation (2.31) equals $dz_2 \wedge dz_3/\hat{z}_4$ in the chart $z_1 = 1$ in Equation (2.16). In particular, there is an isomorphism*

$$\imath: \quad \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})^{(\varepsilon)} \xrightarrow{\cong} \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}})^{(\varepsilon/\lambda_0^4)} , \tag{2.32}$$

such that $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})^{(\varepsilon)}} = \imath^* \omega_{\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}})^{(\varepsilon/\lambda_0^4)}}$.

Proof. For the isomorphism $\imath : (z_1, z_2, z_3, \hat{z}_4) \mapsto (y_1, y_2, y_3, \hat{y}_4) = (z_1, \lambda_0 z_2, \lambda_0^2 z_3, \lambda_0^3 \hat{z}_4)$ the pullback $\imath^*(dy_2 \wedge dy_3/\hat{y}_4)$ equals $dz_2 \wedge dz_3/\hat{z}_4$. □

2.2.2. *Certain elliptic fibrations.* The pencil of lines, given by $tz_1 - z_3 = 0$ in $\mathbb{P}^2 = \mathbb{P}(z_1, z_2, z_3)$, induces an elliptic fibration on the Shioda sextic $\mathcal{W}$. The elliptic fibration, with fibers over $\mathbb{P}^1 = \mathbb{P}(u_1, u_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(X, Y, Z)$, is given by the Weierstrass model

$$\begin{aligned} Y^2 Z = X\Big(X &+ (\lambda_1 - \lambda_3)(\lambda_2 - 1) u_1 u_2 \big(u_1 - \lambda_2 u_2\big)\big(u_1 - \lambda_1\lambda_3 u_2\big) Z\Big) \\ &\times \Big(X + (\lambda_1 - \lambda_2)(\lambda_3 - 1) u_1 u_2 \big(u_1 - \lambda_3 u_2\big)\big(u_1 - \lambda_1\lambda_2 u_2\big) Z\Big) . \end{aligned} \tag{2.33}$$

The discriminant function of the elliptic fibration is given by

$$\begin{aligned} &(\lambda_1 - 1)^2 (\lambda_2 - 1)^2 (\lambda_3 - 1)^2 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2 u_1^6 u_2^6 (u_1 - \lambda_1 u_2)^2 \\ &\times (u_1 - \lambda_2 u_2)^2 (u_1 - \lambda_3 u_2)^2 (u_1 - \lambda_1\lambda_2 u_2)^2 (u_1 - \lambda_1\lambda_3 u_2)^2 (u_1 - \lambda_2\lambda_3 u_2)^2 . \end{aligned}$$

The fibration in Equation (2.33) was denoted fibration (1) in [34]. In [34] the following lemma was proven:

Lemma 2.14. *Equation (2.33) defines a Jacobian elliptic fibration with the singular fibers $6I_2 + 2I_0^*$ and a Mordell-Weil group of sections $\mathbb{Z}/2\mathbb{Z} + \langle 1 \rangle$.*

We also have the following:

Proposition 2.15. *Assume that $\lambda_1, \lambda_2, \lambda_3$ are the Rosenhain roots of a smooth genus-two curve. The surfaces in Equation (2.16) and (2.33) are birational equivalent over $\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)$.*

Proof. In the charts $u_2 = 1, Z = 1$ in Equation (2.33) and $z_1 = 1$ in Equation (2.16), a birational equivalence is given

$$\begin{aligned} z_2 &= \frac{(\lambda_1-1)(\lambda_2-1)(\lambda_3-1)u_1(u_1-\lambda_1)(u_1-\lambda_2)(u_1-\lambda_3)}{X-\lambda_1(\lambda_2-1)(\lambda_3-1)u_1(u_1-\lambda_2)(u_1-\lambda_3)} + u_1 + 1\,, \\ \tilde{z}_4 &= \frac{(\lambda_1-1)(\lambda_2-1)(\lambda_3-1)u_1(u_1-\lambda_1)(u_1-\lambda_2)(u_1-\lambda_3)Y}{\big(X-\lambda_1(\lambda_2-1)(\lambda_3-1)u_1(u_1-\lambda_2)(u_1-\lambda_3)\big)^2}\,, \qquad z_3 = u_1\,. \end{aligned} \tag{2.34}$$

□

The holomorphic two-forms are related as follows:

Corollary 2.16. *The birational equivalence of Proposition 2.15 identifies the holomorphic two-form $dz_2 \wedge dz_3/\tilde{z}_4$ in the chart $z_1 = 1$ in Equation (2.16) with $du_1 \wedge dX/Y$ in the chart $Z = u_2 = 1$ in Equation (2.33).*

Proof. The proof follows by an explicit computation using the transformation provided in the proof of Proposition 2.15. □

On the Shioda sextic $\mathcal{W}$, another elliptic fibration, with fibers over $\mathbb{P}^1 = \mathbb{P}(w_1, w_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(x, y, z)$, is given by the Weierstrass model

$$\begin{aligned} y^2 z = x^3 &+ w_1\Big(w_1 + (\lambda_1 + \lambda_2\lambda_3)w_2\Big)\Big(w_1 + (\lambda_2 + \lambda_1\lambda_3)w_2\Big)\Big(w_1 + (\lambda_3 + \lambda_1\lambda_2)w_2\Big)x^2 z \\ &+ \lambda_1\lambda_2\lambda_3 w_2^2\Big(w_1 + (\lambda_1 + \lambda_2\lambda_3)w_2\Big)^2\Big(w_1 + (\lambda_2 + \lambda_1\lambda_3)w_2\Big)^2\Big(w_1 + (\lambda_3 + \lambda_1\lambda_2)w_2\Big)^2 x z^2. \end{aligned} \tag{2.35}$$

The discriminant function of the elliptic fibration is given by

$$\begin{aligned} &\lambda_1^2\lambda_2^2\lambda_3^2 w_2^4\Big(w_1^2 - 4\lambda_1\lambda_2\lambda_3 w_2^2\Big)\Big(w_1 + (\lambda_1 + \lambda_2\lambda_3)w_2\Big)^6 \\ &\times \Big(w_1 + (\lambda_2 + \lambda_1\lambda_3)w_2\Big)^6\Big(w_1 + (\lambda_3 + \lambda_1\lambda_2)w_2\Big)^6. \end{aligned}$$

We have the immediate:

Lemma 2.17. *Equation (2.35) defines a Jacobian elliptic fibration with the singular fibers $I_4 + 2I_1 + 3I_0^*$ and a Mordell-Weil group of sections $\mathbb{Z}/2\mathbb{Z}$.*

The fibration in Equation (2.35) was denoted fibration (4) in [34]. We have the following:

Proposition 2.18. *Assume that $\lambda_1, \lambda_2, \lambda_3$ are the Rosenhain roots of a smooth genus-two curve. The surfaces in Equation (2.16) and (2.35) are birational equivalent over $\mathbb{Q}(\lambda_1, \lambda_2, \lambda_3)$.*

Proof. In the charts $w_2 = 1, z = 1$ in Equation (2.35) and $z_1 = 1$ in Equation (2.16), a birational equivalence is given by

$$w_1 = \frac{\big(\lambda_1(\lambda_2-\lambda_3)-\lambda_2\lambda_3(\lambda_1+1)\big)z_2z_3+\lambda_1\lambda_2\lambda_3(1+\lambda_1+\lambda_2-\lambda_3)z_2+\lambda_3(1+\lambda_1+\lambda_2-\lambda_3)z_3^2}{\lambda_3z_2z_3-\lambda_1\lambda_2z_2-z_3^2+\big((\lambda_2-\lambda_3)(\lambda_1+1)+\lambda_1-\lambda_2\lambda_3\big)z_3+\lambda_1\lambda_2\lambda_3}$$
$$-\frac{(\lambda_1^2\lambda_2+\lambda_1(1+\lambda_2)(\lambda_2-\lambda_3^2)-\lambda_2\lambda_3^2)z_3+\lambda_1\lambda_2\big(\lambda_2\lambda_3+\lambda_1(\lambda_2\lambda_3-\lambda_2+\lambda_3)\big)}{\lambda_3z_2z_3-\lambda_1\lambda_2z_2-z_3^2+\big((\lambda_2-\lambda_3)(\lambda_1+1)+\lambda_1-\lambda_2\lambda_3\big)z_3+\lambda_1\lambda_2\lambda_3},$$

$$x = \frac{(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2(1-\lambda_3)^2z_3(z_2-z_3-1)(z_3-\lambda_1)(z_3-\lambda_2)(z_3-\lambda_1\lambda_2)(\lambda_1z_2-z_3-\lambda_1^2)(\lambda_2z_2-z_3-\lambda_2^2)}{\big(\lambda_3z_2z_3-\lambda_1\lambda_2z_2-z_3^2+\big((\lambda_2-\lambda_3)(\lambda_1+1)+\lambda_1-\lambda_2\lambda_3\big)z_3+\lambda_1\lambda_2\lambda_3\big)^3},$$

$$y = \frac{(\lambda_3-\lambda_1)^3(\lambda_2-\lambda_3)^3(\lambda_3-1)^3(z_2-z_3-1)(z_3-\lambda_1)^2(z_3-\lambda_2)^2(z_3-\lambda_1\lambda_2)^2(\lambda_1z_2-z_3-\lambda_1^2)(\lambda_2z_2-z_3-\lambda_2^2)\tilde{z}_4}{\big(\lambda_3z_2z_3-\lambda_1\lambda_2z_2-z_3^2+\big((\lambda_2-\lambda_3)(\lambda_1+1)+\lambda_1-\lambda_2\lambda_3\big)z_3+\lambda_1\lambda_2\lambda_3\big)^5}.$$

□

The holomorphic two-forms are related as follows:

Corollary 2.19. *The birational equivalence of Proposition 2.18 identifies the holomorphic two-form $dw_1 \wedge dx/y$ in the chart $z = w_2 = 1$ in Equation (2.35) with $dz_2 \wedge dz_3/\tilde{z}_4$ in the chart $z_1 = 1$ in Equation (2.16).*

Proof. The proof follows by an explicit computation using the transformation provided in the proof of Proposition 2.18. □

2.2.3. *Specialization to Picard rank 18.* We consider the special case of the Kummer surface associated with the Jacobian of the genus-two curve $\mathcal{C}_0$ in Equation (1.17), i.e., a smooth genus-two curve admitting an elliptic involution. We will denote the Shioda sextic in Equation (2.16) for $\lambda_1 = \lambda_2\lambda_3$ by $\mathcal{W}_0$, and the corresponding vanishing locus for the Baker determinant by $\mathcal{U}_0$. Similarly, we use the symbols $\widetilde{\mathcal{C}}_0$, $\mathcal{W}_0^{(\varepsilon)}$, $\widetilde{\mathcal{W}}_0^{(\varepsilon)}$ to refer to the corresponding specialization for $\lambda_1 = \lambda_2\lambda_3$.

Remark 2.20. *The induced involution $\jmath$ on the Shioda sextic $\mathcal{W}_0$ in Equation (2.16) for $\lambda_1 = \lambda_2\lambda_3$ is given by*

$$\jmath\colon \quad \big(z_1,\, z_2,\, z_3,\, \tilde{z}_4\big) \;\mapsto\; \big(z_3,\, \lambda_2\lambda_3 z_2,\, (\lambda_2\lambda_3)^2 z_3,\, (\lambda_2\lambda_3)^3\tilde{z}_4\big). \tag{2.36}$$

In terms of the elliptic fibration already discussed above, the restriction to the curve $\mathcal{C}_0$ in Equation (1.17) is characterized as follows:

Lemma 2.21. *Assume that $\lambda_1, \lambda_2, \lambda_3$ are the Rosenhain roots of a smooth genus-two curve $\mathcal{C}$. We have the following:*

1. *the singular fibers in the fibration of Lemma 2.14 are $I_4 + 4I_2 + 2I_0^*$ if and only if $\lambda_1 = \lambda_2\lambda_3$, $\lambda_2 = \lambda_1\lambda_3$, or $\lambda_3 = \lambda_1\lambda_2$,*
2. *the singular fibers in the fibration of Lemma 2.17 are $I_4 + I_1 + I_1^* + 2I_0^*$ if and only if $\lambda_1 = \lambda_2\lambda_3$, $\lambda_2 = \lambda_1\lambda_3$, or $\lambda_3 = \lambda_1\lambda_2$.*

Proof. We prove the second statement, the first one is analogous. The resultant (with respect to w_1) of the factors in the reduced discriminant corresponding to the fibers

of Kodaira-type I_0^*, i.e.,

$$\Big(w_1 + (\lambda_1 + \lambda_2\lambda_3)w_2\Big)\Big(w_1 + (\lambda_2 + \lambda_1\lambda_3)w_2\Big)\Big(w_1 + (\lambda_3 + \lambda_1\lambda_2)w_2\Big), \tag{2.37}$$

and the factors in the discriminant corresponding to the fibers of Kodaira-type I_1, i.e., $w_1^2 - 4\lambda_1\lambda_2\lambda_3 w_2^2$, yields

$$\Big(\lambda_1 - \lambda_2\lambda_3\Big)^2\Big(\lambda_2 - \lambda_1\lambda_3\Big)^2\Big(\lambda_3 - \lambda_1\lambda_2\Big)^2 . \tag{2.38}$$

□

The elliptic fibrations in Lemma 2.21 and the quadratic twist are related as follows:

Corollary 2.22. *Assume that $\lambda_2, \lambda_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2 \neq \lambda_3^{\pm 1}$. The Weierstrass model, given by*

$$\begin{aligned} Y^2 Z = \varepsilon X\Big(X + \lambda_2(\lambda_3 - 1)^2 u_1 u_2\big(u_1 - \lambda_3 u_2\big)\big(u_1 - \lambda_2^2\lambda_3 u_2\big)Z\Big) \\ \times \Big(X + \lambda_3(\lambda_2 - 1)^2 u_1 u_2\big(u_1 - \lambda_2 u_2\big)\big(u_1 - \lambda_2\lambda_3^2 u_2\big)Z\Big), \end{aligned} \tag{2.39}$$

defines a Jacobian elliptic fibration with the singular fibers $I_4 + 4I_2 + 2I_0^*$ and a Mordell-Weil group of sections $\mathbb{Z}/2\mathbb{Z} + \langle 1\rangle$. The elliptic surface is birational equivalent to $\mathcal{W}_0^{(\varepsilon)}$ over $\mathbb{Q}(\lambda_2\lambda_3, \lambda_2 + \lambda_3)$ and Corollary 2.16 holds.

Corollary 2.23. *Assume that $\lambda_2, \lambda_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2 \neq \lambda_3^{\pm 1}$. The Weierstrass model, given by*

$$\begin{aligned} y^2 z = x^3 + \varepsilon w_1\Big(w_1 + 2\lambda_2\lambda_3 w_2\Big)\Big(w_1 + \lambda_2(1 + \lambda_3^2)w_2\Big)\Big(w_1 + \lambda_3(1 + \lambda_2^2)w_2\Big)x^2 z \\ + \varepsilon^2(\lambda_2\lambda_3)^2 w_2^2\Big(w_1 + 2\lambda_2\lambda_3 w_2\Big)^2\Big(w_1 + \lambda_2(1 + \lambda_3^2)w_2\Big)^2\Big(w_1 + \lambda_3(1 + \lambda_2^2)w_2\Big)^2 x z^2 , \end{aligned} \tag{2.40}$$

defines a Jacobian elliptic fibration with the singular fibers $I_4 + I_1 + I_1^* + 2I_0^*$ and a Mordell-Weil group of sections $\mathbb{Z}/2\mathbb{Z}$. The elliptic surface is birational equivalent to $\mathcal{W}_0^{(\varepsilon)}$ over $\mathbb{Q}(\lambda_2, \lambda_3)$ and Corollary 2.19 holds.

2.3. **Quartic surfaces.** Let us briefly recall the construction of the Göpel-Hudson quartic for a Kummer surface associated with a principally polarized abelian surface $(\mathbf{A}, \mathcal{L})$ based on results in [5]. Considering the rational map $\varphi_{\mathcal{L}^2} : \mathbf{A} \to \mathbb{P}^3$, its image $\varphi_{\mathcal{L}^2}(\mathbf{A})$ is a quartic surface in $\mathbb{P}^3$ which, using the projective coordinates $[\mathbf{w} : \mathbf{x} : \mathbf{y} : \mathbf{z}]$, can be written as

$$\begin{aligned} 0 = \xi_0\left(\mathbf{w}^4 + \mathbf{x}^4 + \mathbf{y}^4 + \mathbf{z}^4\right) + \xi_4\,\mathbf{wxyz} \\ + \xi_1\left(\mathbf{w}^2\mathbf{z}^2 + \mathbf{x}^2\mathbf{y}^2\right) + \xi_2\left(\mathbf{w}^2\mathbf{x}^2 + \mathbf{y}^2\mathbf{z}^2\right) + \xi_3\left(\mathbf{w}^2\mathbf{y}^2 + \mathbf{x}^2\mathbf{z}^2\right), \end{aligned} \tag{2.41}$$

for some parameter set $[\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4] \in \mathbb{P}^4$. A general member of the family (2.41) is smooth. As soon as the surface is singular at a general point, it must have sixteen singular nodal points because of its symmetry. The discriminant turns out to be a homogeneous polynomial of degree eighteen in the parameters $[\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4] \in \mathbb{P}^4$ and was determined in [5, Sec. 7.7 (3)]. Thus, the Kummer surfaces form an open set among these surfaces with parameters $[\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4] \in \mathbb{P}^4$, namely the ones that make the irreducible factor of degree three in the discriminant vanish, i.e.,

$$\xi_0\left(16\xi_0^2 - 4\xi_1^2 - 4\xi_2^2 - 4\xi_3^3 + \xi_4^2\right) + 4\,\xi_1\xi_2\xi_3 = 0 . \tag{2.42}$$

Setting $\xi_0 = 1$ and using the affine moduli $\xi_1 = -A$, $\xi_2 = -B$, $\xi_3 = -C$, $\xi_4 = 2D$, we obtain a normal form for a nodal quartic surface, known as *Göpel-Hudson quartic* (GH-quartic). The Göpel-Hudson quartic is the projective surface in $\mathbb{P}^3 = \mathbb{P}(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ given by

$$\begin{aligned} 0 &= \mathbf{w}^4 + \mathbf{x}^4 + \mathbf{y}^4 + \mathbf{z}^4 + 2D\mathbf{wxyz} \\ &\quad -A\big(\mathbf{w}^2\mathbf{z}^2 + \mathbf{x}^2\mathbf{y}^2\big) - B\big(\mathbf{w}^2\mathbf{x}^2 + \mathbf{y}^2\mathbf{z}^2\big) - C\big(\mathbf{w}^2\mathbf{y}^2 + \mathbf{x}^2\mathbf{z}^2\big)\,, \end{aligned} \tag{2.43}$$

where $A, B, C, D \in \mathbb{C}$ satisfy

$$D^2 = A^2 + B^2 + C^2 + ABC - 4\,. \tag{2.44}$$

By construction, we have the following:

Lemma 2.24. *The minimal resolution of the quartic surface in Equation (2.43), for parameters (A, B, C, D) satisfying (2.44) is isomorphic to the Kummer surface* $\mathrm{Kum}(\mathbf{A})$ *associated with a principally polarized abelian surface* $\mathbf{A}$.

The abelian surface in Lemma 2.24 for generic parameters (A, B, C, D) does *not* admit any additional automorphism. In [14, Thm. 4.46] two of the authors determined explicitly the connection between the parameters (A, B, C, D) and the moduli of the abelian surface $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C})$ for a smooth genus-two curve $\mathcal{C}$. It was also proved in [14] that (A, B, C, D) are modular functions relative to $\Gamma_2(2)$. We have the following:

Proposition 2.25. *The minimal resolution of the Hudson quartic in Equation (2.43), for parameters (A, B, C, D) satisfying Equation (2.44), is isomorphic to the Kummer surface* $\mathrm{Kum}(\mathbf{A})$ *associated with a principally polarized abelian surface* $\mathbf{A} = \mathrm{Jac}\,(\mathcal{C}_0)$ *for a smooth genus-two curve $\mathcal{C}_0$ admitting an elliptic involution if and only if one of the following additional relations holds:*

$$\begin{array}{ll} \textit{case I}: & D = 0\,, \\ \textit{case II}: & B = \pm C\,, \\ \textit{case III}: & A = \pm B\,, \quad \textit{or} \quad A = \pm C\,, \\ \textit{case IV}: & \left\{ \begin{array}{lll} A + B + C \pm D + 6 = 0\,, & \textit{or} & -A - B + C \pm D + 6 = 0\,, \quad \textit{or} \\ A - B - C \pm D + 6 = 0\,, & \textit{or} & -A + B - C \pm D + 6 = 0\,. \end{array} \right. \end{array} \tag{2.45}$$

Remark 2.26. *Table 3 shows the one-to-one correspondence between the 15 components in Equation (2.45) and the components in $\mathbb{H}_2/\Gamma_2(2)$ covering $\mathcal{H}_4$ in the Pringsheim decomposition in Proposition 1.3, using the same marking as before.*

Proof. A linear transformation of variables in Equation (2.22) over a suitable finite extension field yields a Göpel quartic in Equation (2.43). In [14, Thm. 4.46] a corresponding solution for the parameters (A, B, C, D) in terms of the Rosenhain roots of a genus-two curve $\mathcal{C}_0$ in Equation (1.1) was determined explicitly. It is given by

$$\begin{aligned} A &= 2\,\frac{\lambda_1 + 1}{\lambda_1 - 1}, \quad B = 2\,\frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_1 + \lambda_2 + \lambda_3}{(\lambda_2 - \lambda_3)(\lambda_1 - 1)}, \quad C = 2\,\frac{\lambda_3 + \lambda_2}{\lambda_3 - \lambda_2}, \\ D &= 4\,\frac{\lambda_1 - \lambda_2\lambda_3}{(\lambda_2 - \lambda_3)(\lambda_1 - 1)}\,, \end{aligned} \tag{2.46}$$

such that Equation (2.44) is satisfied. All other possible solutions are obtained from this one by the 15 transpositions acting on the roots $(\lambda_1, \lambda_2, \lambda_3, 0, 1, \infty)$. This is precisely the relation between the Pringsheim components for $\mathcal{H}_4$ given in Proposition 1.3. We can then rewrite any relation between $\lambda_1, \lambda_2, \lambda_3$ as an additional relation for A, B, C, D. In this way, we obtain a one-to-one correspondence between the 15 Pringsheim components in Proposition 1.3 and the 15 additional relations for the Göpel-Hudson quartic in Equation (2.45). □

I			
$2\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0$	$\lambda_1 = \lambda_2\lambda_3$	$D = 0$	$\{\mathsf{p}_{15}, \mathsf{p}_{23}, \mathsf{p}_{46}\}$
II			
$\tau_{11} + 2\tau_{12} = 0$	$\lambda_1 - \lambda_2 = \lambda_3(1 - \lambda_2)$	$B + C = 0$	$\{\mathsf{p}_{14}, \mathsf{p}_{23}, \mathsf{p}_{56}\}$
$\tau_{11} + 2\tau_{12} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_1(1 - \lambda_3) = \lambda_2(\lambda_1 - \lambda_3)$	$B - C = 0$	$\{\mathsf{p}_{16}, \mathsf{p}_{23}, \mathsf{p}_{45}\}$
III			
$2\tau_{12} - \tau_{22} = 0$	$\lambda_3 = \lambda_1\lambda_2$	$A - C = 0$	$\{\mathsf{p}_{12}, \mathsf{p}_{35}, \mathsf{p}_{46}\}$
$2\tau_{12} - \tau_{22} + (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_2 = \lambda_1\lambda_3$	$A + C = 0$	$\{\mathsf{p}_{13}, \mathsf{p}_{25}, \mathsf{p}_{46}\}$
$\tau_{11} - \tau_{22} = 0$	$\lambda_1 - \lambda_2 = \lambda_2(\lambda_1 - \lambda_3)$	$A + B = 0$	$\{\mathsf{p}_{15}, \mathsf{p}_{26}, \mathsf{p}_{34}\}$
$\tau_{11} - \tau_{22} + (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_1 - \lambda_3 = \lambda_3(\lambda_1 - \lambda_2)$	$A - B = 0$	$\{\mathsf{p}_{15}, \mathsf{p}_{24}, \mathsf{p}_{36}\}$
IV			
$2\tau_{12} = 1$	$\lambda_2 - \lambda_3 = -\lambda_1(1 - \lambda_2)$	$-A + B - C - D + 6 = 0$	$\{\mathsf{p}_{12}, \mathsf{p}_{34}, \mathsf{p}_{56}\}$
$2\tau_{12} + \tau_{22} = 1$	$\lambda_3(1 - \lambda_2) = -\lambda_1(\lambda_2 - \lambda_3)$	$A + B + C + D + 6 = 0$	$\{\mathsf{p}_{12}, \mathsf{p}_{36}, \mathsf{p}_{45}\}$
$\tau_{11} + 2\tau_{12} = 1$	$\lambda_2 - \lambda_3 = \lambda_1(1 - \lambda_3)$	$-A - B + C + D + 6 = 0$	$\{\mathsf{p}_{13}, \mathsf{p}_{24}, \mathsf{p}_{56}\}$
$\tau_{11} + 2\tau_{12} + \tau_{22} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1(\lambda_2 - \lambda_3) = \lambda_2(1 - \lambda_3)$	$A - B - C - D + 6 = 0$	$\{\mathsf{p}_{13}, \mathsf{p}_{26}, \mathsf{p}_{45}\}$
$4\tau_{12} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_2 - \lambda_3 = \lambda_2(\lambda_1 - \lambda_3)$	$-A + B - C + D + 6 = 0$	$\{\mathsf{p}_{14}, \mathsf{p}_{35}, \mathsf{p}_{26}\}$
$4\tau_{12} + \tau_{22} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1 - \lambda_3 = \lambda_1(\lambda_2 - \lambda_3)$	$A + B + C - D + 6 = 0$	$\{\mathsf{p}_{16}, \mathsf{p}_{24}, \mathsf{p}_{35}\}$
$\tau_{11} + 4\tau_{12} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_2 - \lambda_3 = -\lambda_3(\lambda_1 - \lambda_2)$	$-A - B + C - D + 6 = 0$	$\{\mathsf{p}_{14}, \mathsf{p}_{25}, \mathsf{p}_{36}\}$
$\tau_{11} + 4\tau_{12} + \tau_{22} + 2(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1 - \lambda_2 = -\lambda_1(\lambda_2 - \lambda_3)$	$A - B - C + D + 6 = 0$	$\{\mathsf{p}_{16}, \mathsf{p}_{25}, \mathsf{p}_{34}\}$

TABLE 3. Components of $\mathcal{H}_4$ with matching constraints for the GH-quartic

Proposition 2.25 provides a characterization of the 15 components in $\mathbb{H}_2/\Gamma_2(2)$ covering $\mathcal{H}_4$. Because of Remark 2.10 each of the components also corresponds to a configuration in the Kummer plane where three nodes on $\mathcal{K}_{\mathbf{A}}$ in Table 2 are collinear. On the other hand, for each genus-two curve $\mathcal{C}_0$ with an elliptic involution there is precisely one Göpel group $G \leqslant \mathbf{A}[2]$ with $\mathbf{A} = \mathrm{Jac}(\mathcal{C}_0)$ such that $\mathbf{A}/G \cong \mathcal{E}_1 \times \mathcal{E}_2$; see Corollary 1.16. We then have the following:

Corollary 2.27. *For a smooth genus-two curve $\mathcal{C}_0$ admitting an elliptic involution and $\mathbf{A} = \mathrm{Jac}(\mathcal{C}_0)$, the group $G = \{P_0, P_{ij}, P_{kl}, P_{mn}\} \leqslant \mathbf{A}[2]$ is the unique Göpel group such that $\mathbf{A}/G \cong \mathcal{E}_1 \times \mathcal{E}_2$ if and only if for the corresponding Kummer configuration $(\mathbb{P}^2; \mathsf{T}_1, \ldots, \mathsf{T}_6)$ on $\mathcal{K}_{\mathbf{A}}$ the three nodes $\{\mathsf{p}_{ij}, \mathsf{p}_{mn}, \mathsf{p}_{kl}\}$ are collinear.*

Remark 2.28. *Table 3 shows the 15 components in $\mathbb{H}_2/\Gamma_2(2)$ covering $\mathcal{H}_4$ and the corresponding collinear points in the Kummer plane, using the marking of Weierstrass points $(P_1, \ldots, P_6) = (\lambda_1 = \lambda_2\lambda_3, \lambda_2, \lambda_3, 0, 1, \infty)$ for the curve $\mathcal{C}_0$.*

Proof. The one-to-one correspondence between the possible configurations of collinear points in the Kummer plane and the 15 components in $\mathbb{H}_2/\Gamma_2(2)$ covering $\mathcal{H}_4$ is

computed explicitly using their respective characterizations in terms of Rosenhain roots in Table 2 and Table 1, respectively. □

Remark 2.29. *For $A^2 = B^2 = C^2 = 1$ and $D = 0$ in Equation (2.43) one finds the special solutions in Table 4. The minimal resolution of Equation (2.43) is then isomorphic to the Kummer surface* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)$ *of Picard rank $\rho = 20$ where $\mathcal{C}_0$ is given by Equation (1.17) with λ_2, λ_3 as specified in the table. In this case, the two elliptic-curve quotients are isomorphic to the elliptic curve $\mathcal{E}$, given by*

$$\mathcal{E}: \quad y^2 z = x^3 + z^3 ,$$

admitting a $\mathbb{Z}_3$-symmetry $x \mapsto \omega_3 x$ for $\omega_3^3 = 1$.

x	λ_2	λ_3	A	B	C	D	$\Lambda_1\Lambda_2$	$\Lambda_1+\Lambda_2$	$j(\mathcal{E}_1) = j(\mathcal{E}_2)$	ρ
± 1	-1	3	1	1	1	0	1	1	0	20
± 1	3	-1	1	-1	-1	0	1	1	0	20
± 1	-1	$\frac{1}{3}$	-1	1	-1	0	1	1	0	20
± 1	$\frac{1}{3}$	-1	-1	-1	1	0	1	1	0	20

TABLE 4. Solutions for $A^2 = B^2 = C^2 = 1$ and $D = 0$

2.3.1. *Related normal form.* The Shioda sextic $\mathcal{W}_0$ in Equation (2.29) has a presentation as a simple quartic hypersurface, isomorphic to $\mathcal{W}_0$ over $\mathbb{Q}(\lambda_2+\lambda_3, \lambda_2\lambda_3)$. Notice that this is *not* the case for the Göpel-Hudson quartic in Equation (2.43) – there, a finite field extension is needed to construct such an isomorphism.

Each elliptic curve $\mathcal{E}_l$ for $l = 1, 2$, as given in Legendre form in Equation (1.19), can also be represented as a complete quadric intersection in $\mathbb{P}^3$. Let $\mathcal{I}_1$ be the complete intersection of the two quadric surfaces in $\mathbb{P}^3 = \mathbb{P}(\mathbf{X}_{00}, \mathbf{X}_{01}, \mathbf{X}_{10}, \mathbf{X}_{11})$, given by

$$\mathcal{I}_1 : \quad \begin{cases} \mathbf{X}_{01}^2 &= \mathbf{X}_{10}^2 + \mathbf{X}_{11}^2 , \\ \mathbf{X}_{00}^2 &= \mathbf{X}_{10}^2 + \big(1-\Lambda_1\big)\mathbf{X}_{11}^2 , \end{cases} \tag{2.47}$$

and $\mathcal{I}_2$ be the complete intersection in $\mathbb{P}^3 = \mathbb{P}(\mathbf{Y}_{00}, \mathbf{Y}_{01}, \mathbf{Y}_{10}, \mathbf{Y}_{11})$, given by

$$\mathcal{I}_2 : \quad \begin{cases} \mathbf{Y}_{01}^2 &= \mathbf{Y}_{10}^2 + \mathbf{Y}_{11}^2 , \\ \mathbf{Y}_{00}^2 &= \mathbf{Y}_{10}^2 + \big(1-\Lambda_2\big)\mathbf{Y}_{11}^2 . \end{cases} \tag{2.48}$$

We have the following:

Lemma 2.30. *$\mathcal{E}_l$ is birational equivalent to $\mathcal{I}_l$ for $l = 1, 2$ over $\mathbb{Q}(\Lambda_l)$.*

Proof. A rational map $\mathcal{E}_1 \hookrightarrow \mathbb{P}^3, [x_1 : y_1 : z_1] \mapsto [\mathbf{X}_{00}, \mathbf{X}_{01}, \mathbf{X}_{10}, \mathbf{X}_{11}]$ is given by

$$\Big[\mathbf{X}_{00} : \mathbf{X}_{01} : \mathbf{X}_{10} : \mathbf{X}_{11}\Big] = \Big[x_1^2 - 2\Lambda_1 x_1 z_1 + \Lambda_1 z_1^2 : x_1^2 - \Lambda_1 z_1^2 : x_1^2 - 2x_1 z_1 + \Lambda_1 z_1^2 : \pm 2 y_1 z_1\Big]. \tag{2.49}$$

It has a rational inverse $\mathcal{I}_1 \dashrightarrow \mathcal{E}_1, [\mathbf{X}_{00}, \mathbf{X}_{01}, \mathbf{X}_{10}, \mathbf{X}_{11}] \mapsto [x_1 : y_1 : z_1]$, given by

$$\Big[x_1 : y_1 : z_1\Big] \;=\; \Big[\Lambda_1\big(\mathbf{X}_{1,0} - \mathbf{X}_{0,0}\big) : \; \pm\Lambda_1\big(\Lambda_1 - 1\big)\mathbf{X}_{1,1} : \; \big(1-\Lambda_1\big)\mathbf{X}_{0,1} + \Lambda_1\mathbf{X}_{1,0} - \mathbf{X}_{0,0}\Big]. \tag{2.50}$$

Thus, we obtain a birational equivalence between $\mathcal{E}_1$ and $\mathcal{I}_1$. An analogous argument holds for $\mathcal{E}_2$ and $\mathcal{I}_2$. □

There is a well defined map $\tilde{\pi} : \mathcal{I}_1 \times \mathcal{I}_2 \dashrightarrow \mathbb{P}^3$ with $\mathbb{P}^3 = \mathbb{P}(\mathbf{Z}_{00}, \mathbf{Z}_{01}, \mathbf{Z}_{10}, \mathbf{Z}_{11})$ where one sets

$$\Big[\mathbf{Z}_{00} : \mathbf{Z}_{01} : \mathbf{Z}_{10} : \mathbf{Z}_{11}\Big] = \Big[\mathbf{X}_{00}\mathbf{Y}_{00} : \ \mathbf{X}_{01}\mathbf{Y}_{01} : \ \mathbf{X}_{10}\mathbf{Y}_{10} : \ \mathbf{X}_{11}\mathbf{Y}_{11}\Big], \tag{2.51}$$

for

$$\Big[\mathbf{X}_{00} : \mathbf{X}_{01} : \mathbf{X}_{10} : \mathbf{X}_{11}\Big] \in \mathcal{I}_1\,, \qquad \Big[\mathbf{Y}_{00} : \mathbf{Y}_{01} : \mathbf{Y}_{10} : \mathbf{Y}_{11}\Big] \in \mathcal{I}_1\,. \tag{2.52}$$

We have the following:

Theorem 2.31. *Assume that $\Lambda_1, \Lambda_2 \in \mathbb{P}^1\backslash\{0, 1, \infty\}$ and $\Lambda_1 \neq \Lambda_2$. The image $\tilde{\pi}(\mathcal{I}_1 \times \mathcal{I}_2)$ in $\mathbb{P}^3 = \mathbb{P}(\mathbf{Z}_{00}, \mathbf{Z}_{01}, \mathbf{Z}_{10}, \mathbf{Z}_{11})$ is the quartic projective surface $\mathcal{V}_0$, given by*

$$\mathcal{V}_0 : \quad \begin{cases} \mathbf{Z}_{00}^4 + \big(1-\Lambda_1\big)\big(1-\Lambda_2\big)\mathbf{Z}_{01}^4 + \Lambda_1\Lambda_2\mathbf{Z}_{10}^4 + \Lambda_1\Lambda_2\big(1-\Lambda_1\big)\big(1-\Lambda_2\big)\mathbf{Z}_{11}^4 \\ -\big(2-\Lambda_1-\Lambda_2\big)\Big(\mathbf{Z}_{00}^2\mathbf{Z}_{01}^2 + \Lambda_1\Lambda_2\mathbf{Z}_{10}^2\mathbf{Z}_{11}^2\Big) - \big(2\Lambda_1\Lambda_2 - \Lambda_1 - \Lambda_2\big)\Big(\mathbf{Z}_{00}^2\mathbf{Z}_{11}^2 + \mathbf{Z}_{01}^2\mathbf{Z}_{10}^2\Big) \\ -\big(\Lambda_1+\Lambda_2\big)\Big(\mathbf{Z}_{00}^2\mathbf{Z}_{10}^2 + \big(1-\Lambda_1\big)\big(1-\Lambda_2\big)\mathbf{Z}_{01}^2\mathbf{Z}_{11}^2\Big) = 0\,. \end{cases} \tag{2.53}$$

The minimal resolution is isomorphic to a Kummer surface of Picard rank 18. In particular, the surfaces $\mathcal{V}_0$ and $\mathcal{W}_0$ in Equation (2.29) for parameters satisfying (1.20) are birational equivalent over $\mathbb{Q}(\lambda_2\lambda_3, \lambda_2 + \lambda_3)$.

Proof. We multiply (pairwise) Equations (2.47) and (2.47) and then use the variables in Equation (2.51). Upon eliminating the remaining variables we obtain Equation (2.53). For elliptic moduli K_l and complementary elliptic moduli K_l' with $\Lambda_l = K_l^2 = 1 - (K_l')^2 \in \mathbb{P}^1\backslash\{0, 1, \infty\}$ for $l = 1, 2$, the surface $\mathcal{V}_0$ is isomorphic over $\mathbb{Q}(\sqrt{K_1K_2}, \sqrt{K_1'K_2'})$ to the equation in $\mathbb{P}^3 = \mathbb{P}(\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z})$ given by

$$\begin{aligned} 0 \ = \ \mathbf{w}^4 + \mathbf{x}^4 + \mathbf{y}^4 + \mathbf{z}^4 - \frac{(K_1')^2 + (K_2')^2}{K_1'K_2'}\Big(\mathbf{w}^2\mathbf{z}^2 + \mathbf{x}^2\mathbf{y}^2\Big) \\ -\frac{K_1^2 + K_2^2}{K_1K_2}\Big(\mathbf{w}^2\mathbf{y}^2 + \mathbf{x}^2\mathbf{z}^2\Big) + \frac{(K_1K_2')^2 + (K_1'K_2)^2}{K_1K_2K_1'K_2'}\Big(\mathbf{w}^2\mathbf{x}^2 + \mathbf{y}^2\mathbf{z}^2\Big)\,. \end{aligned} \tag{2.54}$$

In fact, in Equation (2.53) we can rescale

$$\mathbf{Z}_{00} = \mathbf{x}\,, \quad \mathbf{Z}_{01} = \frac{\mathbf{y}}{\sqrt{K_1'K_2'}}\,, \quad \mathbf{Z}_{10} = \frac{\mathbf{z}}{\sqrt{K_1K_2}}\,, \quad \mathbf{Z}_{11} = \frac{\mathbf{w}}{\sqrt{K_1K_2}\sqrt{K_1'K_2'}}\,, \tag{2.55}$$

and obtain Equation (2.54). We check that the latter is a Göpel-Hudson quartic in Equation (2.43) with $D = 0$. According to Proposition 2.25, its minimal resolution is a Jacobian Kummer surface of Picard rank 18.

For parameters satisfying Equations (1.20), a map $\mathcal{V}_0 \to \mathcal{U}_0$, given by

$$[\mathbf{Z}_{00} : \mathbf{Z}_{01} : \mathbf{Z}_{10} : \mathbf{Z}_{11}] \quad \mapsto \quad [\mathbf{W} : \mathbf{X} : \mathbf{Y} : \mathbf{Z}]\,,$$

$$\begin{aligned} \text{with} \qquad \mathbf{W} &= \mathbf{Z}_{0,1} - \mathbf{Z}_{1,0} - \mathbf{Z}_{1,1}\,, \\ \mathbf{X} &= -(1-\lambda_2)(1-\lambda_3)\mathbf{Z}_{0,0} + (1+\lambda_2\lambda_3)\mathbf{Z}_{0,1} - (\lambda_2+\lambda_3)\mathbf{Z}_{1,0}\,, \\ \mathbf{Y} &= \lambda_2\lambda_3\big(\mathbf{Z}_{0,1} - \mathbf{Z}_{1,0} + \mathbf{Z}_{1,1}\big)\,, \\ \mathbf{Z} &= -\lambda_2\lambda_3\big((1-\lambda_2)(1-\lambda_3)\mathbf{Z}_{0,0} + (1+\lambda_2\lambda_3)\mathbf{Z}_{0,1} - (\lambda_2+\lambda_3)\mathbf{Z}_{1,0}\big)\,, \end{aligned} \tag{2.56}$$

is an isomorphism, defined over $\mathbb{Q}(\lambda_2\lambda_3, \lambda_2+\lambda_3)$, between the quartic surface $\mathcal{V}_0$ in Equation (2.53) and $\mathcal{U}_0$ in Equation (2.22) with $\lambda_1 = \lambda_2\lambda_3$. Substituting Equations (2.56) into Equation (2.22), we obtain Equation (2.53) up to a non-vanishing scale factor. Inverting Equations (2.56) yields

$$\begin{aligned} \mathbf{Z}_{0,0} &= \lambda_2\lambda_3\mathbf{X} + \mathbf{Z}\,, \\ \mathbf{Z}_{0,1} &= \lambda_2\lambda_3(\lambda_2+\lambda_3)\mathbf{W} - \lambda_2\lambda_3\mathbf{X} + (\lambda_2+\lambda_3)\mathbf{Y} + \mathbf{Z}\,, \\ \mathbf{Z}_{1,0} &= \lambda_2\lambda_3(\lambda_2\lambda_3+1)\mathbf{W} - \lambda_2\lambda_3\mathbf{X} + (\lambda_2\lambda_3+1)\mathbf{Y} + \mathbf{Z}\,, \\ \mathbf{Z}_{1,1} &= -\lambda_2\lambda_3(1-\lambda_2)(1-\lambda_3)\mathbf{W} + (1-\lambda_2)(1-\lambda_3)\mathbf{Y}\,. \end{aligned} \tag{2.57}$$

Moreover, Equations (2.24) and (2.25) provide a birational equivalence between $\mathcal{U}_0$ and $\mathcal{W}_0$. It is easy to see that for $\lambda_1 = \lambda_2\lambda_3$ in Equations (2.20) this equivalence is well defined over $\mathbb{Q}(\lambda_2\lambda_3, \lambda_2+\lambda_3)$ □

Remark 2.32. *The surface $\mathcal{V}_0$ in Equation (2.53) is the Kummer-surface analogue of the genus-two curve in Equation (1.34) obtained by Legendre's gluing method.*

2.4. **Geometric isogeny.** We proved in Proposition 1.6 that the smooth curve $\mathcal{C}_0$ in Equation (1.17) admits an elliptic involution with the elliptic-curve quotients $\mathcal{E}_l$ for $l = 1, 2$ and rational quotient maps $\pi_{\mathcal{E}_l} : \mathcal{C}_0 \dashrightarrow \mathcal{E}_l$. In this situation, we consider the rational map, given by

$$\psi_0 : \quad \mathcal{C}_0 \times \mathcal{C}_0 \longrightarrow \mathcal{E}_1 \times \mathcal{E}_2\,, \qquad (P,Q) \mapsto \Big(\pi_{\mathcal{E}_1}(P) \oplus \pi_{\mathcal{E}_1}(Q),\ \pi_{\mathcal{E}_2}(P) \oplus \pi_{\mathcal{E}_2}(Q)\Big)\,, \tag{2.58}$$

where the symbol $\oplus$ refers to the addition of two points on the elliptic curve $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. One can easily show that the argument from [55, Sec. 4.6] extends, and the map ψ_0 induces a geometric isogeny ψ between the Shioda sextic $\mathcal{W}_0$ in Equation (2.16) with $\lambda_1 = \lambda_2\lambda_3$ and the double quadric surface $\mathcal{Z}$ in Equation (2.3). If we additionally use the fact that the Jacobian $\mathrm{Jac}\,(\mathcal{C}_0)$ is birational to the symmetric product $\mathrm{Sym}^2(\mathcal{C}_0)$, it follows that the map ψ is related to the $(2,2)$-isogeny in Equation (1.39) by the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{C}_0 \times \mathcal{C}_0 & \longrightarrow & \mathcal{W}_0 & \longleftarrow & \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0) & \longleftarrow & \mathrm{Jac}\,(\mathcal{C}_0) \\ \psi_0 \downarrow & & \downarrow \psi & & \downarrow & & \downarrow \Psi \\ \mathcal{E}_1 \times \mathcal{E}_2 & \longrightarrow & \mathcal{Z} & \longleftarrow & \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2) & \longleftarrow & \mathcal{E}_1 \times \mathcal{E}_2 \end{array} \tag{2.59}$$

Here, the horizontal arrows represent canonical quotient maps. We will now show that, relative to the quartic surface $\mathcal{V}_0$ in Equation (2.53), the geometric isogeny ψ will take a simple form and can be constructed explicitly.

By construction of the map ψ_0 and Proposition 1.6, the isogeny ψ is defined over the field $\mathbb{Q}(q, r^2) \cong \mathbb{Q}(\sqrt{\lambda_2\lambda_3}, \lambda_1 + \lambda_2)$ for $q^2 = \lambda_2\lambda_3$ and $r^2 = (1-\lambda_2)(1-\lambda_3)$. We then have the following:

Lemma 2.33. *For the holomorphic two-forms $\omega_{\mathcal{W}_0} = dz_2 \wedge dz_3/\tilde{z}_4$ in the chart $z_1 = 1$ in Equation (2.16) and $\omega_{\mathcal{Z}} = dx_1 \wedge dx_2/y_{1,2}$ in the chart $z_1 = z_2 = 1$ in Equation (2.3), it follows*

$$\psi^*\left(\frac{dx_1 \wedge dx_2}{y_{12}}\right) = \delta \cdot \frac{dz_2 \wedge dz_3}{\tilde{z}_4} \tag{2.60}$$

with $\delta = 2\sqrt{\lambda_2\lambda_3}(1-\lambda_2)(1-\lambda_3)$.

Proof. The smooth genus-two curve $\mathcal{C}_0$ given by Equation (1.17) is defined by a polynomial of degree five. Thus, in the chart $Z = 1$, the regular differentials from a vector space with a basis dX/Y and XdX/Y. Similarly, in the charts $z_l = 1$ on the elliptic curves $\mathcal{E}_l$ holomorphic one-forms are given by dx_l/y_l for $l = 1, 2$. One checks by a direct computation that their pullbacks are give by

$$p_l^*\left(\frac{dx_l}{y_l}\right) = \left(p_{l1}X + p_{l0}\right)\frac{dX}{Y} = \left(rX - (-1)^l rq\right)\frac{dX}{Y}\,. \tag{2.61}$$

The independence of the pullbacks is equivalent to the non-vanishing of the quantity

$$\delta = p_{21}p_{10} - p_{20}p_{11} = 2r^2q \neq 0\,. \tag{2.62}$$

The regular two-from $dx_1 \wedge dx_2/(y_1y_2)$ in Equation (2.4) on $\mathcal{E}_1 \times \mathcal{E}_2$ is the pullback (along the horizontal arrow in Equation (2.59)) of the holomorphic two-form $\omega_{\mathcal{Z}}$. It was shown in [55, Sec. 4.6] that one has

$$\psi_0^*\left(\frac{dx_1}{y_1} \wedge \frac{dx_2}{y_2}\right) = \delta\big(X^{(2)} - X^{(1)}\big)\frac{dX^{(1)} \wedge dX^{(2)}}{Y^{(1)}Y^{(2)}} = \delta\frac{d\big(X^{(1)} + X^{(2)}\big) \wedge d\big(X^{(1)}X^{(2)}\big)}{Y^{(1)}Y^{(2)}}\,,$$

where $\left[X^{(l)} : Y^{(l)} : Z^{(l)}\right]$ for $l = 1, 2$ are the coordinates of the two copies of $\mathcal{C}_0$. □

One avoids the pre-factor δ in Equation (2.60) by using quadratic twists:

Corollary 2.34. *The map ψ extends to a rational map between the surfaces $\mathcal{W}_0^{(\varepsilon)}$ in Equation (2.29) and $\mathcal{Z}^{(2^4)}$ in Equation (2.6), i.e.,*

$$\psi: \quad \mathcal{W}_0^{(\varepsilon)} \dashrightarrow \mathcal{Z}^{(2^4)}\,, \qquad \textit{with} \quad \varepsilon = \frac{4}{\lambda_2\lambda_3(1-\lambda_2)^2(1-\lambda_3)^2}\,, \tag{2.63}$$

such that the holomorphic two-forms $\omega_{\mathcal{W}_0^{(\varepsilon)}} = dz_2 \wedge dz_3/\hat{z}_4$ in the chart $z_1 = 1$ and $\omega_{\mathcal{Z}^{(2^4)}} = dx_1 \wedge dx_2/\hat{y}_{1,2}$ in the chart $z_1 = z_2 = 1$ satisfy $\omega_{\mathcal{W}_0^{(\varepsilon)}} = \psi^*\omega_{\mathcal{Z}^{(2^4)}}$.

Proof. An isomorphism between $\mathcal{W}_0$ in Equation (2.16) and $\mathcal{W}_0^{(\varepsilon)}$ in Equation (2.29) is given by $\tilde{z}_4 = \hat{z}_4/\sqrt{\epsilon}$. Similarly, an isomorphism between $\mathcal{Z}$ in Equation (2.16) and $\mathcal{Z}^{(2^4)}$ in Equation (2.29) is given by $y_{1,2} = \hat{y}_{1,2}/4$. Equation (2.60) then becomes $\omega_{\mathcal{W}_0^{(\varepsilon)}} = \psi^*\omega_{\mathcal{Z}^{(2^4)}}$. □

On the other hand, the double quadric $\mathcal{Z}$ in Equation (2.3) is related to the surface $\mathcal{V}_0$ in Equation (2.53) as follows:

Lemma 2.35. *In the commutative diagram*

$$\begin{array}{ccc} \mathcal{E}_1 \times \mathcal{E}_2 & \xrightarrow{\cong} & \mathcal{I}_1 \times \mathcal{I}_2 \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ \mathcal{Z} & \xrightarrow{\phi_\pm} & \mathcal{V}_0 \end{array} \tag{2.64}$$

the maps $\phi_\pm : \mathcal{Z} \dashrightarrow \mathcal{V}_0$ *are rational maps of degree two given by*

$$\begin{aligned} \phi_\pm : \quad & (x_1, z_1, x_2, z_2, y_{1,2}) \mapsto [\mathbf{Z}_{00} : \mathbf{Z}_{01} : \mathbf{Z}_{10} : \mathbf{Z}_{11}] , \\ \textit{with} \quad & \mathbf{Z}_{00} = \left(x_1^2 - 2\Lambda_1 x_1 z_1 + \Lambda_1 z_1^2\right)\left(x_2^2 - 2\Lambda_2 x_2 z_2 + \Lambda_2 z_2^2\right), \\ & \mathbf{Z}_{01} = \left(x_1^2 - \Lambda_1 z_1^2\right)\left(x_2^2 - \Lambda_2 z_2^2\right), \\ & \mathbf{Z}_{10} = \left(x_1^2 - 2x_1 z_1 + \Lambda_1 z_1^2\right)\left(x_2^2 - 2x_2 z_2 + \Lambda_2 z_2^2\right), \\ & \mathbf{Z}_{11} = \pm 4 y_{1,2} , \end{aligned} \tag{2.65}$$

such that $\phi_\pm \circ \pi \circ \imath_{\mathcal{E}_l} = \phi_\mp \circ \pi$ *for* $l = 1, 2$ *where* $\imath_{\mathcal{E}_l}$ *is the hyperelliptic involution on* $\mathcal{E}_l$.

Proof. Equation (2.65) follows immediately from the equivalence $\mathcal{E}_1 \times \mathcal{E}_2 \cong \mathcal{I}_1 \times \mathcal{I}_2$ given by Lemma 2.30 (with a choice ± 1 for the relative sign of y_1 and y_2) and the fact that the projection map $\pi : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{Z}$ is given by $y_{1,2} = z_1 z_2 y_1 y_2$. The fact that the map is a rational map of degree two follows directly from the fact that Equation (2.3) is a double quadric surface. □

For the elliptic curve $\mathcal{E}_l$ with $\Lambda_l \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ for $l = 1, 2$, multiplication by two is given by the composition of two-isogenies $\chi'_{\mathcal{E}_l} \circ \chi_{\mathcal{E}_l}$, or, equivalently, by the map

$$\begin{aligned} x_l &\mapsto 2\left(x_l^2 - \Lambda_l z_l\right)^2 y_l z_l , \\ y_l &\mapsto \left(x_l^2 - 2\Lambda_l x_l z_l + \Lambda_l z_l^2\right)\left(x_l^2 - 2x_l z_l + \Lambda_l z_l^2\right)\left(x_l^2 - \Lambda_l z_l^2\right), \\ z_l &\mapsto 8 y_l^3 z_l^3 . \end{aligned} \tag{2.66}$$

The two-isogenies $\chi_{\mathcal{E}_l}$ can then be expressed as rational maps originating from the complete intersection of quadrics $\mathcal{I}_1$ in Equation (2.47) (resp. $\mathcal{I}_2$ in Equation (2.48)). It follows that

$$\chi'_{\mathcal{E}_1} \circ \chi_{\mathcal{E}_1} : \quad [\mathbf{X}_{00}, \mathbf{X}_{01}, \mathbf{X}_{10}, \mathbf{X}_{11}] \mapsto [x_1 : y_1 : z_1] = [\mathbf{X}_{0,1}^2 \mathbf{X}_{1,1} : \mathbf{X}_{0,0}\mathbf{X}_{0,1}\mathbf{X}_{1,0} : \mathbf{X}_{1,1}^3] , \tag{2.67}$$

and a similar formula holds for $\chi'_{\mathcal{E}_2} \circ \chi_{\mathcal{E}_2}$. The composition of their Cartesian products, given by

$$\mathcal{I}_1 \times \mathcal{I}_2 \xrightarrow{\chi_{\mathcal{E}_1} \times \chi_{\mathcal{E}_2}} \mathcal{E}'_1 \times \mathcal{E}'_2 \xrightarrow{\chi'_{\mathcal{E}_1} \times \chi'_{\mathcal{E}_2}} \mathcal{E}_1 \times \mathcal{E}_2 , \tag{2.68}$$

induces a rational map from $\mathcal{V}_0$ to $\mathcal{Z}$. We have the following:

Proposition 2.36. *The family of rational maps $\psi_\pm : \mathcal{V}_0 \dashrightarrow \mathcal{Z}$ of degree two given by*

$$\begin{aligned} \psi_\pm : \quad & [\mathbf{Z}_{00} : \mathbf{Z}_{01} : \mathbf{Z}_{10} : \mathbf{Z}_{11}] \quad \mapsto \quad (x_1, z_1, x_2, z_2, y_{1,2}) \,, \\ \textit{with} \quad & x_1 = Q \,, \qquad\qquad z_1 = (\Lambda_1 - \Lambda_2)\mathbf{Z}_{1,1}^2 \,, \\ & x_2 = (\Lambda_1 - \Lambda_2)\mathbf{Z}_{0,1}^2 \,, \qquad z_2 = Q \,, \\ & y_{1,2} = \pm(\Lambda_1 - \Lambda_2)^2 Q^2 \mathbf{Z}_{0,0}\mathbf{Z}_{0,1}\mathbf{Z}_{1,0}\mathbf{Z}_{1,1} \,, \end{aligned} \tag{2.69}$$

with

$$Q = \mathbf{Z}_{0,0}^2 - (1-\Lambda_1)\mathbf{Z}_{0,1}^2 - \Lambda_1 \mathbf{Z}_{1,0}^2 + \Lambda_1(1-\Lambda_2)\mathbf{Z}_{1,1}^2 \,, \tag{2.70}$$

satisfies that the composition $\psi_\pm \circ \phi_\pm : \mathcal{Z} \dashrightarrow \mathcal{Z}$ is the diagonal action of multiplication by two on $\mathcal{E}_1 \times \mathcal{E}_2$ in Equation (2.66).

Proof. The composition of elliptic-curve isogenies in Equation (2.68) is given by

$$\left(\begin{array}{c} [\mathbf{X}_{00} : \mathbf{X}_{01} : \mathbf{X}_{10} : \mathbf{X}_{11}] \\ [\mathbf{Y}_{00} : \mathbf{Y}_{01} : \mathbf{Y}_{10} : \mathbf{Y}_{11}] \end{array} \right) \mapsto \left(\begin{array}{c} [x_1 : z_1 : y_1] = [\mathbf{X}_{0,1}^2\mathbf{X}_{1,1} : \mathbf{X}_{1,1}^3 : \mathbf{X}_{0,0}\mathbf{X}_{0,1}\mathbf{X}_{1,0}] \\ [x_2 : z_2 : y_2] = [\mathbf{Y}_{0,1}^2\mathbf{Y}_{1,1} : \mathbf{Y}_{1,1}^3 : \mathbf{Y}_{0,0}\mathbf{Y}_{0,1}\mathbf{Y}_{1,0}] \end{array} \right). \tag{2.71}$$

The relation, given by

$$\Big[\mathbf{Z}_{00} : \mathbf{Z}_{01} : \mathbf{Z}_{10} : \mathbf{Z}_{11}\Big] \;=\; \Big[\mathbf{X}_{00}\mathbf{Y}_{00} : \;\mathbf{X}_{01}\mathbf{Y}_{01} : \;\mathbf{X}_{10}\mathbf{Y}_{10} : \;\mathbf{X}_{11}\mathbf{Y}_{11}\Big], \tag{2.72}$$

induces a correspondence between $\mathcal{Z}$ and $\mathcal{V}_0$, given by

$$\begin{aligned} x_1x_2 = \mathbf{Z}_{0,1}^2 \,, \qquad z_1z_2 = \mathbf{Z}_{1,1}^2 \,, \qquad y_{1,2} = \mathbf{Z}_{0,0}\mathbf{Z}_{0,1}\mathbf{Z}_{1,0}\mathbf{Z}_{1,1} \,, \\ (\Lambda_1 - \Lambda_2)x_1z_2 = Q \,, \qquad (\Lambda_2 - \Lambda_1)x_2z_1 = R \,, \qquad QR = -(\Lambda_1 - \Lambda_2)^2\mathbf{Z}_{0,1}^2\mathbf{Z}_{1,1}^2 \,, \end{aligned} \tag{2.73}$$

with $Q = (\Lambda_1 - \Lambda_2)\mathbf{X}_{01}^2\mathbf{Y}_{11}^2$ and $R = (\Lambda_2 - \Lambda_1)\mathbf{X}_{11}^2\mathbf{Y}_{01}^2$. One then obtains

$$\begin{aligned} Q &= \mathbf{Z}_{0,0}^2 - (1-\Lambda_1)\mathbf{Z}_{0,1}^2 - \Lambda_1\mathbf{Z}_{1,0}^2 + \Lambda_1(1-\Lambda_2)\mathbf{Z}_{1,1}^2 \,, \\ R &= \mathbf{Z}_{0,0}^2 - (1-\Lambda_2)\mathbf{Z}_{0,1}^2 - \Lambda_2\mathbf{Z}_{1,0}^2 + \Lambda_2(1-\Lambda_1)\mathbf{Z}_{1,1}^2 \,. \end{aligned} \tag{2.74}$$

When solving for an equivalence class, given by

$$\Big(x_1, z_1, x_2, z_2, y_{1,2}\Big) \;\sim\; \Big(\mu x_1,\; \mu z_1,\; \nu x_2,\; \nu z_2,\; \mu^2\nu^2 y_{1,2}\Big) \text{ with } \mu, \nu \in \mathbb{C}^\times \,, \tag{2.75}$$

we obtain Equations (2.69). Using the relation $QR = -(\Lambda_1 - \Lambda_2)^2\mathbf{Z}_{0,1}^2\mathbf{Z}_{1,1}^2$ the solution can also be written as

$$\begin{aligned} x_1 = (\Lambda_2 - \Lambda_1)\mathbf{Z}_{0,1}^2 \,, \quad z_1 = R \,, \quad x_2 = R \,, \quad z_2 = (\Lambda_2 - \Lambda_1)\mathbf{Z}_{1,1}^2 \,, \\ y_{1,2} = (\Lambda_1 - \Lambda_2)^2 R^2 \mathbf{Z}_{0,0}\mathbf{Z}_{0,1}\mathbf{Z}_{1,0}\mathbf{Z}_{1,1} \,. \end{aligned} \tag{2.76}$$

Equation (2.66) provides an explicit formula for the (diagonal) action of multiplication by two on $\mathcal{E}_1 \times \mathcal{E}_2$. We apply the natural projection map $\pi : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{Z}$ given by $y_{1,2} = z_1z_2y_1y_2$ to obtain the induced action on the the double quadric $\mathcal{Z}$ in Equation (2.3). We check that this agrees with the composition of maps $\psi_\pm \circ \phi_\pm : \mathcal{Z} \dashrightarrow \mathcal{Z}$, i.e., we have

$$\psi_\pm \circ \phi_\pm \;=\; \pi\Big(\big(\chi'_{\mathcal{E}_1} \times \chi'_{\mathcal{E}_2}\big) \circ \big(\chi_{\mathcal{E}_1} \times \chi_{\mathcal{E}_2}\big)\Big). \tag{2.77}$$

□

The map ψ_- in Equation (2.69) can be related to the geometric isogeny ψ in Equation (2.59). We have the following:

Proposition 2.37. *The map ψ in Equation (2.59) with the normalization determined by Equation (2.60) coincides with ψ_- in Proposition 2.36.*

Proof. Considering the finite field extension $\mathbb{Q}(k_1, k_2)$ with

$$\Lambda_1 = -\frac{(k_2 - k_3)^2}{(1-k_2^2)(1-k_3^2)}\,, \qquad \Lambda_2 = -\frac{(k_2 + k_3)^2}{(1-k_2^2)(1-k_3^2)}\,, \tag{2.78}$$

the maps $\psi_\pm : \mathcal{V}_0 \dashrightarrow \mathcal{Z}$ in Proposition 2.36 are equivalently given by

$$\psi_\pm : \qquad [\mathbf{Z}_{00} : \mathbf{Z}_{01} : \mathbf{Z}_{10} : \mathbf{Z}_{11}] \quad \mapsto \quad (x_1, z_1, x_2, z_2, y_{1,2})\,, \tag{2.79}$$

with

$$\begin{aligned} x_1 &= S_+\,, \qquad z_1 = 4k_2k_3(1-k_2^2)(1-k_3^2)\mathbf{Z}_{1,1}^2\,,\\ x_2 &= S_-\,, \qquad z_2 = 4k_2k_3(1-k_2^2)(1-k_3^2)\mathbf{Z}_{1,1}^2\,,\\ y_{1,2} &= \pm 256 k_2^4 k_3^4 (1-k_2^2)^2(1-k_3^2)^2 \mathbf{Z}_{0,0}\mathbf{Z}_{0,1}\mathbf{Z}_{1,0}\mathbf{Z}_{1,1}^5\,, \end{aligned} \tag{2.80}$$

where $S_\pm$ are the quadrics given by

$$\begin{aligned} S_\pm = {} & \pm(1-k_2^2)^2(1-k_3^2)\mathbf{Z}_{0,0}^2 \mp (1 \mp k_2k_3)^2(1-k_2^2)(1-k_3^2)\mathbf{Z}_{0,1}^2\\ & \pm (k_2 \mp k_3)^2(1-k_2^2)(1-k_3^2)\mathbf{Z}_{1,0}^2 \mp (k_2 \mp k_3)^2(1 \pm k_2k_3)^2\mathbf{Z}_{1,1}^2\,, \end{aligned} \tag{2.81}$$

such that $S_+S_- = 16k_2^2k_3^2(1-k_2^2)^2(1-k_3^2)^2\mathbf{Z}_{0,1}^2\mathbf{Z}_{1,1}^2$. The proof follows from a tedious computation using `Maple`: one first constructs the rational map ψ in Equation (2.59) as a map from $\mathcal{W}_0$ in the chart $z_1 = 1$ in Equation (2.29) to $\mathcal{Z}$ in the chart $z_1 = z_2 = 1$ in Equation (2.3) over $\mathbb{Q}(k_2, k_3)$. One then uses the identification $\mathcal{W}_0 \cong \mathcal{V}_0$ from the proof of Proposition 2.31. The map ψ then simplifies considerably and coincides with ψ_- in Equation (2.80). □

Legendre's gluing method in Section 1.3.4 determines the smooth genus-two curve in Equation (1.34), admitting an elliptic involution with elliptic-curve quotients $\mathcal{E}_1$ and $\mathcal{E}_2$. The quotient maps determine the $(2,2)$-isogeny $\Psi : \mathrm{Jac}(\mathcal{C}_0) \to \mathcal{E}_1 \times \mathcal{E}_2$ in Equation (1.39) and its dual isogeny Φ. The construction descends and yields the following Kummer sandwich theorem:

Theorem 2.38. *The surface $\mathcal{V}_0$ in Equation (2.53) and rational maps $\phi_\pm, \psi_\pm$ in Equations (2.65) and (2.69) fit into the following Kummer sandwich:*

$$\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2) \xrightarrow{\phi_\pm} \mathcal{V}_0 \xrightarrow{\psi_\pm} \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$$

The maps $\phi_\pm, \psi_\pm$ are defined as families over $\mathbb{Q}(\Lambda_1, \Lambda_2)$ and induced (up to the action of the minus identity involution) by the $(2,2)$-isogeny in Equation (1.7) and its dual $(2,2)$-isogeny in Equation (1.39) of principally polarized abelian surfaces, respectively, i.e., the isogenies in the following diagram:

$$\mathcal{E}_1 \times \mathcal{E}_2 \xrightarrow{\Phi} \mathrm{Jac}(\mathcal{C}_0) \xrightarrow{\Psi} \mathcal{E}_1 \times \mathcal{E}_2 \tag{2.82}$$

$\mathcal{C}_0$ is the smooth genus-two curve admitting an elliptic involution, and $\mathcal{E}_l$ are the elliptic-curve quotients determined in Proposition 1.6.

Proof. General results in [5, Sec. 1] and Theorem 1.1 show that there is a $(2,2)$-isogeny Φ in Equation (1.7), given by

$$\Phi: \quad \mathcal{E}_1 \times \mathcal{E}_2 \longrightarrow \mathrm{Jac}\,(\mathcal{C}_0)\,. \tag{2.83}$$

The dual isogeny $\Psi : \mathrm{Jac}\,(\mathcal{C}_0) \to \mathcal{E}_1 \times \mathcal{E}_2$ was constructed in Equation (1.39). Up to isomorphisms, the composition is given by multiplication by two on each factor of $\mathcal{E}_1 \times \mathcal{E}_2$. Proposition 2.37 shows that Ψ induces the map $\psi_\pm$. The minimal resolution of the double quadric surface $\mathcal{Z}$ is isomorphic to the Kummer surface $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$. Moreover, the induced action of $\Psi \circ \Phi$ on the associated Kummer surfaces coincides with the one of $\psi_\pm \circ \phi_\pm$ due to Proposition 2.36. On the other hand, the map $\psi_\pm \circ \phi_\pm$ factors though $\mathcal{V}_0$ by construction. Proposition 2.31 proves that $\mathcal{V}_0$ is isomorphic to $\mathcal{W}_0$ whose minimal resolution is $\mathrm{Kum}(\mathcal{C}_0)$ due to Proposition 2.11. □

3. K3 surfaces related to the Legendre pencil

The family of projective surfaces, given by the equation

$$y^2 = z_1 z_2 z_3 (z_1 + z_2)(z_2 + z_3)(z_1 + t z_3) \tag{3.1}$$

with $t \in \mathbb{Q}$ is a family of double sextic surfaces considered by Van Geemen and Top in [55]. The minimal resolutions yield K3 surfaces of Picard rank 19. A natural generalization, referred to as the *two-parameter twisted Legendre pencil* in [22–26], is the family $\mathcal{X}$, given by

$$\mathcal{X}: \quad y^2 = z_1 z_2 (z_1 - z_2)(z_1 - z_3)\,(z_3^2 + 2\rho_1 z_2 z_3 + \rho_2^2 z_2^2) \tag{3.2}$$

with $\rho_1, \rho_2 \in \mathbb{Q}$ and $\rho_1 \neq \pm\rho_2$. If fact, changing variables according to $z_1 \mapsto -z_1$, $z_3 \mapsto 2\rho_1 z_3$, and $y \mapsto 2i\rho_1 y$, the family $\mathcal{X}$ is isomorphic to the family in Equation (3.1) for $(\rho_1, \rho_2) = (t/2, 0)$ over $\mathbb{Q}(i)$. A natural holomorphic two-form $\omega_{\mathcal{X}}$ on $\mathcal{X}$ is given by $dz_1 \wedge dz_3/y$ in the chart $z_2 = 1$ in Equation (3.2).

3.1. **Configurations of six lines.** The general member of the family in Equation (3.2) yields a K3 surface of Picard rank 18, and thus its transcendental lattice $\mathrm{T}_{\mathcal{X}}$ has rank 4. Hoyt showed in [23, 26] that

$$\mathrm{T}_{\mathcal{X}} = \langle 2 \rangle \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle\,. \tag{3.3}$$

There is also a further generalization, known as three-parameter twisted Legendre pencil, given by

$$y^2 = z_1 (z_1 - z_2)(z_1 - z_3)(A z_2 - z_3)(B z_2 - z_3)(C z_2 - z_3)\,. \tag{3.4}$$

The branch locus consists of six lines, three of which are coincident in a point; see [12, 23, 37]. The minimal resolutions are K3 surfaces of Picard rank greater than or equal to 17. In [26] it was shown that the transcendental lattice of the K3 surface associated with a general member in Equation (3.4) is $\langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 3}$.

The case $C = \infty$ in Equation (3.4) after rescaling is given by

$$\mathcal{X}: \quad y^2 = z_1 z_2 (z_1 - z_2)(z_1 - z_3)(A z_2 - z_3)(B z_2 - z_3)\,. \tag{3.5}$$

This pencil is isomorphic to Equation (3.2) over $\mathbb{Q}(\rho_1, \rho_2, \sqrt{\rho_1^2 - \rho_2^2})$ with $AB = \rho_2^2$ and $A + B = -2\rho_1$. We have the following:

Proposition 3.1. *For $A \neq B$ the general member in Equation (3.5) is branched on the union of six lines, one of which is coincident with two pairs of lines in two distinct points and generic otherwise. The minimal resolution has Picard rank 18. For the following specializations the minimal resolution has Picard rank 19:*

$$(1)\ AB = 0, \qquad (2)\ (A+B-4)^2 - 4AB = 0, \qquad (3)\ (A-B+1)(A-B-1) = 0.$$

Remark 3.2. *In terms of Equation (3.2) the subsets of the parameter space in Proposition 3.1 are given by*

$$(1)\ \rho_2 = 0, \qquad (2)\ (\rho_1 - \rho_2 + 2)(\rho_1 + \rho_2 + 2) = 0, \qquad (3)\ 4\rho_1^2 - 4\rho_2^2 - 1 = 0.$$

Remark 3.3. *For case(1) in Proposition 3.1 the branch locus of the sextic surface is the union of six lines such that three lines are coincident with two pairs of lines in two distinct points and the remaining three lines are coincident with one pair of lines in one point and generic otherwise.*

Proof. The branch locus of Equation (3.5) is given by the six lines

$$\ell_1 = z_1, \quad \ell_2 = z_2, \quad \ell_3 = z_1 - z_2, \quad \ell_4 = z_1 - z_3, \quad \ell_5 = Az_2 - z_3, \quad \ell_6 = Bz_2 - z_3, \tag{3.6}$$

whose intersection pattern looks as follows:

(3.7)

	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6
ℓ_1	–	$[0:0:1]$	$[0:0:1]$	$[0:1:0]$	$[0:1:A]$	$[0:1:B]$
ℓ_2	$[0:0:1]$	–	$[0:0:1]$	$[1:0:1]$	$[1:0:0]$	$[1:0:0]$
ℓ_3	$[0:0:1]$	$[0:0:1]$	–	$[1:1:1]$	$[1:1:A]$	$[1:1:B]$
ℓ_4	$[0:1:0]$	$[1:0:1]$	$[1:1:1]$	–	$[A:1:A]$	$[B:1:B]$
ℓ_5	$[0:1:A]$	$[1:0:0]$	$[1:1:A]$	$[A:1:A]$	–	$[1:0:0]$
ℓ_6	$[0:1:B]$	$[1:0:0]$	$[1:1:B]$	$[B:1:B]$	$[1:0:0]$	–

Statement (1) is immediate. For (2) one can check that the double sextic in Equation (3.5) admits a non-torsion section: set $z_1 = X$, $z_2 = t$, $z_3 = 1$, then for $x = -\frac{4(At-1)(Bt-1)}{(A-B)^2 t}$ we have

$$y^2 = -\frac{4(At-1)^2(Bt-1)^2((A+B)t-2)^2\big(4ABt^2 + (A^2+B^2-2AB-4A-4B)t+4\big)}{(A-B)^6 t^2}.$$

The discriminant for the quadratic term in the numerator of y^2 factors into the product of $(A-B)^2$ times $A^2+B^2-2AB-8A-8B-16$. Since we assumed $A \neq B$ we obtain a non-torsion section if $A^2+B^2-2AB-8A-8B-16 = 0$. For (3) one checks that $X = \frac{At-1}{A-B}$ and $y^2 = \frac{A(At-1)^2(Bt-1)^2t^2}{(A-B)^3}$ is a section if and only if $A-B+1 = 0$. Switching the roles of A and B the statement follows. □

In Equation (3.5) the pencil of lines through the point $[1:0:0]$ (or, equivalently, the point $[0:0:1]$) induces an elliptic fibration on $\mathcal{X}$. A simple coordinate transformation, defined over $\mathbb{Q}(\rho_1, \rho_2^2)$, yields a Weierstrass model with fibers over $\mathbb{P}^1 = \mathbb{P}(v_1, v_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(x, y, z)$, given by

$$\mathcal{X}: \quad y^2 z = x\Big(x - v_1v_2\big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2\big)z\Big)\Big(x - v_2^2\big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2\big)z\Big). \tag{3.8}$$

This change of coordinates also maps via pullback the holomorphic two-form $\omega_{\mathcal{X}} = dv_1 \wedge dx/y$ in the chart $v_2 = z = 1$ in Equation (3.8) to the holomorphic two-form $dz_1 \wedge dz_3/y$ in the chart $z_2 = 1$ in Equation (3.2). The discriminant function of the fibration is given by $v_2^8 v_1^2 (v_1 - v_2)^2 (v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2)^6$. We have the immediate:

Lemma 3.4. *Equation (3.8) defines a Jacobian elliptic fibration with the singular fibers $I_2^* + 2I_2 + 2I_0^*$ and a Mordell-Weil group of sections $(\mathbb{Z}/2\mathbb{Z})^2$.*

Remark 3.5. *The pencil of lines through the point $[z_1 : z_2 : z_3] = [1 : 0 : 1]$ induces a second Jacobian elliptic fibration on $\mathcal{X}$ with the singular fibers $I_4^* + 4I_2 + I_0^*$ and a Mordell-Weil group of sections $(\mathbb{Z}/2\mathbb{Z})^2$.*

Remark 3.6. *Hoyt proved in [25] that there are countably many algebraic curves $W_N \subset \mathbb{C}^2$ in the parameter space $\{(A, B) \mid A \neq B \text{ and } A, B \neq 0, 1\}$ for Equation (3.5) such that each W_N corresponds to a modular correspondence between two elliptic curves. Moreover, it was shown that the rank of the Mordell-Weil group in Lemma 3.4 is greater than zero if and only if $(A, B) \in \bigcup_N W_N$. We will provide the modular correspondences for the subsets of the parameter space in Proposition 3.1 in Section 3.5.*

3.2. **Double coverings by Kummer surfaces.** We recall that an *even eight* on a K3 surface $\mathcal{X}$ is a set of eight disjoint (-2)-rational smooth curves $E_1, \ldots, E_8$ such that there is a divisor $D \in \mathrm{NS}(\mathcal{X})$ with

$$E_1 + \cdots + E_8 \sim 2D\,, \tag{3.9}$$

where the symbol $\sim$ denotes linear equivalence. Choosing an even eight as the branch locus on a K3 surface for a two-fold ramified covering, *Nikulin's construction* yields a new K3 surface $\mathcal{Y}$ together with a double covering $f\colon \mathcal{Y} \to \mathcal{X}$; see [46]. Equivalently, there is a Nikulin involution on $\mathcal{Y}$ such that the minimal resolution of the quotient surface of $\mathcal{Y}$ by the involution yields $\mathcal{X}$. A *Nikulin involution* is an analytic involution of $\mathcal{Y}$ which acts trivially on $H^{2,0}(\mathcal{Y})$. It is well known that such an involution always has eight fixed points and that by blowing them up and taking the quotient we recover the K3 surface $\mathcal{X}$ [44].

In our situation, reducible fibers of type D_6 or D_4 arise in Lemma 3.4 as minimal resolutions of the fibers located over $v_2 = 0$ and $v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2 = 0$, respectively. Configurations of eight non-central components of these fibers then form even eights on the minimal resolution of $\mathcal{X}$. Since the Mordell-Weil group of sections in Lemma 3.4 is $(\mathbb{Z}/2\mathbb{Z})^2$, we can always choose the four marked components in the reducible fibers in such a way that each of them is met by the zero section or a two-torsion section. In our situation, there are then the following even eights to consider:

(1) non-central components of D_6 and D_4 over $v_2 = 0$ and $v_1 = (-\sqrt{\rho_1^2 - \rho_2^2} + \rho_1)v_2$ meeting sections,
(2) non-central components of D_6 and D_4 over $v_2 = 0$ and $v_1 = (+\sqrt{\rho_1^2 - \rho_2^2} + \rho_1)v_2$ meeting sections,
(3) non-central components of D_4's over $v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2 = 0$.

Choosing any of these even eights as the branch locus of a double cover, Nikulin's construction yields a new K3 surface obtained as the minimal resolution of a Jacobian elliptic surface $\mathcal{Y}_n$, together with a rational double cover $f_n : \mathcal{Y}_n \dashrightarrow \mathcal{X}$ for $n = 1, \ldots, 3$.

3.2.1. *Double coverings by Kummer surfaces associated with two elliptic curves.* Let us first construct the Jacobian elliptic surface $\mathcal{Y}_3$ and the double cover $f_3 : \mathcal{Y}_3 \to \mathcal{X}$. A double cover between two rational curves, defined over $\mathbb{Q}(\rho_1, \rho_2)$, is given by

$$\mathbb{P}^1 \dashrightarrow \mathbb{P}^1, \qquad [u_1 : u_2] \mapsto [v_1 : v_2] = [(\rho_1 - \rho_2)u_1^2 - 2\rho_1 u_1 u_2 + (\rho_1 + \rho_2)u_2^2 : \ 2u_1 u_2]. \tag{3.10}$$

The pullback of Equation (3.8) along the double cover in Equation (3.10) yields an elliptic fibration, with fibers over $\mathbb{P}^1 = \mathbb{P}(u_1, u_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(X, Y, Z)$, given by the Weierstrass model

$$\begin{aligned} \mathcal{Y}_3 : \quad Y^2 Z = X \Big(X + \frac{1}{2} u_1 u_2 \big(u_1 - u_2\big)\big((\rho_1 - \rho_2)u_1 - (\rho_1 + \rho_2)u_2\big) Z \Big) \\ \times \Big(X + \frac{1}{2} u_1 u_2 \big((\rho_1 - \rho_2)u_1^2 - 2(\rho_1 + 1)u_1 u_2 + (\rho_1 + \rho_2)u_2^2\big) Z \Big) . \end{aligned} \tag{3.11}$$

In fact, Equation (3.11) is obtained by setting

$$\begin{aligned} x = \big((\rho_1 - \rho_2)u_1^2 - (\rho_1 + \rho_2)u_2^2\big)^2 \Big(4X + 2u_1 u_2 \big(u_1 - u_2\big)\big((\rho_1 - \rho_2)u_1 - (\rho_1 + \rho_2)u_2\big) Z\Big), \\ y = 8\big((\rho_1 - \rho_2)u_1^2 - (\rho_1 + \rho_2)u_2^2\big)^3 Y, \qquad z = Z, \end{aligned} \tag{3.12}$$

in Equation (3.8). Together, Equations (3.12) and (3.10) determine the double cover

$$f_3 : \quad \mathcal{Y}_3 \dashrightarrow \mathcal{X}, \qquad \Big([u_1 : u_2], [X : Y : Z]\Big) \mapsto \Big([v_1 : v_2], [x : y : z]\Big), \tag{3.13}$$

which is branched along the even eight on $\mathcal{X}$ that consists of the non-central components of the two reducible fibers of type D_4 obtained as the minimal resolution of the fibers of Kodaira-type I_0^* located over $v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2 = 0$. Conversely, on $\mathcal{Y}_3$ in the chart $Z = 1$ and $u_2 = 1$ a compatible Nikulin involution is given by

$$\Big(u_1,\ X,\ Y\Big) \mapsto \left(u_1' = \frac{\rho_1 + \rho_2}{(\rho_1 - \rho_2)u_1},\ X' = \frac{(\rho_1 + \rho_2)^2 X}{(\rho_1 - \rho_2)^2 u_1^4},\ Y' = -\frac{(\rho_1 + \rho_2)^3 Y}{(\rho_1 - \rho_2)^3 u_1^6}\right), \tag{3.14}$$

and it leaves the holomorphic two-forms $\omega_{\mathcal{Y}_3} = du_1 \wedge dX/Y$ invariant. We have the following:

Lemma 3.7. *For the holomorphic two-form $\omega_{\mathcal{X}} = dw_1 \wedge dx/y$ in the chart $z = w_2 = 1$ on $\mathcal{X}$ in Equation (3.8) and the holomorphic two-form $\omega_{\mathcal{Y}_3} = du_1 \wedge dX/Y$ in the chart $Z = u_2 = 1$ on $\mathcal{Y}_3$ in Equation (3.11), we have $\omega_{\mathcal{Y}_3} = f_3^* \omega_{\mathcal{X}}$.*

Proof. The proof follows by an explicit computation using the transformation in Equations (3.12) and (3.10). □

It turns out that the minimal resolution of $\mathcal{Y}_3$ is isomorphic to the Kummer surface $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$ associated with the product surface of two non-isogenous elliptic curves. We have the following:

Proposition 3.8. *For $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ let parameters ρ_1, ρ_2 be given by*

$$\rho_1 = \Lambda_1 + \Lambda_2 - 2\Lambda_1\Lambda_2 - 1\,, \qquad \rho_2 = 1 - \Lambda_1 - \Lambda_2\,. \tag{3.15}$$

The surfaces $\mathcal{Y}_3$ in Equation (3.11) and $\mathcal{Z}$ in Equation (2.3) are birational equivalent over $\mathbb{Q}(\Lambda_1, \Lambda_2)$. The birational equivalence identifies the holomorphic two-form $\omega_{\mathcal{Y}_3} = du_1 \wedge dX/Y$ in the chart $Z = u_2 = 1$ with $\omega_{\mathcal{Z}} = dx_1 \wedge dx_2/y_{1,2}$ in the chart $z_1 = z_2 = 1$.

Proof. Equation (3.11) is then precisely Equation (2.11), describing a Jacobian elliptic fibration on the double quadric surface $\mathcal{Z}$ whose minimal resolution is $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$. The proof then follows from Proposition 2.5 and Corollary 2.9. □

3.2.2. *Double coverings by Jacobian Kummer surfaces.* Next we construct the K3 surface $\mathcal{Y}_1$ and the double cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}$. We introduce new parameters μ_2, μ_3 such that in Equation (3.8) we have

$$\rho_1 = \frac{(\mu_3 - 1)^2(\mu_2^2 + 6\mu_2 + 1)}{8(\mu_2\mu_3 - 1)(\mu_2 - \mu_3)}\,, \qquad \rho_2^2 = \frac{(\mu_3 - 1)^4(\mu_2 + 1)^2\mu_2}{4(\mu_2\mu_3 - 1)^2(\mu_2 - \mu_3)^2}\,. \tag{3.16}$$

Equations (3.16) are sufficient to obtain a family of Weierstrass models from Equation (3.8) over $\mathbb{Q}(\mu_2, \mu_3)$ since the latter family only depends on ρ_2^2.

A double cover $\mathbb{P}^1 \to \mathbb{P}^1$ between two rational curves, defined over $\mathbb{Q}(\mu_2, \mu_3)$, is given by $[u_1 : u_2] \mapsto [v_1 : v_2]$ with

$$[v_1 : v_2] = [\mu_2(\mu_3 - 1)^2(u_1 - \mu_3 u_2)(u_1 - \mu_2^2\mu_3 u_2) : (\mu_2 - \mu_3)(1 - \mu_2\mu_3)(u_1 - \mu_2\mu_3 u_2)^2]\,. \tag{3.17}$$

The pullback of Equation (3.8) along the double cover in Equation (3.17) then yields an elliptic fibration, with fibers over $\mathbb{P}^1 = \mathbb{P}(u_1, u_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(X, Y, Z)$, given by the Weierstrass model

$$\begin{aligned} \mathcal{Y}_1 : \quad Y^2 Z = \epsilon X \big(X + \mu_2(\mu_3 - 1)^2 u_1 u_2 \big(u_1 - \mu_3 u_2\big)\big(u_1 - \mu_2^2\mu_3 u_2\big) Z\big) \\ \times \big(X + \mu_3(\mu_2 - 1)^2 u_1 u_2 \big(u_1 - \mu_2 u_2\big)\big(u_1 - \mu_2\mu_3^2 u_2\big) Z\big)\,, \end{aligned} \tag{3.18}$$

with $1/\epsilon = 4\mu_2\mu_3(1 - \mu_2\mu_3)(\mu_2 - \mu_3)$. In fact, Equation (3.18) is obtained by setting

$$\begin{aligned} x = \frac{1}{4}\mu_2\mu_3(1 - \mu_2\mu_3)(\mu_2 - \mu_3)(\mu_2 - 1)^4(\mu_3 - 1)^4(u_1 - \mu_2\mu_3 u_2)^2(u_2 + \mu_2\mu_3 u_2)^2 \\ \times \Big(X + \mu_2(\mu_3 - 1)^2 u_1 u_2 (u_1 - \mu_3 u_2)(u_1 - \mu_2^2\mu_3 u_2) Z\Big)\,, \\ y = \frac{1}{4}\mu_2^2\mu_3^2(\mu_2 - \mu_3)^2(1 - \mu_2\mu_3)^2(\mu_2 - 1)^6(\mu_3 - 1)^6(u_1 - \mu_2\mu_3 u_2)^3(u_1 + \mu_2\mu_3 u_2)^3 Y\,, \end{aligned} \tag{3.19}$$

and $z = Z$ in Equation (3.8). Together, Equations (3.19) and (3.17) then determine a double cover

$$f_1 : \quad \mathcal{Y}_1 \dashrightarrow \mathcal{X}\,, \qquad \Big([u_1 : u_2], [X : Y : Z]\Big) \mapsto \Big([v_1 : v_2], [x : y : z]\Big)\,, \tag{3.20}$$

which is branched along the even eight on $\mathcal{X}$ that consists of non-central components of the reducible fibers of type D_6 and D_4 meeting the sections. In terms of the

parameters μ_2, μ_3, the reducible fibers are the minimal resolution of the fibers of Kodaira-type I_2^* and I_0^* located over

$$v_2 = 0\,, \quad \text{and} \quad v_1 = \frac{(\mu_2+1)(1-\mu_3)^2}{4(1-\mu_2\mu_3)(\mu_2-\mu_3)} v_2\,, \tag{3.21}$$

whereas the remaining fiber of Kodaira-type I_0^* is located over

$$v_1 = \frac{\mu_2(1-\mu_3)^2}{(1-\mu_2\mu_3)(\mu_2-\mu_3)} v_2\,. \tag{3.22}$$

A compatible Nikulin involution on $\mathcal{Y}_1$ can be constructed in the same way as already demonstrated for the case of $\mathcal{Y}_3$. We have the following:

Lemma 3.9. *For the holomorphic two-form $\omega_{\mathcal{X}} = dw_1 \wedge dx/y$ in the chart $z = w_2 = 1$ on $\mathcal{X}$ in Equation (3.8) and the holomorphic two-form $\omega_{\mathcal{Y}_1} = du_1 \wedge dX/Y$ in the chart $Z = u_2 = 1$ on $\mathcal{Y}_1$ in Equation (3.18), we have $\omega_{\mathcal{Y}_1} = f_1^* \omega_{\mathcal{X}}$.*

Proof. The proof follows by an explicit computation using the transformation in Equations (3.19) and (3.17). □

Remark 3.10. *For the construction of the double cover $f_2 : \mathcal{Y}_2 \to \mathcal{X}$ the induced map on the base curves is branched over $v_2 = 0$ and $v_1 = (\sqrt{\rho_1^2 - \rho_2^2} + \rho_1) v_2$. We introduce new Rosenhain roots $\widetilde{\mu}_2 = \widetilde{k}_2^2$ and $\widetilde{\mu}_3 = \widetilde{k}_3^2$ such that*

$$\begin{aligned} \frac{(\mu_2+1)(1-\mu_3)^2}{4(1-\mu_2\mu_3)(\mu_2-\mu_3)} &= \frac{\widetilde{\mu}_2(1-\widetilde{\mu}_3)^2}{(1-\widetilde{\mu}_2\widetilde{\mu}_3)(\widetilde{\mu}_2-\widetilde{\mu}_3)}\,, \\ \frac{\mu_2(1-\mu_3)^2}{(1-\mu_2\mu_3)(\mu_2-\mu_3)} &= \frac{(\widetilde{\mu}_2+1)(1-\widetilde{\mu}_3)^2}{4(1-\widetilde{\mu}_2\widetilde{\mu}_3)(\widetilde{\mu}_2-\widetilde{\mu}_3)}\,. \end{aligned} \tag{3.23}$$

With respect to the new Rosenhain roots the positions of the two fibers of Kodaira-type I_0^* in Equations (3.21) and (3.22) are interchanged, whereas the equations for the parameters ρ_1, ρ_2^2 remain unchanged, i.e.,

$$\rho_1 = \frac{(\widetilde{\mu}_3-1)^2(\widetilde{\mu}_2^2+6\widetilde{\mu}_2+1)}{8(\widetilde{\mu}_2\widetilde{\mu}_3-1)(\widetilde{\mu}_2-\widetilde{\mu}_3)}\,, \qquad \rho_2^2 = \frac{(\widetilde{\mu}_3-1)^4(\widetilde{\mu}_2+1)^2\widetilde{\mu}_2}{4(\widetilde{\mu}_2\widetilde{\mu}_3-1)^2(\widetilde{\mu}_2-\widetilde{\mu}_3)^2}\,. \tag{3.24}$$

The solutions of Equation (3.23) are given by

$$\mu_2 = -\frac{(i \pm \widetilde{k}_2)^2}{(i \mp \widetilde{k}_2)^2}\,, \qquad \mu_3 = -\frac{(i \pm \widetilde{k}_3)^2}{(i \mp \widetilde{k}_3)^2}\,, \tag{3.25}$$

where the sign choices in the formulas for μ_2 and μ_3 are independent. Thus, all results related to $f_2 : \mathcal{Y}_2 \to \mathcal{X}$ can be derived from those for f_1 by replacing the moduli according to Equation (3.25).

Let us also consider the double sextic surface given by

$$\mathcal{W}_0'^{(\varepsilon')}: \quad \hat{z}_4^2 = \varepsilon' z_1 z_3 \big(z_1 - z_2 + z_3\big)\big((\mu_2\mu_3)^2 z_1 - \mu_2\mu_3\, z_2 + z_3\big) \prod_{i=2}^{3} \big(\mu_i^2\, z_1 - \mu_i\, z_2 + z_3\big)\,. \tag{3.26}$$

It follows from Proposition 2.11 that $\mathcal{W}_0'^{(\varepsilon')}$ is birational equivalent to the twisted Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')^{(\varepsilon')}$ associated with the Jacobian of the smooth genus-two curve given by

$$\mathcal{C}_0' : \quad Y^2 = XZ\big(X-Z\big)\big(X-\mu_2\mu_3 Z\big)\big(X^2-(\mu_2+\mu_3)XZ+\mu_2\mu_3 Z^2\big)\,. \tag{3.27}$$

It turns out that the minimal resolution of $\mathcal{Y}_1$ is isomorphic to the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')$. We have the following:

Proposition 3.11. *Assume that $\mu_2,\mu_3\in\mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\mu_2\neq\mu_3^{\pm1}$, and a quadric-twist factor is given by*

$$\varepsilon' = \frac{1}{4\mu_2\mu_3(1-\mu_2\mu_3)(\mu_2-\mu_3)}\,. \tag{3.28}$$

The surfaces $\mathcal{Y}_1$ in Equation (3.18) and $\mathcal{W}_0'^{(\varepsilon')}$ in Equation (3.26) are birational equivalent over $\mathbb{Q}(\mu_2\mu_3,\mu_2+\mu_3)$. The birational equivalence identifies the holomorphic two-form $\omega_{\mathcal{Y}_1}=du_1\wedge dX/Y$ in the chart $Z=u_2=1$ with $\omega_{\mathcal{W}_0'^{(\varepsilon')}}=dz_2\wedge dz_3/\hat{z}_4$ in the chart $z_1=1$.

Proof. Equation (3.18) is precisely Equation (2.39), with λ_l's replaced by μ_l's for $l=2,3$. The proof then follows from Corollary 2.22. The constraint that the birational equivalence identifies $\omega_{\mathcal{Y}_1}$ and $\omega_{\mathcal{W}_0'^{(\varepsilon')}}$ determines ε'. □

3.3. **Kummer surfaces as quotients.** For the affine Weierstrass equation for $\mathcal{X}$ of the form $y^2=x^3+p_1x^2+p_2x$, the translation by the two-torsion section $T:(x,y)=(0,0)$ in each fiber constitutes a Nikulin involution $\imath$, which is given by addition $\oplus$ of T with respect to the group law in each smooth elliptic fiber, i.e.,

$$\imath : \quad (x,y)\mapsto(x,y)\ \oplus\ T\ =\ \left(\frac{p_2}{x},-\frac{p_2 y}{x^2}\right). \tag{3.29}$$

By resolving the eight nodes of $\mathcal{X}/\langle\imath\rangle$ we obtain a new K3 surface as a Jacobian elliptic surface $\mathcal{X}^\vee$, given by the affine Weierstrass model

$$Y^2 = X^3-2\,p_1X^2+\big(p_1^2-4p_2\big)\,X\,. \tag{3.30}$$

One constructs a dual Van Geemen-Sarti involution $\imath^\vee$ analogously. The explicit formulas for the isogeny and the dual isogeny are well known[3] and given by

$$\varphi : \quad \mathcal{X}\dashrightarrow\mathcal{X}^\vee\,, \qquad (x,y)\ \mapsto\ \left(\frac{y^2}{x^2},\frac{y\left(x^2-p_2\right)}{x^2}\right), \tag{3.31}$$

and

$$\varphi^\vee : \quad \mathcal{X}^\vee\dashrightarrow\mathcal{X}, \qquad (X,Y)\ \mapsto\ \left(\frac{Y^2}{4X^2},\frac{Y\left(X^2-p_1^2+4\,p_2\right)}{8\,X^2}\right). \tag{3.32}$$

We have the following:

[3] They were given explicitly in [13]

Proposition 3.12. *The K3 surfaces $\mathcal{X}$ and $\mathcal{X}^\vee$ admit dual Van Geemen-Sarti involutions $\imath$ and $\imath^\vee$ associated with fiberwise translations by the two-torsion section $T : (x, y) = (0, 0)$ and $T^\vee : (X, Y) = (0, 0)$, respectively, and the following pair of dual geometric two-isogenies:*

$$\begin{tikzcd} \mathcal{X}^\vee \arrow[loop left, "\imath^\vee"] \arrow[rr, dashed, shift right, "\varphi^\vee"'] & & \mathcal{X} \arrow[loop right, "\imath"] \arrow[ll, dashed, shift right, "\varphi"'] \end{tikzcd} \tag{3.33}$$

Moreover, it follows $\omega_{\mathcal{X}} = \varphi^* \omega_{\mathcal{X}^\vee}$ and $2\,\omega_{\mathcal{X}^\vee} = (\varphi^\vee)^* \omega_{\mathcal{X}}$ for the holomorphic two-forms $\omega_{\mathcal{X}} = dv \wedge dx/y$ and $\omega_{\mathcal{X}^\vee} = dv \wedge dX/Y$ on $\mathcal{X}$ and $\mathcal{X}^\vee$, respectively.

Applying Proposition 3.12 to Equation (3.8) using the Nikulin involution $\imath_3$ associated with translations by the two-torsion section

$$T_3 : \quad [X : Y : Z] \;=\; [v_1 v_2 \big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2\big) : \; 0 : \; -1]\,, \tag{3.34}$$

we obtain a new K3 surface as the minimal resolution of $\mathcal{X}/\langle \imath_3 \rangle$ together with a quotient map $\varphi_3 : \mathcal{X} \dashrightarrow \mathcal{X}_3^\vee$ defined over $\mathbb{Q}(\rho_1, \rho_2^2)$. Here, a Weierstrass model for $\mathcal{X}_3^\vee$ is given by

$$\begin{aligned} \mathcal{X}_3^\vee : \quad Y^2 Z = X^3 &- 2v_2\big(2v_1 - v_2\big)\Big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2\Big) X^2 Z \\ &+ v_2^4 \Big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2\Big)^2 X Z^2\,. \end{aligned} \tag{3.35}$$

It turns out that the minimal resolution of $\mathcal{X}_3^\vee$ is isomorphic to the Kummer surface $\mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2')$ associated with the product surface of the two-isogenous elliptic curves in Equation (1.42). We have the following:

Proposition 3.13. *For $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ let parameters ρ_1, ρ_2 satisfy*

$$\rho_1 = \Lambda_1 + \Lambda_2 - 2\Lambda_1\Lambda_2 - 1\,, \qquad \rho_2^2 = (1 - \Lambda_1 - \Lambda_2)^2\,. \tag{3.36}$$

The surfaces $\mathcal{X}_3^\vee$ in Equation (3.35) and $\mathcal{Z}'$ in Equation (2.8) are birationally equivalent over $\mathbb{Q}(\kappa_1, \kappa_2)$ with $\Lambda_l = 1/(1 - \kappa_l^2)$ for $l = 1, 2$. The birational equivalence identifies the holomorphic two-form $\omega_{\mathcal{X}_3^\vee} = dv_1 \wedge dX/Y$ in the chart $Z = v_2 = 1$ with $\omega_{\mathcal{Z}'} = dX_1 \wedge dX_2/Y_{1,2}$ in the chart $Z_1 = Z_2 = 1$.

Proof. Equation (3.35) is precisely Equation (2.14), describing a Jacobian elliptic fibration on the Kummer surface $\mathcal{Z}'$ whose minimal resolution is $\mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2')$. The proof then follows from Proposition 2.5 and Corollary 2.9. □

Applying Proposition 3.12 to Equation (3.8) using the Nikulin involution $\imath_1$ associated with translations by the two-torsion section $T_1 : [X : Y : Z] = [0 : 0 : 1]$, we obtain a new K3 surface as the minimal resolution of $\mathcal{X}/\langle \imath_1 \rangle$ together with a rational quotient map $\varphi_1 : \mathcal{X} \dashrightarrow \mathcal{X}_1^\vee$ defined over $\mathbb{Q}(\rho_1, \rho_2^2)$. Here, a Weierstrass model for $\mathcal{X}_1^\vee$, with fibers over $\mathbb{P}^1 = \mathbb{P}(v_1, v_2)$ embedded into $\mathbb{P}^2 = \mathbb{P}(X, Y, Z)$, is

$$\begin{aligned} \mathcal{X}_1^\vee : \quad Y^2 Z = X^3 &+ 2v_2\big(v_1 + v_2\big)\Big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^2\Big) X^2 Z \\ &+ v_2^2 \big(v_1 - v_2\big)^2 \Big(v_1^2 + 2\rho_1 v_1 v_2 + \rho_2^2 v_2^3\Big)^2 X Z^2\,. \end{aligned} \tag{3.37}$$

The discriminant function of the fibration is $16v_1v_2^7(v_1-v_2)^4(v_1^2+2\rho_1v_1v_2+\rho_2^2v_2^2)^6$.

Remark 3.14. *Applying Proposition 3.12 using the (different) two-torsion section in Equation (3.8), namely*

$$T_2: \quad [X:Y:Z] \;=\; [v_2^2\big(v_1^2+2\rho_1v_1v_2+\rho_2^2v_2^2\big):\; 0:\; -1]\,, \tag{3.38}$$

yields a K3 surface with the same singular fibers as $\mathcal{X}_2^\vee$, but for different moduli. In fact, the Weierstrass equation coincides with Equation (3.37) if we replace

$$\Big(\rho_1,\rho_2^2\Big) \;\mapsto\; \Big(-\rho_1-1,\;\rho_2^2+2\rho_1+1\Big). \tag{3.39}$$

In terms of Lemma 3.4 the change of parameters is equivalent to the interchange of the two fibers of Kodaira-type I_2 while keeping the fiber of Kodaira-type I_2^* fixed.

It turns out that the minimal resolution of $\mathcal{X}_1^\vee$ in Equation (3.37) is isomorphic to the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)$ associated with the Jacobian of a smooth genus-two curve admitting an elliptic involution. We have the following:

Proposition 3.15. *Assume that $\lambda_2,\lambda_3\in\mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2\neq\lambda_3^{\pm1}$. Further assume that parameters ρ_1,ρ_2 satisfy*

$$\rho_1=-\frac{\lambda_2^2\lambda_3^2+\lambda_2^2+\lambda_3^2-4\lambda_2\lambda_3+1}{(\lambda_2-1)^2(\lambda_3-2)^2}\,,\qquad \rho_2^2=\frac{(\lambda_2+1)^2(\lambda_3+1)^2}{(\lambda_2-1)^2(\lambda_3-1)^2}\,, \tag{3.40}$$

and a quadric-twist factor is given by

$$\varepsilon=\frac{4}{\lambda_2\lambda_3(1-\lambda_2)^2(1-\lambda_3)^2}\,. \tag{3.41}$$

The surfaces $\mathcal{X}_1^\vee$ in Equation (3.37) and $\mathcal{W}_0^{(\varepsilon)}$ in Equation (2.29) are birational equivalent over $\mathbb{Q}(\lambda_2+\lambda_3,\lambda_2\lambda_3)$. The rational equivalence identifies the two-forms $\omega_{\mathcal{X}_1^\vee}=dv_1\wedge dX/Y$ in the chart $Z=v_2=1$ and $\omega_{\mathcal{W}_0^{(\varepsilon)}}=dz_2\wedge dz_3/\hat{z}_4$ in the chart $z_1=1$.

Proof. Since the genus-two curve $\mathcal{C}_0$ is smooth, we have $\lambda_2,\lambda_3\in\mathbb{P}^1\backslash\{0,1,\infty\}$ and $\lambda_2\neq\lambda_3^{\pm1}$. A birational equivalence relating Equation (2.40) and Equation (3.37) is given by

$$w_1=-\lambda_2\lambda_3\big(v_1+v_2\big)\,,\quad w_2=\frac{1}{2}\big(v_1-v_2\big)\,,$$

$$x=\frac{\varepsilon}{4}(\lambda_2\lambda_3)^3\big(\lambda_2\lambda_3-\lambda_2-\lambda_3+1\big)^2X\,,\;\; y=\frac{\varepsilon^{\frac{3}{2}}}{8}(\lambda_2\lambda_2)^{\frac{9}{2}}\big(\lambda_2\lambda_3-\lambda_2-\lambda_3+1\big)^3Y\,,\;\; z=Z\,.$$

We observe that for $\varepsilon=\varepsilon_1^2/(\lambda_2\lambda_3)$ the above transformation is well defined over $\mathbb{Q}(\lambda_2+\lambda_3,\lambda_2\lambda_3)$. The remainder of the proof follows from Corollary 2.23 and by an explicit computation using the transformations provided above. The constraint that the birational equivalence identifies $\omega_{\mathcal{X}_1^\vee}$ and $\omega_{\mathcal{W}_0^{(\varepsilon)}}$ determines ε. □

3.4. **Kummer sandwich theorems.** In this section we combine the results of the previous sections to obtain several *Kummer sandwich theorems*, a term introduced by Shioda [52], that relate the Legendre pencil to various Kummer surfaces. In [9, 41] it was demonstrated that Kummer sandwich theorems allow to construct rational point counting functions on the K3 surface sandwiched.

The first Kummer sandwich relates the twisted Legendre pencil $\mathcal{X}$ to two Kummer surfaces associated with the product of two elliptic curves. The first one is the Kummer surface $\mathrm{Kum}\,(\mathcal{E}_1 \times \mathcal{E}_2)$ associated with the product abelian surface $\mathcal{E}_1 \times \mathcal{E}_2$, where the elliptic curves $\mathcal{E}_l$ with modular parameter Λ_l is given by Equations (1.19). The second is the Kummer surface $\mathrm{Kum}\,(\mathcal{E}_1' \times \mathcal{E}_2')$ where $\mathcal{E}_l'$ for $l = 1, 2$ are the two-isogenous elliptic curves in Equation (1.42). Models for the Kummer surfaces are the double quadric surfaces in Equation (2.3) and Equation (2.8), respectively. We have the following:

Proposition 3.16. *For $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ let parameters ρ_1, ρ_2 satisfy*

$$\rho_1 = \Lambda_1 + \Lambda_2 - 2\Lambda_1\Lambda_2 - 1\,, \qquad \rho_2^2 = (1 - \Lambda_1 - \Lambda_2)^2\,. \tag{3.42}$$

The twisted Legendre pencil $\mathcal{X}$ in Equation (3.2) fits into the following Kummer sandwich of rational maps:

$$\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2) \;\longrightarrow\; \mathcal{X} \;\longrightarrow\; \mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2') \;\longrightarrow\; \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)^{(2^4)} \tag{3.43}$$

The maps are defined as families over $\mathbb{Q}(\kappa_1, \kappa_2)$ with $\Lambda_l = 1/(1 - \kappa_l^2)$ for $l = 1, 2$. The holomorphic two-form $\omega_{\mathcal{X}}$ equals the pullback of $\omega_{\mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2')}$, resp. its pullback equals $\omega_{\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)}$ and the pullback of $\omega_{\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)^{2^4}}$ equals $\omega_{\mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2')}$.

Proof. We relate the twisted Legendre pencil $\mathcal{X}$ to the double quadric surfaces $\mathcal{Z}$ and $\mathcal{Z}'$ in Equation (2.3) and (2.8), respectively, by relating it to the Jacobian elliptic K3 surfaces $\mathcal{Y}_3$ and $\mathcal{X}_3^\vee$ in Equations (3.11) and (3.35) first. As a reminder, the minimal resolution of $\mathcal{Z}$ is the Kummer surface $\mathrm{Kum}\,(\mathcal{E}_1 \times \mathcal{E}_2)$. Similarly, the minimal resolution of $\mathcal{Z}'$ is the Kummer surface $\mathrm{Kum}\,(\mathcal{E}_1' \times \mathcal{E}_2')$.

First we assume $\rho_2 = 1 - \Lambda_1 - \Lambda_2$. We combine Lemma 3.7, Proposition 3.8, Proposition 3.12, Proposition 3.13, Lemma 2.3 to obtain the following diagram of rational maps:

$$\begin{array}{ccccccc} \mathcal{Z} \;\cong\; \mathcal{Y}_3 & \xrightarrow{f_3} & \mathcal{X} & \xrightarrow{\varphi_3} & \mathcal{X}_3^\vee \;\cong\; \mathcal{Z}' & \xrightarrow{\chi} & \mathcal{Z}^{(2^4)} \\ \wr\!\shortparallel & & & & \wr\!\shortparallel & & \wr\!\shortparallel \\ \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2) & \longrightarrow & \mathcal{X} & \longrightarrow & \mathrm{Kum}(\mathcal{E}_1' \times \mathcal{E}_2') & \longrightarrow & \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)^{(2^4)} \end{array} \tag{3.44}$$

By construction, all maps and equivalences are defined over $\mathbb{Q}(\Lambda_1, \Lambda_2)$, except the equivalence $\mathcal{X}_3^\vee \cong \mathcal{Z}'$ which is defined only over the field extension $\mathbb{Q}(\kappa_1, \kappa_2)$. The holomorphic two-forms are related by pullback as follows:

$$\omega_{\mathcal{Z}} = \omega_{\mathcal{Y}_3}\,, \quad \omega_{\mathcal{Y}_3} = f_3^* \omega_{\mathcal{X}}\,, \quad \omega_{\mathcal{X}} = \varphi_3^* \omega_{\mathcal{X}_3^\vee}\,, \quad \omega_{\mathcal{X}_3^\vee} = \omega_{\mathcal{Z}'}\,, \quad \omega_{\mathcal{Z}^\vee} = \chi^* \omega_{\mathcal{Z}^{(2^4)}}\,. \tag{3.45}$$

Using Equation (2.5) the statement regarding the two-forms follows.

Finally, the modular transformation $(\Lambda_1, \Lambda_2) \mapsto (1-\Lambda_1, 1-\Lambda_2)$ leaves the j-functions of $\mathcal{E}_1$ and $\mathcal{E}_2$ invariant and induces a sign change $(\rho_1, \rho_2) \mapsto (\rho_1, -\rho_2)$. Thus, the same statements hold for $(\rho_1, -\rho_2)$. □

The second Kummer sandwich relates the twisted Legendre pencil $\mathcal{X}$ to certain Jacobian Kummer surfaces. The first is the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)$ associated with the Jacobian of the smooth genus-two curve $\mathcal{C}_0$ admitting an elliptic involution in Equation (1.17) (with Rosenhain roots λ_2, λ_3). The second is the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')$ where $\mathcal{C}_0'$ is the $(2,2)$-isogenous Richelot-curve in Equation (1.46) (with Rosenhain roots μ_2, μ_3). That is, we have $\mathrm{Jac}\,(\mathcal{C}_0) \cong \mathrm{Jac}\,(\mathcal{C}_0')/G$ for a certain Göpel group determined in Proposition 1.13. Models for the Kummer surfaces are the double sextic surfaces in Equation (2.29) and (3.26), respectively. We have the following:

Proposition 3.17. *Assume that $\lambda_2, \lambda_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2 \neq \lambda_3^{\pm 1}$ and $\lambda_2 \neq -1$. Further assume that parameters μ_2, μ_3 are given by*

$$\begin{aligned} \mu_2 &= \frac{\lambda_2\lambda_3 + \lambda_2 - \lambda_3 - 1 + 2d}{\lambda_2\lambda_3 + \lambda_2 - \lambda_3 - 1 - 2d}, \\ \mu_3 &= \frac{\big(\lambda_2\lambda_3 + \lambda_2 - \lambda_3 - 1 + 2d\big)\big((1+\lambda_2)^2 k_3 + (1-\lambda_2)d - 2(1+\lambda_3)\lambda_2\big)}{\big(\lambda_2\lambda_3 + \lambda_2 - \lambda_3 - 1 - 2d\big)\big((1+\lambda_2)^2 k_3 - (1-\lambda_2)d - 2(1+\lambda_3)\lambda_2\big)}, \end{aligned} \tag{3.46}$$

with $k_3^2 = \lambda_3$ and $d^2 = (\lambda_2 - \lambda_3)(\lambda_2\lambda_3 - 1)$, and parameters (ρ_1, ρ_2) satisfy

$$\rho_1 = -\frac{\lambda_2^2\lambda_3^2 + \lambda_2^2 + \lambda_3^2 - 4\lambda_2\lambda_3 + 1}{(\lambda_2 - 1)^2(\lambda_3 - 2)^2}, \qquad \rho_2^2 = \frac{(\lambda_2+1)^2(\lambda_3+1)^2}{(\lambda_2-1)^2(\lambda_3-1)^2}, \tag{3.47}$$

and quadratic-twist factors are given by

$$\varepsilon = \frac{4}{\lambda_2\lambda_3(1-\lambda_2)^2(1-\lambda_3)^2}, \qquad \varepsilon' = \frac{1}{4\mu_2\mu_3(1-\mu_2\mu_3)(\mu_2-\mu_3)}. \tag{3.48}$$

The twisted Legendre pencil $\mathcal{X}$ in Equation (3.2) fits into the following Kummer sandwich of rational maps:

$$\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')^{(\varepsilon')} \longrightarrow \mathcal{X} \longrightarrow \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)} \longrightarrow \mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)^{(2^4)} \tag{3.49}$$

The maps are defined as families over a finite field extension of $\mathbb{Q}(\lambda_2, \lambda_3)$. The holomorphic two-form $\omega_{\mathcal{X}}$ equals the pullback of $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}}$, resp. its pullback equals $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')^{(\varepsilon')}}$ and the pullback of $\omega_{\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)^{(2^4)}}$ equals $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}}$.

Proof. We relate the twisted Legendre pencil $\mathcal{X}$ to the (twisted) double sextic surfaces $\mathcal{W}_0^{(\varepsilon)}$ and $\mathcal{W}_0'^{(\varepsilon')}$ in Equation (2.29) and (3.26), respectively, by relating it to the Jacobian elliptic K3 surface $\mathcal{Y}_1$ and $\mathcal{X}_1^\vee$ in Equations (3.18) and (3.37) first.

As a reminder, the minimal resolution of $\mathcal{W}_0^{(\varepsilon)}$ is isomorphic to the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}$. Similarly, the minimal resolution of $\mathcal{W}_0'^{(\varepsilon')}$ is isomorphic to the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')^{(\varepsilon')}$.

We combine Lemma 3.9, Proposition 3.11, Proposition 3.12, Proposition 3.15, Corollary 2.34 to obtain the following diagram of rational maps:

$$\begin{array}{ccccccccc} \mathcal{W}_0'^{(\varepsilon')} & \cong & \mathcal{Y}_1 & \xrightarrow{f_1} & \mathcal{X} & \xrightarrow{\varphi_1} & \mathcal{X}_1^\vee & \cong & \mathcal{W}_0^{(\varepsilon)} & \xrightarrow{\psi} & \mathcal{Z}^{(2^4)} \\ \wr\!\shortparallel & & & & & & & & \wr\!\shortparallel & & \wr\!\shortparallel \\ \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0')^{(\varepsilon')} & & & \longrightarrow & \mathcal{X} & \longrightarrow & & & \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)} & \longrightarrow & \mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)^{(2^4)} \end{array} \tag{3.50}$$

Comparing Equations (3.16) and (3.40), one obtains a relation between the Rosenhain roots (λ_2,λ_3) and (μ_2,μ_3). There are eight solutions for (μ_2,μ_3); see Remark 1.14. Since we only assumed $\lambda_2 \neq -1$ the four solutions in Proposition 1.13 – with the sign for k_3 and d chosen independently – determine a smooth genus-two curve $\mathcal{C}_0'$.

By construction, the maps and equivalences in Equation (3.50) to the left of $\mathcal{X}$ are defined over $\mathbb{Q}(\mu_2,\mu_3)$. The map ψ is defined over the field extension $\mathbb{Q}(\lambda_2+\lambda_3, q)$ with $q^2 = \lambda_2\lambda_3$. The remaining maps and equivalences in Equation (3.50) are families defined over $\mathbb{Q}(\lambda_2,\lambda_3)$. The holomorphic two-forms are related by pullback as follows:

$$\omega_{\mathcal{W}_0'^{(\varepsilon')}} = \omega_{\mathcal{Y}_1}\,, \quad \omega_{\mathcal{Y}_1} = f_1^*\omega_{\mathcal{X}}\,, \quad \omega_{\mathcal{X}} = \varphi_1^*\omega_{\mathcal{X}_1^\vee}\,, \quad \omega_{\mathcal{X}_1^\vee} = \omega_{\mathcal{W}_0^{(\varepsilon)}}\,, \quad \omega_{\mathcal{W}_0^{(\varepsilon)}} = \psi^*\omega_{\mathcal{Z}^{(2^4)}}\,. \tag{3.51}$$

Using Equation (2.28) the statement regarding the two-forms follows. □

We combine the results of Proposition 3.16 and Proposition 3.17 to obtain one of the main results of this article:

Theorem 3.18. *Assume that $\lambda_2,\lambda_3 \in \mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2 \neq \lambda_3^{\pm 1}$. Further assume that Λ_1,Λ_2 satisfy*

$$\Lambda_1\Lambda_2 = \frac{(\lambda_2+\lambda_3)^2 - 4\lambda_2\lambda_3}{(1-\lambda_2)^2(1-\lambda_3)^2}\,, \qquad \Lambda_1+\Lambda_2 = -\frac{2(\lambda_2+\lambda_3)}{(1-\lambda_2)(1-\lambda_3)}\,. \tag{3.52}$$

and the quadratic-twist factor ε is given by

$$\varepsilon = \frac{4}{\lambda_2\lambda_3(1-\lambda_2)^2(1-\lambda_3)^2}\,. \tag{3.53}$$

The twisted Legendre pencil, given by

$$\mathcal{X}: \quad y^2 = z_1z_2(z_1-z_2)(z_1-z_3)\left(z_3-\left(\frac{1+\lambda_2}{1-\lambda_2}\right)^2 z_2\right)\left(z_3-\left(\frac{1+\lambda_3}{1-\lambda_3}\right)^2 z_2\right), \tag{3.54}$$

fits into the following Kummer sandwich of rational maps:

$$\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2) \longrightarrow \mathcal{X} \longrightarrow \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)} \longrightarrow \mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)^{(2^4)} \tag{3.55}$$

The maps are defined as families over $\mathbb{Q}(k_2,k_3)$ with $k_l^2 = \lambda_l$ for $l = 2,3$. The holomorphic two-form $\omega_{\mathcal{X}}$ equals the pullback of $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}}$, resp. its pullback equals $\omega_{\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)}$ and the pullback of $\omega_{\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)^{(2^4)}}$ equals $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}}$.

Proof. For the given relations in Equations (3.47) the discriminant $\rho_1^2 - \rho_2^2$ is a perfect square of a rational function in $\mathbb{Q}(\lambda_2, \lambda_3)$. The twisted Legendre pencil $\mathcal{X}$ in Equation (3.2) then is equivalent to Equation (3.54).

First assume that $\rho_2 = 1 - \Lambda_1 - \Lambda_2$. Comparing Equations (3.15) and (3.40) one obtains a relation between the Rosenhain roots (λ_2, λ_3) and the modular parameters (Λ_1, Λ_2). This relation is precisely Equation (1.20), relating a genus-two curve $\mathcal{C}_0$ with an elliptic involution and its elliptic-curve quotients $\mathcal{E}_l$ for $l = 1, 2$ in Proposition 1.6. The second solution, obtained by changing $\rho_2 \mapsto -\rho_2$, is related to the first one by a modular transformation, e.g., by applying $(\Lambda_1, \Lambda_2) \mapsto (1 - \Lambda_1, 1 - \Lambda_2)$. Following Remark 1.7 we set $\lambda_l = k_l^2$ for $l = 2, 3$ and obtain the modular parameters of the elliptic-curve quotients $\mathcal{E}_l$ in Lemma 1.5 as the algebraic solutions in Equations (3.52). This also implies that $\Lambda_1, \Lambda_2 \in \mathbb{P}^1 \backslash \{0, 1, \infty\}$ so that the elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ are smooth.

Combining Proposition 3.16 and Proposition 3.17 we obtain the following diagram of rational maps:

$$\begin{array}{ccccccc} \mathcal{Z} \cong \mathcal{Y}_3 & \xrightarrow{f_3} & \mathcal{X} & \xrightarrow{\varphi_1} & \mathcal{X}_1^\vee \cong \mathcal{W}_0^{(\varepsilon)} & \xrightarrow{\psi} & \mathcal{Z}^{(2^4)} \\ \wr\!\shortparallel & & & & \wr\!\shortparallel & & \wr\!\shortparallel \\ \mathrm{Kum}\,(\mathcal{E}_1 \times \mathcal{E}_2) & \longrightarrow & \mathcal{X} & \longrightarrow & \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)} & \longrightarrow & \mathrm{Kum}\,(\mathcal{E}_1 \times \mathcal{E}_2)^{(2^4)} \end{array} \tag{3.56}$$

By construction, the maps and equivalences in Equation (3.56) to the left of $\mathcal{X}$ are defined over $\mathbb{Q}(\Lambda_1, \Lambda_2)$. The map ψ is defined over $\mathbb{Q}(\lambda_2 + \lambda_3, q)$ with $q^2 = \lambda_2\lambda_3$. The map φ_1 is defined over $\mathbb{Q}(\rho_1, \rho_2)$, and the equivalence $\mathcal{X}_1^\vee \cong \mathcal{W}_0^{(\varepsilon)}$ over $\mathbb{Q}(\lambda_2 + \lambda_3, \lambda_2\lambda_3)$. Thus, the smallest common finite extension field is $\mathbb{Q}(k_1, k_2)$. Thus, all maps and equivalences are families defined over $\mathbb{Q}(k_2, k_3)$, and the holomorphic two-forms are related by pullback as follows:

$$\omega_{\mathcal{Z}} = \omega_{\mathcal{Y}_3}\,, \quad \omega_{\mathcal{Y}} = f_3^*\omega_{\mathcal{X}}\,, \quad \omega_{\mathcal{X}} = \varphi_1^*\omega_{\mathcal{X}_1^\vee}\,, \quad \omega_{\mathcal{X}_1^\vee} = \omega_{\mathcal{W}_0^{(\varepsilon)}}\,, \quad \omega_{\mathcal{W}_0^{(\varepsilon)}} = \psi^*\omega_{\mathcal{Z}^{(2^4)}}\,. \tag{3.57}$$

The remainder of the statement follows easily. □

3.5. **One-parameter subfamilies.** In this section we will construct five subfamilies associated with certain modular correspondences for the two elliptic curves in Theorem 3.18. The first four families are associated with the specializations of the six-line configurations of Picard rank 19 found in Proposition 3.1. Since two algebraic K3 surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ with a rational map of finite degree $f : \mathcal{S}_1 \dashrightarrow \mathcal{S}_2$ have the same Picard number [32, Cor. 1.2], all K3 surfaces occurring in the corresponding Kummer sandwiches have Picard rank 19. The last family is obtained by making one elliptic-curve factor constant with complex multiplication and j-invariant 12^3.

3.5.1. *Family with isomorphic elliptic curves.* We will consider the three specializations of the families of K3 and abelian surfaces in Theorem 3.18 obtained by requiring the two elliptic curves to be isomorphic. The two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic if and only if their j-invariants coincide. One easily checks that this is the case if and only if the following expression vanishes:

$$(\Lambda_1 - \Lambda_2)(1 - \Lambda_1 - \Lambda_2)(\Lambda_1\Lambda_2 - \Lambda_1 - \Lambda_2)(1 - \Lambda_1\Lambda_2)(\Lambda_1\Lambda_2 - \Lambda_1 + 1)(\Lambda_1\Lambda_2 - \Lambda_2 + 1). \tag{3.58}$$

However, it follows from Equation (3.52) that

$$(\Lambda_1 - \Lambda_2)^2 = \frac{16\lambda_2\lambda_3}{(1-\lambda_2)^2(1-\lambda_3)^2} \neq 0\,,$$

whence we must have $\Lambda_1 \neq \Lambda_2$ if $\mathcal{C}_0$ is a smooth genus-two curve. However, the two elliptic curves in Theorem 3.18 can still be isomorphic. We have the following:

Lemma 3.19. *The elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ in Theorem 3.18 are isomorphic if the following polynomial vanishes:*

$$\begin{aligned} 0 = {} & \big(\lambda_2+1\big)\big(\lambda_3+1\big)\big(2-\lambda_2-\lambda_3\big)\big(\lambda_2\lambda_3-2\lambda_2+1\big)\big(2\lambda_2\lambda_3-\lambda_2-\lambda_3\big)\big(\lambda_2\lambda_3-2\lambda_3+1\big) \\ & \times\Big((\lambda_2+\lambda_3)^4 - 2(\lambda_2\lambda_3)(\lambda_2+\lambda_3)^3 + 3(\lambda_2\lambda_3)^2(\lambda_2+\lambda_3)^2 - 2(\lambda_2\lambda_3)^3(\lambda_2+\lambda_3) \\ & \quad + (\lambda_2\lambda_3)^4 - 2(\lambda_2+\lambda_3)^3 - 6(\lambda_2\lambda_3)(\lambda_2+\lambda_3)^2 + 10(\lambda_2\lambda_3)^2(\lambda_2+\lambda_3) - 8(\lambda_2\lambda_3)^3 \\ & \quad + 3(\lambda_2\lambda_3)^2 + 10(\lambda_2\lambda_3)(\lambda_2+\lambda_3) - 2(\lambda_2\lambda_3)^2 - 2(\lambda_2+\lambda_3) - 8(\lambda_2\lambda_3) + 1\Big)\,. \end{aligned} \tag{3.59}$$

Proof. Equation (3.59) is obtained by setting $j_1 - j_2 = 0$. That is, we start with

$$j_1 - j_2 = j(\mathcal{E}_1) - j(\mathcal{E}_2) = \frac{256(\Lambda_1^2-\Lambda_1+1)^3}{\Lambda_1^2(\Lambda_1-1)^2} - \frac{256(\Lambda_2^2-\Lambda_2+1)^3}{\Lambda_2^2(\Lambda_2-1)^2}\,, \tag{3.60}$$

use Equations (3.52) to express the right hand side as a rational function in λ_2, λ_3, and then select the numerator. □

Thus, we consider the one-parameter subfamilies in Theorem 3.18 with a linear (more precisely, linear or fractional linear) relation between the Rosenhain roots:

(1) $\lambda_3 = -1$ (or $\lambda_2 = -1$) such that $1 - \Lambda_1 - \Lambda_2 = 0$,
(2) $\lambda_3 = 2 - \lambda_2$ such that $\Lambda_1\Lambda_2 - \Lambda_1 - \Lambda_2 = 0$, or,
$\lambda_3^{-1} = 2 - \lambda_2$ such that $1 - \Lambda_1\Lambda_2 = 0$,
(3) $\lambda_3^{-1} = 2 - \lambda_2^{-1}$ such that $\Lambda_1\Lambda_2 - \Lambda_1 - \Lambda_2 = 0$, or,
$\lambda_3 = 2 - \lambda_2^{-1}$ such that $1 - \Lambda_1\Lambda_2 = 0$.

The case $\big(\Lambda_1\Lambda_2-\Lambda_1+1\big)\big(\Lambda_1\Lambda_2-\Lambda_2+1\big) = 0$ occurs exactly if the irreducible polynomial of degree $(4,4)$ in Equation (3.59) vanishes. We will not consider this case further.

For each remaining case we introduce the following one-parameter families of elliptic curves, genus-two curves, double sextic and double quadric surfaces:

(1) for $\lambda_3 = -1$ or $\lambda_3 = 2 - \lambda_2$ we set

$$\begin{aligned} \widetilde{\mathcal{E}}_1 : \quad & y^2 z &&= \big(x-z\big)\big(x^2 - \tfrac{1}{t+1}z^2\big)\,, \\ \widetilde{\mathcal{C}}_1 : \quad & Y^2 &&= XZ\big(X^2+4XZ+4(t+1)Z^2\big)\big(X^2-4XZ+4(t+1)Z^2\big)\,, \\ \widetilde{\mathcal{X}}_1 : \quad & y^2 &&= z_1z_2z_3\big(z_1-z_2\big)\big(z_1-z_3\big)\big(tz_2+z_3\big)\,, \\ \widetilde{\mathcal{Z}}_1^{(\varepsilon_1)} : \quad & \hat{y}_{12}^2 &&= \varepsilon_1 \prod_{k=1,2}\big(x_k-z_k\big)\big(x_k^2 - \tfrac{1}{t+1}z_k^2\big)z_k\,, \end{aligned} \tag{3.61}$$

(2) for $\lambda_3 = 2-\lambda_2$ or $\lambda_3^{-1} = 2-\lambda_2$ we set

$$
\begin{aligned}
\widetilde{\mathcal{E}}_2: \qquad & y^2 z &&= \big(x-z\big)\Big(x^2 - \tfrac{(t-1)^2}{(t+1)(t-3)} z^2\Big)\,,\\
\widetilde{\mathcal{C}}_2: \qquad & Y^2 &&= XZ\big(X+(t^2-1)Z\big)\big(X+(t-1)^2 Z\big)\\
& &&\quad \times\big(X+(t+1)(t-3)Z\big)\big(X+(t-1)(t-3)Z\big)\,,\\
\widetilde{\mathcal{X}}_2: \qquad & y^2 &&= z_1 z_2\big(z_1-z_2\big)\big(z_1-z_3\big)\big(t^2 z_2 - z_3\big)\big((t-2)^2 z_2 - z_3\big)\,,\\
\widetilde{\mathcal{Z}}_2^{(\varepsilon_2)}: \qquad & \hat{y}_{12}^2 &&= \varepsilon_2 \prod_{k=1,2}\big(x_k - z_k\big)\Big(x_k^2 - \tfrac{(t-1)^2}{(t+1)(t-3)} z_k^2\Big) z_k\,,
\end{aligned}
\tag{3.62}
$$

(3) for $\lambda_3 = 2-\lambda_2^{-1}$ or $\lambda_3^{-1} = 2-\lambda_2^{-1}$ we set

$$
\begin{aligned}
\widetilde{\mathcal{E}}_3: \qquad & y^2 z &&= x\big(x-z\big)\Big(x + \tfrac{(t+1)^2}{(t-1)^2} z\Big)\,,\\
\widetilde{\mathcal{C}}_3: \qquad & Y^2 &&= XZ\big(X+4t^2 Z\big)\big(X+2(t^2+1)Z\big)\\
& &&\quad \times\big(X+2t^2(t^2+1)Z\big)\big(X+(t^2+1)^2 Z\big)\,,\\
\widetilde{\mathcal{X}}_3: \qquad & y^2 &&= z_1 z_2\big(z_1-z_2\big)\big(z_1-z_3\big)\Big(\tfrac{(t^2+3)^2}{(t^2-1)^2} z_2 - z_3\Big)\Big(\tfrac{(3t^2+1)^2}{(t^2-1)^2} z_2 - z_3\Big)\,,\\
\widetilde{\mathcal{Z}}_3^{(\varepsilon_3)}: \qquad & \hat{y}_{12}^2 &&= \varepsilon_3 \prod_{k=1,2} x_k z_k\big(x_k - z_k\big)\Big(x_k + \tfrac{(t+1)^2}{(t-1)^2} z_k\Big)\,.
\end{aligned}
\tag{3.63}
$$

By construction, the minimal resolution of $\widetilde{\mathcal{Z}}_n$ is $\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)$ for $n = 1,2,3$. We have the following:

Corollary 3.20. *For $t \in \mathbb{P}^1 \backslash S_n$ with*

$$
\begin{array}{llllllll}
S_1 &= \{0,-1,\infty\}\,, & \varepsilon_1 &= \frac{1}{t+1}\,, & \delta_{1,1} &= \frac{t+1}{2}\,, & \delta_{2,1} &= \frac{1}{8}\,,\\
S_2 &= \{\pm 1, 3, \infty\}\,, & \varepsilon_2 &= \frac{1}{(t+1)(t-3)}\,, & \delta_{1,2} &= \frac{(t+1)(t-3)}{2}\,, & \delta_{2,2} &= \frac{1}{2(t-1)}\,,\\
S_3 &= \{0, \pm 1, \pm i, \infty\}\,, & \varepsilon_3 &= 1\,, & \delta_{1,3} &= \frac{t-1}{t+1}\,, & \delta_{2,3} &= \frac{1}{t(t^2+1)(t^2-1)^2}\,,
\end{array}
$$

and $\delta_{3,n} = 4\,\delta_{1,n}$, the twisted Legendre pencil $\widetilde{\mathcal{X}}_n$ for $n = 1,2,3$ fits into the following smooth Kummer sandwich of rational maps:

$$
\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)^{(-\varepsilon_n \delta_{1,n}^2)} \longrightarrow \widetilde{\mathcal{X}}_n \longrightarrow \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)^{(\varepsilon_n \delta_{2,n}^2)} \longrightarrow \mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)^{(-\varepsilon_n \delta_{3,n}^2)}
\tag{3.64}
$$

The maps are defined as families over a finite field extension of $\mathbb{Q}(t)$, and the holomorphic two-form $\omega_{\widetilde{\mathcal{X}}_n}$ equals the pullback of $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)(\bullet)}$, resp. its pullback equals $\omega_{\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)(\bullet)}$ and the pullback of $\omega_{\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)(\bullet)}$ equals $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)(\bullet)}$.

Proof. In the three cases we introduce a parameter t in the following way:

(1) for $\lambda_3 = -1$ set $t = -\frac{(1+\lambda_2)^2}{(1-\lambda_2)^2} = 4\Lambda_1(\Lambda_1 - 1)$,

(2) for $\lambda_3 = 2-\lambda_2$ set $\lambda_2 = \frac{t+1}{t-1}$ and $\frac{\Lambda_1^2}{(\Lambda_1-2)^2} = \frac{(t-1)^2}{(t+1)(t+3)}$, or,
for $\lambda_3^{-1} = 2-\lambda_2$ set $\lambda_2 = \frac{t+1}{t-1}$ and $\frac{(\Lambda_1-1)^2}{(\Lambda_1+1)^2} = \frac{(t-1)^2}{(t+1)(t+3)}$,

(3) for $\lambda_3 = 2-\lambda_2^{-1}$ set $\lambda_2 = \frac{1+t^2}{2}$ and $\Lambda_1 = -\frac{(t+1)^2}{(t-1)^2}$, or,
for $\lambda_3^{-1} = 2-\lambda_2^{-1}$ set $\lambda_2 = \frac{1+t^2}{2}$ and $\Lambda_1 = \frac{2(t^2+1)}{(t+1)^2}$.

The field of definition $\mathbb{Q}(\lambda_2, \lambda_3)$ is related to $\mathbb{Q}(t)$ at worst by a quadratic field extension. In fact, in the three cases above, we have $\mathbb{Q}(t) \subseteq \mathbb{Q}(\lambda_2, \lambda_3)$, $\mathbb{Q}(\lambda_2, \lambda_3) \cong \mathbb{Q}(t)$, and $\mathbb{Q}(\lambda_2, \lambda_3) \subseteq \mathbb{Q}(t)$, respectively.

One checks by an explicit computation that the double quadric surface $\widetilde{\mathcal{Z}}_n^{(-\epsilon_n \delta_{1,n}^2)}$ for $n = 1, 2, 3$ is isomorphic to $\mathcal{Z}$ in Equation (2.3) over $\mathbb{Q}(\Lambda_1, \Lambda_2)$ which is (at worst) a finite field extension of $\mathbb{Q}(t)$. Moreover, the pullback by this isomorphism leaves the holomorphic two-form invariant. Taking the minimal resolution we obtain the desired isomorphism between $\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)^{(-\varepsilon_n \delta_{1,n}^2)}$ and $\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)$. One also checks that the curve $\widetilde{\mathcal{C}}_n$ for $n = 1, 2, 3$ is isomorphic to $\mathcal{C}_0$ for the given (fractional) linear relation between the Rosenhain roots. We use Equation (2.30) and Proposition 2.13 with a suitable parameter λ_0, namely,

$$n = 1 : \lambda_0 = \frac{4}{\lambda_2 - 1}, \quad n = 2 : \lambda_0 = -\frac{4}{(\lambda_2 - 1)^2}, \quad n = 3 : \lambda_0 = -4(2\lambda_2 - 1). \tag{3.65}$$

It then follows that the (twisted) double sextic surfaces $\mathcal{W}_0^{(\varepsilon)}$ in Equation (2.29) is isomorphic to the quadratic twist of the double sextic defined by $\widetilde{\mathcal{C}}_n$ with twist factor $\varepsilon_n \delta_{2,n}^2$. By construction, the isomorphism is defined over a finite field extension of $\mathbb{Q}(t)$. Proposition 2.13 guarantees that the pullback by the isomorphism leaves the holomorphic two-form invariant. The proof then follows from applying the constructed isomorphisms to the situation in Theorem 3.18. The set S_n is obtained from the vanishing locus of the discriminant for $\widetilde{\mathcal{C}}_n$. □

We make the following two remarks:

Remark 3.21. *For generic parameters λ_2, λ_3 in Theorem 3.18 the transcendental lattices of the corresponding K3 surfaces are given by*

$$\mathrm{T}_{\mathrm{Kum}(\mathcal{E}_1 \times \mathcal{E}_2)} = H(2) \oplus H(2), \quad \mathrm{T}_{\mathcal{X}} = \langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle^{\oplus 2}, \quad \mathrm{T}_{\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)} = H(2) \oplus \langle 4 \rangle \oplus \langle -4 \rangle.$$

This follows from Remarks 2.2, 2.10, and Equation (3.3). For the surfaces in Corollary 3.20 the transcendental lattices specialize to

$$\mathrm{T}_{\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{E}}_n)} = H(2) \oplus \langle 4 \rangle, \quad \mathrm{T}_{\widetilde{\mathcal{X}}_n} = \langle 2 \rangle^{\oplus 2} \oplus \langle -2 \rangle, \quad \mathrm{T}_{\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)} = \langle 4 \rangle^{\oplus 2} \oplus \langle -4 \rangle.$$

This follows from Corollary 3.20 and results in [24, 55].

Remark 3.22. *In the two-dimensional parameter space $\{(A, B) \,|\, A \neq B\}$ of Equation (3.5) we considered the conic $(A + B - 4)^2 - 4AB = 0$ in Proposition 3.1. One checks that for*

$$A = \left(\frac{1 + \lambda_2}{1 - \lambda_2}\right)^2, \qquad B = \left(\frac{1 + \lambda_3}{1 - \lambda_3}\right)^2, \tag{3.66}$$

the vanishing locus of the conic becomes

$$\prod_{\sigma_2, \sigma_3 \in \{\pm 1\}} \left(2 - \lambda_2^{\sigma_2} - \lambda_3^{\sigma_3}\right) = 0. \tag{3.67}$$

Thus, the double sextic surfaces associated with these four components realize Equations (3.62) and (3.63).

3.5.2. *Family with isogenous elliptic curves.* As another example, we consider the classical modular curve $X_0(2)$, i.e., the irreducible plane algebraic curve of genus zero, given by

$$\begin{aligned} 0 = &- j_1^2 j_2^2 + j_1^3 + j_2^3 + 1488 j_1 j_2 (j_1 + j_2) - 2^4 3^4 5^3 (j_1^2 + j_2^2) \\ &+ 3^4 5^3 4027 j_1 j_2 + 2^7 3^7 5^5 (j_1 + j_2) - 2^{12} 3^9 5^9 . \end{aligned} \tag{3.68}$$

This is the relation between the j-invariants $j_1 = j(\tau_1) = j(\mathcal{E}_1)$ and $j_2 = j(\tau_2) = j(\mathcal{E}_2)$ of two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, such that $\tau_2 = 2 \cdot \tau_1$; see [21]. We have the following:

Lemma 3.23. *Assume $t \in \mathbb{P}^1 \backslash \{0, \pm 1, \infty\}$. For the two smooth elliptic curves*

$$\mathcal{E}_1 : \quad y_1^2 z_1 = x_1 \big(x_1 - z_1\big)\big(x_1 - t^2 z_1\big) , \qquad \mathcal{E}_2 : \quad y_2^2 z_2 = \big(x_2 - (t^2+1) z_2\big)\big(x_2^2 - 4t^2 z_2^2\big) , \tag{3.69}$$

with j-invariants $j_1 = j(\tau_1) = j(\mathcal{E}_1)$ and $j_2 = j(\tau_2) = j(\mathcal{E}_2)$, respectively, one has $\tau_2 = 2 \cdot \tau_1$. Moreover, for the two-isogenous elliptic curves – as defined in Equation (1.42) – one has $\mathcal{E}_1' \cong \mathcal{E}_2$ and $\mathcal{E}_2' \cong \mathcal{E}_1$.

Proof. For two elliptic curves in Legendre normal form as in Equation (1.19) one irreducible component of Equation (3.68) is given by

$$0 = 16 \Lambda_1^2 \Lambda_2^2 - 16 \Lambda_1 \Lambda_2 (\Lambda_1 + \Lambda_2) + 16 \Lambda_1 \Lambda_2 - 1 . \tag{3.70}$$

$X_0(2)$ has a rational parametrization given by $j_1 = (h + 256)^3 / h^2$ and $j_2 = (h + 16)^3 / h$ for $h \in \mathbb{C}^\times$; see [21]. We may take a suitable covering by setting $h = 16(t^2 - 1)^2 / t^2$ to obtain rational modular parameters for elliptic curves in Legendre form; see [39]. Then, the elliptic curves in Legendre form with $\Lambda_1 = -1/(t^2 - 1)$ and $\Lambda_2 = (t+1)^2/(4t)$, i.e., the curves

$$\mathcal{E}_1 : \quad y^2 z = x\Big(x - z\Big)\left(x + \frac{1}{t^2 - 1} z\right) , \qquad \mathcal{E}_2 : \quad y^2 z = x\Big(x - z\Big)\left(x - \frac{(t+1)^2}{4t} z\right) , \tag{3.71}$$

satisfy $j_1 = j(\mathcal{E}_1)$ and $j_2 = j(\mathcal{E}_2)$. In particular, the given parameters Λ_1, Λ_2 satisfy Equation (3.70). The two-isogenous elliptic curves in Equation (1.42) are then given by

$$\mathcal{E}_1' : \; Y_1^2 Z_1 = X_1 \left(X_1^2 + \frac{2(t^2+1)}{(t^2-1)} X_1 Z_1 + Z_1^2 \right) , \quad \mathcal{E}_2' : \; Y_2^2 Z_2 = X_2 \left(X_2^2 - \frac{t^2+1}{t} X_2 Z_2 + Z_2^2 \right) .$$

The remainder of the statement is checked by an explicit computation. □

We make the following:

Remark 3.24. *The proof of Lemma 3.23 shows that the double sextic surfaces for case (4) in Proposition 3.1 are covered by Kummer surfaces associated with two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\tau_2 = 2 \cdot \tau_1$. In particular, the conic $(A - B + 1)(A - B - 1) = 0$ is equivalent to the relation*

$$0 = 16 \Lambda_1^2 \Lambda_2^2 - 16 \Lambda_1 \Lambda_2 (\Lambda_1 + \Lambda_2) + 16 \Lambda_1 \Lambda_2 - 1 \tag{3.72}$$

between the elliptic modular parameters of the elliptic curves. For the two-isogenous elliptic curves – as defined by Equation (1.42) – we then have $j(\mathcal{E}_1) = j(\mathcal{E}_2')$ and $j(\mathcal{E}_2) = j(\mathcal{E}_1')$. Using Equations (3.52) this is also equivalent to the relation

$$\begin{aligned} 0 =& \big(\lambda_2^2\lambda_3^2 - 6\lambda_2^2\lambda_3 + 2\lambda_2\lambda_3^2 + \lambda_2^2 + \lambda_3^2 + 4\lambda_2\lambda_3 + 2\lambda_2 - 6\lambda_3 + 1\big) \\ &\times\big(\lambda_2^2\lambda_3^2 + 2\lambda_2^2\lambda_3 - 6\lambda_2\lambda_3^2 + \lambda_2^2 + \lambda_3^2 + 4\lambda_2\lambda_3 - 6\lambda_2 + 2\lambda_3 + 1\big)\,, \end{aligned} \tag{3.73}$$

between the Rosenhain roots of a genus-two curve $\mathcal{C}_0$ with an elliptic involution.

We define the following one-parameter families of elliptic curves, genus-two curves, double sextic and double quadric surfaces:

$$\begin{aligned} \widetilde{\mathcal{E}}_4: \quad & y^2 z &&= x\big(x-z\big)\big(x-t^2 z\big)\,, \\ \widetilde{\mathcal{E}}_4': \quad & Y^2 Z &&= \big(X-(t^2+1)Z\big)\big(X^2-4t^2Z^2\big)\,, \\ \widetilde{\mathcal{C}}_4: \quad & Y^2 &&= XZ\Big(X^4 - 4X^3Z + \tfrac{2(3t^4+8t^3+6t^2-8t+3)}{(t^2+1)^2}X^2Z^2 \\ & && \quad -\tfrac{4(t^2+2t-1)^2}{(t^2-2t-1)^2}XZ^3 + \tfrac{(t^2+2t-1)^4}{(t^2-2t-1)^4}Z^4\Big)\,, \\ \widetilde{\mathcal{X}}_4: \quad & y^2 &&= z_1z_2\big(z_1-z_2\big)\big(z_1-z_3\big)\Big(z_3+\tfrac{(t^2+1)^2}{4t(t^2-1)}z_2\Big)\Big(z_3+\tfrac{(t^2-2t-1)^2}{4t(t^2-1)}z_2\Big)\,, \\ \widetilde{\mathcal{Z}}_4^{(\varepsilon_4)}: \quad & \hat{y}_{12}^2 &&= \varepsilon_4 x_1z_1\big(x_1-z_1\big)\big(x_1-t^2z_1\big)\big(x_2-(t^2+1)z_2\big)\big(x_2^2-4t^2z_2^2\big)z_2\,. \end{aligned} \tag{3.74}$$

By construction, the minimal resolution of $\widetilde{\mathcal{Z}}_4^{(\varepsilon_4)}$ is $\mathrm{Kum}(\widetilde{\mathcal{E}}_4\times\widetilde{\mathcal{E}}_4')^{(\varepsilon_4)}$. Moreover, the j-invariants of $\widetilde{\mathcal{E}}_4$ and $\widetilde{\mathcal{E}}_4'$ satisfy Equation (3.68). We have the following:

Corollary 3.25. *For $t\in\mathbb{P}^1\backslash S_4$ with $S_4=\{0,\pm(\sqrt{2}\pm1),\pm1,\pm i,\infty\}$ and*

$$\varepsilon_4 = \frac{1}{t(t^2-1)}\,, \qquad \delta_{1,4}=\frac{1}{2}\,, \qquad \delta_{2,4}=\frac{(t^2-2t-1)^2(t^2+1)}{8t(t^2-1)(t^2+2t-1)}\,, \qquad \delta_{3,4}=2\,, \tag{3.75}$$

the twisted Legendre pencil $\widetilde{\mathcal{X}}_4$ fits into the following smooth Kummer sandwich of rational maps:

$$\mathrm{Kum}(\widetilde{\mathcal{E}}_4\times\widetilde{\mathcal{E}}_4')^{(\varepsilon_4\delta_{1,4}^2)} \longrightarrow \widetilde{\mathcal{X}}_4 \longrightarrow \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_4)^{(\delta_{2,4}^2)} \longrightarrow \mathrm{Kum}(\widetilde{\mathcal{E}}_4\times\widetilde{\mathcal{E}}_4')^{(\varepsilon_4\delta_{3,4}^2)} \tag{3.76}$$

The maps are defined as families over a finite field extension of $\mathbb{Q}(t)$, and the holomorphic two-form $\omega_{\widetilde{\mathcal{X}}_4}$ equals the pullback of $\omega_{\mathrm{Kum}(\mathrm{Jac}\widetilde{\mathcal{C}}_4)^{(\bullet)}}$, resp. its pullback equals $\omega_{\mathrm{Kum}(\widetilde{\mathcal{E}}_4\times\widetilde{\mathcal{E}}_4')^{(\bullet)}}$ and the pullback of $\omega_{\mathrm{Kum}(\widetilde{\mathcal{E}}_4\times\widetilde{\mathcal{E}}_4')^{(\bullet)}}$ equals $\omega_{\mathrm{Kum}(\mathrm{Jac}\widetilde{\mathcal{C}}_4)^{(\bullet)}}$.

Proof. We assume that two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ in Theorem 3.18 are given by $\Lambda_1=-1/(t^2-1)$ and $\Lambda_2=(t+1)^2/(4t)$. It follows from Lemma 3.23 that $\tau_1=2\tau_2$, and for the two-isogenous elliptic curves in Equation (1.42) we have $j(\mathcal{E}_1)=j(\mathcal{E}_2')=j(\widetilde{\mathcal{E}}_4)$ and $j(\mathcal{E}_2)=j(\mathcal{E}_1')=j(\widetilde{\mathcal{E}}_4')$. By an explicit computation one checks that the isomorphisms

$$\mathcal{E}_1\cong\big(\widetilde{\mathcal{E}}_4\big)^{\left(\frac{1}{t^2-1}\right)}\,, \qquad \mathcal{E}_2\cong\big(\widetilde{\mathcal{E}}_4'\big)^{\left(\frac{1}{4t}\right)}\,, \qquad \mathcal{E}_1'\cong\big(\widetilde{\mathcal{E}}_4'\big)^{\left(\frac{1}{t}\right)}\,, \qquad \mathcal{E}_2'\cong\big(\widetilde{\mathcal{E}}_4\big)^{\left(\frac{1}{t^2-1}\right)}\,, \tag{3.77}$$

are defined over $\mathbb{Q}(t)$ and identify the corresponding holomorphic one-forms via pullback. Furthermore, we use Equation (2.30) and Proposition 2.13 with the parameter

$$\lambda_0 = \frac{(T^4+1)(T^4+2T^2-1)^2}{(T^4-2T^2-1)(T^8+4DT^5-6T^6-4DT^3+2T^2-1)}\,, \tag{3.78}$$

where $t = T^2$ and $D^2 = 1 - T^4$. The remainder of the proof is analogous to the proof of Corollary 3.20. □

3.5.3. *Family with trivial factor.* The last family is not obtained by a fixed modular correspondence between two elliptic curves. Instead we make one elliptic-curve factor constant with complex multiplication and j-invariant 12^3. We introduce the following one-parameter families of elliptic curves, genus-two curves, double sextic and double quadric surfaces:

$$\begin{aligned}
\widetilde{\mathcal{E}}_5 : \qquad & y^2 z &&= \big(x - (t^2+1)z\big)\big(x^2 - 4t^2 z^2\big)\,, \\
\widetilde{\mathcal{F}}_5 : \qquad & Y^2 Z &&= X\big(X^2 - Z^2\big)\,, \\
\widetilde{\mathcal{C}}_5 : \qquad & Y^2 &&= XZ\big(X^2 - 2(t^2-4t+1)XZ + (t^2+1)^2 Z^2\big) \\
& && \quad \times\big(X^2 - 2(t^2+4t+1)XZ + (t^2+1)^2 Z^2\big)\,, \\
\widetilde{\mathcal{X}}_5 : \qquad & y^2 &&= z_1 z_2 \big(z_1 - z_2\big)\big(z_1 - z_3\big)\big(z_3^2 - z_2 z_3 + \tfrac{(t^2+1)^2}{16t^2} z_2^2\big)\,, \\
\widetilde{\mathcal{Z}}_5^{(\varepsilon_5)} : \qquad & \hat{y}_{12}^2 &&= \varepsilon_5 \big(x_1 - (t^2+1)z_1\big)\big(x_1^2 - 4t^2 z_1^2\big)\, z_1 x_2 z_2 \big(x_2^2 - z_2^2\big)\,.
\end{aligned} \tag{3.79}$$

By construction, the minimal resolution of $\widetilde{\mathcal{Z}}_5^{(\varepsilon_5)}$ is $\mathrm{Kum}(\widetilde{\mathcal{E}}_5 \times \widetilde{\mathcal{F}}_5)^{(\varepsilon_5)}$. Moreover, one checks that $\widetilde{\mathcal{F}}_5$ is constant with complex multiplication and $j(\widetilde{\mathcal{F}}_5) = 12^3$. We have the following:

Corollary 3.26. *For $t \in \mathbb{P}^1 \backslash S_5$ with $S_5 = \{0, \pm 1, \pm i, \infty\}$ and*

$$\varepsilon_5 = \frac{1}{4t}\,, \qquad \delta_{1,5} = \frac{1}{2}\,, \qquad \delta_{2,5} = \frac{1}{8t(t^2+1)}\,, \qquad \delta_{3,5} = 2\,, \tag{3.80}$$

the twisted Legendre pencil $\widetilde{\mathcal{X}}_5$ fits into the following smooth Kummer sandwich of rational maps:

$$\mathrm{Kum}(\widetilde{\mathcal{E}}_5 \times \widetilde{\mathcal{F}}_5)^{(\varepsilon_5 \delta_{1,5}^2)} \longrightarrow \widetilde{\mathcal{X}}_5 \longrightarrow \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_5)^{(\delta_{2,5}^2)} \longrightarrow \mathrm{Kum}(\widetilde{\mathcal{E}}_5 \times \widetilde{\mathcal{F}}_5)^{(\varepsilon_4 \delta_{3,5}^2)} \tag{3.81}$$

The maps are defined as families over a finite field extension of $\mathbb{Q}(t)$, and the holomorphic two-form $\omega_{\widetilde{\mathcal{X}}_5}$ equals the pullback of $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_5)^{(\bullet)}}$, resp. its pullback equals $\omega_{\mathrm{Kum}(\widetilde{\mathcal{E}}_5 \times \widetilde{\mathcal{F}}_5)^{(\bullet)}}$ and the pullback of $\omega_{\mathrm{Kum}(\widetilde{\mathcal{E}}_5 \times \widetilde{\mathcal{F}}_5)^{(\bullet)}}$ equals $\omega_{\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_5)^{(\bullet)}}$.

Proof. We assume that the two elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$ in Theorem 3.18 are given by $\Lambda_1 = (t+1)^2/(4t)$ and $\Lambda_2 = 1/2$. Furthermore, we use Equation (2.30) and Proposition 2.13 with the parameter

$$\lambda_0 = (u + i\omega)^2 (u - \omega)^2\,, \tag{3.82}$$

where $t = u^2$ and $\omega = \exp(-\pi i/4)$. The remainder of the proof is analogous to the proof of Corollary 3.20. □

3.6. **Construction of correspondences.** In Theorem 3.18 we constructed a Kummer sandwich for the Legendre pencil in Equation 3.54. The surfaces involved are defined over $\mathbb{Q}(\lambda_2, \lambda_3)$, the maps are defined as families over the finite field extension $\mathbb{Q}(k_2, k_3)$ with $k_l^2 = \lambda_l$ for $l = 2, 3$.

More generally, let us consider two families of Jacobian elliptic K3 surfaces V_λ and W_λ defined over $\mathbb{Q}(\lambda_2, \lambda_3)$ whose generic members have Picard rank 18. Here, we

denote the tuple (λ_2, λ_3) by λ. Since we assume that the K3 surfaces are Jacobian elliptic K3 surfaces, they have Weierstrass models over $\mathbb{Q}(\lambda_2, \lambda_3)$. For example, for V_λ we assume this Weierstrass model to be of the form

$$V_\lambda : \quad y^2 z = 4x^3 - g_2(v)\, xz^2 - g_3(v) z^3 \,, \tag{3.83}$$

where v is a suitable coordinate on the base curve $\mathbb{P}^1$, and g_2 and g_3 are polynomials of degree 8 and 12, respectively, with coefficients in $\mathbb{Q}(\lambda_2, \lambda_3)$. There is a natural notion of twisted K3 surfaces $V_\lambda^{(\varepsilon)}$ for $\varepsilon \in \mathbb{Q}(\lambda_2, \lambda_3)$ by taking the minimal resolution of the twisted Weierstrass model

$$V_\lambda^{(\varepsilon)} : \quad \hat{y}^2 z = \varepsilon \Big(4x^3 - g_2(v)\, xz^2 - g_3(v) z^3 \Big) . \tag{3.84}$$

Let $F : V_\lambda \dashrightarrow W_\lambda$ be a rational map of finite degree. An *algebraic correspondence* from V_λ to W_λ is a subset $\Gamma \subset V_\lambda \times W_\lambda$ such that Γ is finite and surjective over each component of V_λ. Then, the graph of $F : V_\lambda \dashrightarrow W_\lambda$ is clearly such a correspondence $\Gamma = \Gamma_F$. We have the following:

Lemma 3.27. *Assume $\varepsilon = \delta^2 \neq 0$ with $\delta \in \mathbb{Q}(\lambda_2, \lambda_3)$. Then, there is an algebraic correspondence from V_λ to $V_\lambda^{(\varepsilon)}$ over $\mathbb{Q}(\lambda_2, \lambda_3)$ that induces a nonzero map $H^{2,0}(V_\lambda^{(\varepsilon)}) \to H^{2,0}(V_\lambda)$.*

Proof. Since $\varepsilon = \delta^2$ the map from Equation (3.83) to Equation (3.84) is given by $F : y \mapsto \hat{y} = \delta y$. Generators of $H^{2,0}(V_\lambda)$ and $H^{2,0}(V_\lambda^{(\varepsilon)})$ are given in the charts $z = 1$ by the regular two-forms $\omega = dv \wedge dx/y$ and $\omega' = dv \wedge dx/\hat{y}$, respectively. We have $F^*\omega' = \omega/\delta \neq 0$. □

For a map $F : V_\lambda \dashrightarrow W_\lambda$ which is defined over $\mathbb{Q}(k_2, k_3)$, the graph Γ_F determines a correspondence, but only over $\mathbb{Q}(k_2, k_3)$. However, in our situation F has an additional property: for generators of $H^{2,0}(V_\lambda)$ and $H^{2,0}(W_\lambda)$ given in local coordinates by the regular two-forms ω_{V_λ} and ω_{W_λ}, respectively, we have $F^*\omega_{W_\lambda} = \omega_{V_\lambda}$. We consider the $\mathrm{Gal}\,(\mathbb{Q}(k_2, k_3)/\mathbb{Q}(\lambda_2, \lambda_3))$-conjugate maps $F_{(\pm,\pm)} : V_\lambda \dashrightarrow W_\lambda$ obtained by replacing $(k_1, k_2) \mapsto (\pm k_1, \pm k_2)$ such that $F_{(+,+)} = F$. Since the surfaces and the two-forms are defined over $\mathbb{Q}(\lambda_2, \lambda_3)$, we obtain

$$\sum_{\sigma_2, \sigma_3 \in \{\pm\}} F^*_{(\sigma_2, \sigma_3)} \omega_{W_\lambda} = 4\, \omega_{V_\lambda} \neq 0 \,. \tag{3.85}$$

The sum of the graphs

$$\Gamma \;=\; \Gamma_{F_{(+,+)}} + \Gamma_{F_{(+,-)}} + \Gamma_{F_{(-,+)}} + \Gamma_{F_{(-,-)}} \tag{3.86}$$

then defines a correspondence on $V_\lambda \times W_\lambda$ over $\mathbb{Q}(\lambda_2, \lambda_3)$ that induces a nonzero map $H^{2,0}(W_\lambda) \to H^{2,0}(V_\lambda)$. Thus, it must induce an isomorphism on the transcendental lattices of W_λ and V_λ when tensored with $\mathbb{Q}$. The Artin comparison theorem for complex and ℓ-adic cohomology implies that the same is true for the corresponding Galois representations [2]. Therefore, the correspondence Γ induces an isomorphism of $\mathrm{G}_{\mathbb{Q}(\lambda_2, \lambda_3)}$-representations, denoted by $[\Gamma]$, between the part of $H^2_{\text{ét}}(V_{\lambda,\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ that is orthogonal to the eighteen algebraically independent étale cycle classes and the

corresponding part of $H^2_{\text{ét}}(W_{\lambda,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. This isomorphism produces, via the Künneth formula and Poincaré duality, a Galois invariant class in $H^4_{\text{ét}}(V_{\lambda,\overline{\mathbb{Q}}}\times W_{\lambda,\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$. Of course, the existence of such a class is already expected due to the Tate conjecture.

In the generic situation of Theorem 3.18 the K3 surfaces have Picard rank 18. Their transcendental lattices generate continuous representations of $\mathrm{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, there are isomorphisms of $\mathrm{G}_{\mathbb{Q}}$-representations, namely

$$\begin{aligned} H^2_{\text{ét}}\Big(\mathcal{X}_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) &\cong \mathrm{T}_\ell^{(1)}\oplus\mathbb{Q}_\ell(-1)^{\oplus 18},\\ H^2_{\text{ét}}\Big(\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) &\cong \mathrm{T}_\ell^{(2)}\oplus\mathbb{Q}_\ell(-1)^{\oplus 18},\\ H^2_{\text{ét}}\Big(\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) &\cong \mathrm{T}_\ell^{(3)}\oplus\mathbb{Q}_\ell(-1)^{\oplus 18}, \end{aligned} \tag{3.87}$$

where $\mathrm{T}_\ell^{(m)}$ are the four-dimensional ℓ-adic representations obtained from the corresponding transcendental lattices using the comparison theorem for $m = 1,2,3$. We have the following:

Theorem 3.28. *Assume that $\lambda_2,\lambda_3\in\mathbb{P}^1\backslash\{0,1,\infty\}$ satisfy $\lambda_2\neq\lambda_3^{\pm1}$ and are generic otherwise. Further assume that Λ_1,Λ_2 satisfy*

$$\Lambda_1\Lambda_2 = \frac{(\lambda_2+\lambda_3)^2-4\lambda_2\lambda_3}{(1-\lambda_2)^2(1-\lambda_3)^2}\,,\qquad \Lambda_1+\Lambda_2 = -\frac{2(\lambda_2+\lambda_3)}{(1-\lambda_2)(1-\lambda_3)}\,, \tag{3.88}$$

and $\varepsilon=\lambda_2\lambda_3$. Then, there are explicit algebraic correspondences

$$\begin{aligned} \Gamma^{(1,2)} &\subset \mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)\times\mathcal{X}\,,\\ \Gamma^{(2,3)} &\subset \mathcal{X}\times\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}\,,\\ \Gamma^{(3,1)} &\subset \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C}_0)^{(\varepsilon)}\times\mathrm{Kum}(\mathcal{E}_1\times\mathcal{E}_2)\,, \end{aligned} \tag{3.89}$$

defined over $\mathbb{Q}(\lambda_2,\lambda_3)$, which induce isomorphisms of the ℓ-adic representations

$$\Big[\Gamma^{(i,j)}\Big]:\quad \mathrm{T}_\ell^{(i)}\xrightarrow{\cong}\mathrm{T}_\ell^{(j)}\,, \tag{3.90}$$

for $i,j\in\{1,2,3\}$ with $i\neq j$.

Proof. It follows from Remark 3.21 that for generic parameters $\lambda_2,\lambda_3\in\mathbb{Q}$, the K3 surfaces in Theorem 3.18 have Picard rank 18. Using the families of maps in Equation (3.55) defined over $\mathbb{Q}(k_2,k_3)$, we construct correspondences between the different Kummer surfaces as explained above. Lemma 3.27 allows us to replace the twist factor $\varepsilon=\frac{4}{\lambda_2\lambda_3(1-\lambda_2)^2(1-\lambda_3)^2}$ by $\varepsilon'=\lambda_2\lambda_3$ as $\varepsilon'/\varepsilon=(\lambda_2\lambda_3(1-\lambda_2)(1-\lambda_3)/2)^2$. Similarly for $\Gamma^{(3,1)}$ we eliminate the twist factor 2^4. □

In the situation of Corollary 3.20 the K3 surfaces have Picard rank 19. Then, there are isomorphisms of $\mathrm{G}_{\mathbb{Q}}$-representations, namely

$$\begin{aligned} H^2_{\text{ét}}\Big(\widetilde{\mathcal{X}}_{n,\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) &\cong \widetilde{\mathrm{T}}^{(1)}_{n,\ell}\oplus\mathbb{Q}_\ell(-1)^{\oplus 19}\,,\\ H^2_{\text{ét}}\Big(\mathrm{Kum}(\widetilde{\mathcal{E}}_n\times\widetilde{\mathcal{E}}_n)^{(-\varepsilon_n)}_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) &\cong \widetilde{\mathrm{T}}^{(2)}_{n,\ell}\oplus\mathbb{Q}_\ell(-1)^{\oplus 19}\,,\\ H^2_{\text{ét}}\Big(\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)^{(\varepsilon_n)}_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big) &\cong \widetilde{\mathrm{T}}^{(3)}_{n,\ell}\oplus\mathbb{Q}_\ell(-1)^{\oplus 19}\,, \end{aligned}$$

where $\widetilde{\mathrm{T}}^{(m)}_{n,\ell}$ are the three-dimensional ℓ-adic representations obtained from the corresponding transcendental lattices using the comparison theorem for $m,n=1,2,3$. In the cases above, the representation $\widetilde{\mathrm{T}}^{(2)}_{n,\ell}$ are naturally isomorphic to a symmetric square, namely $\mathrm{Sym}^2(H^1_{\text{ét}}(\widetilde{\mathcal{E}}_{n,\overline{\mathbb{Q}}},\mathbb{Q}_\ell))(\chi)$ where χ is the Dirichlet character of the quadratic twist given by ε_n; see [55]. We have the following:

Theorem 3.29. *In Equations (3.61), (3.62), (3.63) let $t\in\mathbb{P}^1\backslash S_n$ with*

$$S_1=\{0,-1,\infty\}\,,\qquad S_2=\{\pm1,3,\infty\}\qquad S_3=\{0,\pm1,\pm i,\infty\}\,,$$

and $\varepsilon_1=\frac{1}{t+1}$, $\varepsilon_2=\frac{1}{(t+1)(t-3)}$, $\varepsilon_3=1$. Then, there are explicit algebraic correspondences

$$\begin{aligned} \widetilde{\Gamma}^{(1,2)}_n &\subset \mathrm{Kum}(\widetilde{\mathcal{E}}_n\times\widetilde{\mathcal{E}}_n)^{(-\varepsilon_n)}\times\widetilde{\mathcal{X}}_n\,,\\ \widetilde{\Gamma}^{(2,3)}_n &\subset \widetilde{\mathcal{X}}_n\times\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)^{(\varepsilon_n)}\,,\\ \widetilde{\Gamma}^{(3,1)}_n &\subset \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)^{(\varepsilon_n)}\times\mathrm{Kum}(\widetilde{\mathcal{E}}_n\times\widetilde{\mathcal{E}}_n)^{(-\varepsilon_n)}\,, \end{aligned}\tag{3.91}$$

defined over $\mathbb{Q}(t)$, which induce isomorphisms of the ℓ-adic representations

$$\Big[\widetilde{\Gamma}^{(i,j)}_n\Big]:\quad \widetilde{\mathrm{T}}^{(i)}_{n,\ell}\xrightarrow{\cong}\widetilde{\mathrm{T}}^{(j)}_{n,\ell}\,,\tag{3.92}$$

for $i,j,n\in\{1,2,3\}$ with $i\neq j$.

Proof. It follows from Remark 3.21 that the K3 surfaces in Corollary 3.20 have Picard rank 19. Using the families of maps in Equation (3.64) defined over a finite field extension of $\mathbb{Q}(t)$, we construct correspondences between the different Kummer surfaces as explained above. Lemma 3.27 then allows us to replace the quadratic-twist factors $\pm\varepsilon_n\delta^2_{m,n}$ by $\pm\varepsilon_n$. □

Remark 3.30. *For $n=1$ the statement of Theorem 3.29 is precisely the main result of [55]. In fact, Equations (3.61) recover the families that appeared in [55][4]. To our knowledge, the other two cases have not appeared in the literature.*

In the situation of Corollary 3.25 and 3.26 the K3 surfaces have Picard rank 19 and 18, respectively, and there are isomorphisms of $\mathrm{G}_{\mathbb{Q}}$-representations, namely

$$H^2_{\text{ét}}\Big(\widetilde{\mathcal{X}}_{n,\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big)\cong\widetilde{\mathrm{T}}^{(1)}_{n,\ell}\oplus\mathbb{Q}_\ell(-1)^{\oplus\mu_n},\qquad H^2_{\text{ét}}\Big(\mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n)_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell\Big)\cong\widetilde{\mathrm{T}}^{(3)}_{n,\ell}\oplus\mathbb{Q}_\ell(-1)^{\oplus\mu_n},$$

[4] *All* surfaces in Theorem 3.29 have been twisted by (-1) compared to [55]. This is due to a different sign choice in the definition of the twisted Legendre pencil in Equation (3.2).

for $n = 4, 5$ and $\mu_4 = 19$, $\mu_5 = 18$. Moreover, we have the isomorphisms of $\mathrm{G}_{\mathbb{Q}}$-representations

$$H^2_{\text{ét}}\Big(\mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{F}}_n)^{(\varepsilon_n)}_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell\Big) \;\cong\; \widetilde{\mathrm{T}}^{(2)}_{n,\ell} \oplus \mathbb{Q}_\ell(-1)^{\oplus \mu_n}.$$

Here, $\widetilde{\mathcal{F}}_4 = \widetilde{\mathcal{E}}'_4$ is the two-isogenous elliptic curve such that the j-invariants of $\widetilde{\mathcal{E}}_4$ and $\widetilde{\mathcal{E}}'_4$ satisfy Equation (3.68), i.e., the equation of the classical modular curve $X_0(2)$. And $\widetilde{\mathcal{F}}_5$ is the constant elliptic curve with complex multiplication and $j(\widetilde{\mathcal{F}}_5) = 12^3$. We have the following:

Theorem 3.31. *In Equations (3.74), (3.79) let $t \in \mathbb{P}^1 \backslash S_n$ with*

$$S_4 = \{0, \pm(\sqrt{2} \pm 1), \pm 1, \pm i, \infty\}, \qquad S_5 = \{0, \pm 1, \pm i, \infty\},$$

and $\varepsilon_4 = \frac{1}{t(t^2-1)}$, $\varepsilon_5 = \frac{1}{4t}$. Then, there are explicit algebraic correspondences

$$\begin{aligned} \widetilde{\Gamma}^{(1,2)}_n &\subset \mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{F}}_n)^{(\varepsilon_n)} \times \widetilde{\mathcal{X}}_n, \\ \widetilde{\Gamma}^{(2,3)}_n &\subset \widetilde{\mathcal{X}}_n \times \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n), \\ \widetilde{\Gamma}^{(3,1)}_n &\subset \mathrm{Kum}(\mathrm{Jac}\,\widetilde{\mathcal{C}}_n) \times \mathrm{Kum}(\widetilde{\mathcal{E}}_n \times \widetilde{\mathcal{F}}_n)^{(\varepsilon_n)}, \end{aligned} \tag{3.93}$$

defined over $\mathbb{Q}(t)$, which induce isomorphisms of the ℓ-adic representations

$$\Big[\widetilde{\Gamma}^{(i,j)}_n\Big]: \quad \widetilde{\mathrm{T}}^{(i)}_{n,\ell} \xrightarrow{\cong} \widetilde{\mathrm{T}}^{(j)}_{n,\ell}, \tag{3.94}$$

for $i, j \in \{1, 2, 3\}$ with $i \neq j$ and $n = 4, 5$.

Proof. The proof is analogous to the proof of Theorem 3.29. □

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