Mathematics Underlying the F-Theory/Heterotic String Duality in Eight Dimensions

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Abstract

We give an analytic description of the moduli space of classical vacua for F-theory compactified on elliptic $K3$ surfaces, on open tubular regions near the two Type II boundary divisors. The structure of these open sets is related to the total spaces of certain holomorphic theta fibrations over the corresponding boundary divisors. As the two Type II divisors can be naturally identified as moduli spaces of elliptic curves and flat G-bundles with $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$, one is led to an analytic isomorphism between these open domains and regions of the moduli spaces of heterotic string theory compactified on the two-torus corresponding to large volumes of the torus. This provides a proof for the classical version of F-theory/Heterotic String Duality in eight dimensions. A description of the Type II boundary points in terms of elliptic stable $K3$ surfaces is also given.

1   Introduction

One of the dualities in string theory, the F-theory/heterotic string duality in eight dimensions [34] [31], predicts an interesting correspondence between two seemingly disparate geometrical objects. On one side of the duality there are elliptically fibered $K3$ surfaces with section. On the other side, one finds elliptic curves endowed with certain flat connections and complexified Kahler classes.

The F-theory [30] [34] is a 12-dimensional string theory which generally exists on elliptically fibered ambient manifolds with section. Heterotic string theory, on the other hand, exists on a 10-dimensional space-time. In order to obtain effective 8-dimensional models, one compactifies the two theories along elliptic $K3$ surfaces and elliptic curves, respectively. The duality mentioned above predicts then that the two theories are equivalent at the quantum level. In particular, their moduli spaces of quantum vacua should be isomorphic. As it is generally believed, in certain ranges of parameters the quantum corrections should be small and the quantum vacua should be well approximated by classical vacua. This leads one to expect that, the moduli spaces of classical vacua of the two theories should resemble each other, at least on regions corresponding to insignificant quantum effects.

The classical vacua for the heterotic string theory compactified along a two-torus E (there are two distinct such theories, one with structure Lie group $G_1 = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ and the other with $G_2 = \mathrm{Spin}(32)/\mathbb{Z}_2$) consists of a flat G_i-connection on E, a flat metric and an extra one-dimensional field, the B-field. In the original physics formulation [24] [25], the B-field appears as a globally defined two-form B. The metric and B fit together to form the imaginary and, respectively, the real part of the so-called complexified Kahler class. Each triplet (A, g, B) determines a lattice of momenta $\mathcal{L}_{(A,g,B)}$ (after K. Narain [24]) governing the associated physical theory. The lattices $\mathcal{L}_{(A,g,B)}$, turn out to be even, unimodular and of rank 20. They are well-defined up to $O(2) \times O(18)$ rotations and vary, according to the triplet parameter, in a fixed ambient

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real space $\mathbb{R}^{2,18}$. The real group $O(2,18)$ acts transitively on the set of all $\mathcal{L}_{(A,g,B)}$ and, in this light, one can regard the physical momenta as parameterized by the 36-dimensional real homogeneous space:

$$O(2,18)/O(2) \times O(18). \tag{1}$$

One identifies then the configurations in (1) determining equivalent quantum theories. This amounts to factoring out the left-action of the group Γ of integral isometries of the lattice. However, not all identifications so created are accounted for by classical geometry. Part of the Γ-action models the so-called quantum corrections [2] and results in identifying momenta for pairs of triplets (A, g, B) which are not isomorphic from the geometric point of view. The quantum (Narain) moduli space of distinct heterotic string theories compactified on the two-torus appears as:

$$\mathcal{M}_{\text{het}}^{\text{quantum}} = \Gamma \backslash O(2,18)/O(2) \times O(18). \tag{2}$$

The above physics-inspired Narain construction has a major flaw, though. It does not provide a holomorphic description. Technically, one can endow the homogeneous quotient (2) with a natural complex structure, but holomorphic families of elliptic curves and flat connections do not embed as holomorphic sub-varieties in $\mathcal{M}_{\text{het}}^{\text{quantum}}$. An outline of these issues is provided in the Appendix.

In the recent years, it has been noted by a number of authors (see for example [35] or [8]) that, in order to fulfill various anomaly cancellation conditions required by heterotic string theory, the B-field has to be understood within a gerbe-like formalism. In [8], D. Freed introduces B as a cochain in differential cohomology. Taking this point of view, one can ask then for a description of the space of triplets (A, g, B) up to natural geometric isomorphism. This is the moduli space $\mathcal{M}_{\text{het}}^{G_i}$ of classical vacua for G_i-heterotic string theory compactified over the two-torus. Freed's approach can be used to describe $\mathcal{M}_{\text{het}}^{G_i}$ in an explicit holomorphic framework. It was shown in [6] that:

Theorem 1.

1. *The classical G_i-heterotic moduli space $\mathcal{M}_{\text{het}}^{G_i}$ can be given the structure of a 18-dimensional complex variety with orbifold singularities.*

2. *$\mathcal{M}_{\text{het}}^{G_i}$ represents the total space of a holomorphic Seifert $\mathbb{C}^*$-fibration*

$$\mathcal{M}_{\text{het}}^{G_i} \to \mathcal{M}_{E,G_i}, \tag{3}$$

where $\mathcal{M}_{E,G_i}$ is the moduli space of isomorphism classes of pairs of elliptic curves and flat G_i-bundles.

The holomorphic orbifold structure of $\mathcal{M}_{E,G_i}$ is described in [15]. If one denotes by $\mathbb{H}$ the upper half-plane, and by Λ the co-root lattice of G_i, then $\mathcal{M}_{E,G_i}$ is represented by a quotient of $\mathbb{H} \times \Lambda_{\mathbb{C}}$ through the action of a discrete group. Under this description, the fibration (3) can be recovered from a well-known fibration with complex lines over $\mathcal{M}_{E,G_i}$. This is, roughly speaking, the line fibration supporting the holomorphic theta function:

$$B_G \colon \mathbb{H} \times \Lambda_{\mathbb{C}} \to \mathbb{C}, \quad B_G(\tau, \, z) = \frac{1}{\eta(\tau)^{16}} \left(\sum_{\gamma \in \Lambda} e^{\pi i (2(z,\gamma) + \tau(\gamma,\gamma))} \right) \tag{4}$$

where $\eta(\tau)$ is Dedekind's eta-function. In this setting, one can prove:

Theorem 2. *([6]) The $\mathbb{C}^*$-fibration (3) is holomorphically identified with the complement of the zero-section in the complex line fibration induced by (4).*

Hence, the heterotic classical moduli space $\mathcal{M}_{\text{het}}^{G_i}$ can be holomorphically identified with the total space of the theta fibration with the zero-section divisor removed.

We turn now to the other side of the duality. The classical vacua for 8-dimensional F-theory are simply elliptically fibered $K3$ surfaces with section. Using the period map and global Torelli theorem [4] [9], one can regard the moduli space $\mathcal{M}_{K3}$ of such structures as a moduli space of Hodge structures of weight two,

i.e. as a quotient of an open 18-dimensional hermitian symmetric domain Ω by an arithmetic group of integral automorphisms. In order to identify all equivalent classical vacua, there is one more factorization to be taken into account, identifying the complex conjugate structures. The classical 8-dimensional F-theory moduli space obtained, denoted $\mathcal{M}_F$, is then double-covered by $\mathcal{M}_{K3}$ and can be seen to be isomorphic to an arithmetic quotient of a symmetric domain:

$$\mathcal{M}_F \;\simeq\; \Gamma \backslash O(2,18)/O(2) \times O(18). \tag{5}$$

The identification between the above description and (2) is the usual physics literature formulation of the F-theory/heterotic string duality in eight dimensions.

The goal of this paper is to establish a rigorous geometric comparison between the classical moduli spaces $\mathcal{M}_F$ and $\mathcal{M}_{\mathrm{het}}^{G_i}$. Our construction provides a natural holomorphic identifications between these classical moduli spaces, exactly on the regions where physics predicts the quantum effects are insignificant.

The paper is structured as follows. In section 2 we review various facts pertaining to the construction of the moduli space $\mathcal{M}_{K3}$ of elliptic $K3$ surfaces with section. This space is not compact. However, using a special case of Mumford toroidal compactification [1] [11], one can perform an arithmetic partial compactification:

$$\mathcal{M}_{K3} \;\subset\; \overline{\mathcal{M}}_{K3}$$

by adding two divisors at infinity $\mathcal{D}_1$ and $\mathcal{D}_2$, related to the two possible kinds of Type II maximal rational parabolic subgroups of $O(2,18)$. The arithmetic machinery producing the partial compactification is reviewed in section 2.3. In section 3 we discuss the geometrical interpretation of the compactification. The points of $\mathcal{D}_1$ and $\mathcal{D}_2$ correspond to semi-stable degenerations of $K3$ surfaces given by either a union of two rational elliptic surfaces glued together along a smooth fiber or a union of rational surfaces glued along an elliptic curve with elliptic fibration degeneration into two rational curves meeting at two points. Each of the two configurations exhibits an elliptic curve E (the double curve of the degeneration) and endows this elliptic curve with a flat G-connection. For $\mathcal{D}_1$ the resulting Lie group G turns out to be $G_1 = (E_8 \times E_8) \rtimes \mathbb{Z}_2$ whereas for $\mathcal{D}_2$ one obtains $G_2 = \mathrm{Spin}(32)/\mathbb{Z}_2$. In the second case the flat connections obtained carry "vector structure", in the sense that they can be lifted to flat $\mathrm{Spin}(32)$-connections. Under this geometrically defined correspondence, one obtains a holomorphic isomorphism:

$$\mathcal{D}_i \;\simeq\; \mathcal{M}_{E,G_i}. \tag{6}$$

Next, each of the two types of parabolic groups determining the boundary components $\mathcal{D}_i$ produces an infinite sheeted non-normal parabolic cover $p\colon \mathcal{P}_i \to \mathcal{M}_{K3}$. The total space $\mathcal{P}_i$ fibers holomorphically $\pi\colon \mathcal{P}_i \to \mathcal{D}_i$ over the corresponding divisor at infinity, all fibers being copies of $\mathbb{C}^*$. Under identification (6), one obtains therefore a diagram:

$$
\begin{array}{ccccc}
\mathcal{P}_i & \xrightarrow{\ \pi\ } & \mathcal{D}_i & \simeq & \mathcal{M}_{E,G_i} \\
\downarrow & & & & \\
\mathcal{M}_{K3}. & & & &
\end{array}
\tag{7}
$$

It turns out that, there exists an open subset $\mathcal{V}_i \subset \mathcal{P}_i$ on which the restriction of the parabolic projection $p\colon \mathcal{P}_i \to \mathcal{M}_{K3}$ becomes a holomorphic isomorphism to an open neighborhood of infinity near the cusp $\mathcal{D}_i$ in $\mathcal{M}_{K3}$. Moreover, one can realize $\mathcal{V}_i$ as a punctured tubular neighborhood of the zero-section in the holomorphic complex line fibration naturally associated to

$$\pi\colon \mathcal{P}_i \to \mathcal{M}_{E,G_i}. \tag{8}$$

Thus, a neighborhood of the boundary component $\mathcal{D}_i$ in $\mathcal{M}_{K3}$ can then be identified with an open punctured neighborhood of the zero-section of the line fibration associated to (8).

In section 5 we give an explicit description of this line fibration.

Theorem 3. *The fibration with complex lines associated to (8) is holomorphically isomorphic with the theta $\mathbb{C}$-fibration induced by (4).*

In the light of theorems 1 and 2, there is then a holomorphic isomorphism of $\mathbb{C}^*$-fibrations, unique up to twisting with a unitary complex number:

$$
\begin{array}{ccc}
\mathcal{P}_i & \simeq & \mathcal{M}_{\mathrm{het}}^{G_i} \\
\downarrow & & \downarrow \\
\mathcal{M}_{E,G_i} & = & \mathcal{M}_{E,G_i}
\end{array}
\tag{9}
$$

and that gives a natural explicit mathematical identification between the region in $\mathcal{M}_{\mathrm{het}}^{G_i}$ corresponding to large volumes with a region of $\mathcal{M}_F$ in the vicinity of the boundary component D_i. These are exactly the regions that the physics duality predicts should be isomorphic.

This paper belongs to a long project begun by the second author jointly with R. Friedman and E. Witten [13] in 1996 and continued jointly with R. Friedman afterwards. The initial aim of the project was to give precise mathematical descriptions of various moduli spaces of principal G-bundles over elliptic curves in order to verify conjectures arising out of the F-theory/heterotic string theory duality in physics. Building on this earlier work, the present paper and [6] establish the mathematical results allowing one to describe the duality completely when the two theories in question are compactified to eight dimensions.

2 Review of the Compactification Procedure

A coarse moduli space $\mathcal{M}_{K3}$ for isomorphism classes of elliptically fibered $K3$ surfaces with section can be described using the period map. In this section we review the Type II partial compactification of $\mathcal{M}_{K3}$.

2.1 Period Space

It is well-known that any two $K3$ surfaces are diffeomorphic. The second cohomology group over integers is torsion-free of rank 22 and, when endowed with the symmetric bilinear form given by cup product, is an even unimodular lattice of signature $(3, 19)$. Up to isometry, there exists a unique lattice with these properties. We pick a lattice of this type and denote it by $\mathbb{L}$. It happens then that for any $K3$ surface X there always exists an isometry:

$$
\varphi \colon H^2(X, \mathbb{Z}) \to \mathbb{L}.
\tag{10}
$$

Such a map is called a marking.

An elliptic structure with section on X induces naturally two particular line bundles $\mathcal{F}, \mathcal{S} \in \mathrm{Pic}(X)$ corresponding to the elliptic fiber and section. Let $f, s \in H^2(X, \mathbb{Z})$ be the cohomology classes corresponding $\mathcal{F}$ and $\mathcal{S}$. These special classes intersect as $f^2 = 0$, $f.s = 1$, $s^2 = -2$ and therefore span a hyperbolic type sub-lattice Q inside $H^2(X, \mathbb{Z})$. The notion of marking can be adapted for this framework. Let H be a choice of hyperbolic sub-lattice in $\mathbb{L}$. All such choices are equivalent under the action of the group of isometries of $\mathbb{L}$. Choose a basis $\{F, S\}$ for H with $(F, F) = 0$, $(F, S) = 1$ and $(S, S) = -2$. A marking φ as in (10) is said to be compatible with the elliptic structure if $\varphi(f) = F$ and $\varphi(s) = S$. In particular, a compatible marking transports the hyperbolic sub-lattice $Q \subset H^2(X, \mathbb{Z})$ isomorphically to H. Two marked pairs (X, φ) and (X', φ') are called isomorphic if there exists an isomorphism of surfaces $g \colon X \to X'$ such that $\varphi' = \varphi \circ g^*$.

Let $\mathbb{L}_o$ be the sub-lattice of $\mathbb{L}$ orthogonal to H. The lattice $\mathbb{L}_o$ is even, unimodular, and of signature $(2, 18)$. By standard arguments, a marked pair (X, φ) determines a polarized Hodge structure of weight two on $\mathbb{L}_o \otimes \mathbb{C}$ which is esentially determined by the period $(2,0)$-line $[\omega] \subset \mathbb{L}_o \otimes \mathbb{C}$. The periods satisfy the Hodge-Riemann bilinear relations $(\omega, \omega) = 0$, $(\omega, \bar{\omega}) > 0$. The classifying space of polarized Hodge structures of weight two on $\mathbb{L}_o \otimes \mathbb{C}$ is then given by the period domain

$$
\Omega \;=\; \{\, \omega \in \mathbb{P}\left(\mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}\right) \mid (\omega, \omega) = 0, \; (\omega, \bar{\omega}) > 0 \,\}.
\tag{11}
$$

This is an open 18-dimensional complex analytic variety embedded inside the compact complex quadric:

$$\Omega^\vee = \{\omega \mid (\omega, \omega) = 0\} \subset \mathbb{P}\left(\mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}\right).$$

One can equivalently regard the periods $\omega \in \Omega$ as space-like, oriented two-planes in $\mathbb{L}_o \otimes \mathbb{R}$. The real Lie group $O(2, 18)$ of real isometries of $\mathbb{L}_o \otimes \mathbb{R}$ acts then transitively on Ω leading to a description of the period domain in the form of a symmetric bounded domain:

$$\Omega \simeq O(2, 18)/SO(2) \times O(18). \tag{12}$$

An elliptic structure with section on a $K3$ surface X can be naturally regarded as a (pseudo-ample) lattice polarization of X with a hyperbolic type lattice. Following arguments of [28] [21] and [26], one can prove then the existence of a fine moduli space of marked elliptically fibered $K3$ surfaces with section, which is a 18-dimensional complex manifold $\mathcal{M}_{K3}^{\mathrm{mark}}$. It follows then that a marked elliptic $K3$ surface with section is uniquely determined by its period. The period map:

$$\mathrm{per} \colon \mathcal{M}_{K3}^{\mathrm{mark}} \to \Omega \tag{13}$$

is a holomorphic isomorphism. However, in this setting, the period $\omega \in \Omega$ clearly depends on the choice of marking. One removes the markings from the picture by dividing out the period domain by the action of the isometry group of the lattice.

Let Γ be the group of isometries of $\mathbb{L}_o$. Two periods correspond to isomorphic marked surfaces if and only if they can be transformed one into the other through an isometry in Γ. The arguments of global Torelli theorem allow one to conclude that:

$$\mathcal{M}_{K3} = \Gamma\backslash\Omega \tag{14}$$

is a coarse moduli space for elliptic $K3$ surfaces with section, without regard to marking.

Let us briefly analyze the quotient (14). First of all, $\mathcal{M}_{K3}$ is connected. The period domain Ω consists of two connected components, corresponding to the choice of orientation in the set of positive two-planes in $\mathbb{L}_o \otimes \mathbb{R}$. The two components are mapped into each other by complex conjugation. We choose either one and denote it by D. Thus $\Omega = D \sqcup \overline{D}$. However, there are isometries in Γ which exchange D and $\overline{D}$ and therefore (14) is connected. Secondly, the space $\mathcal{M}_{K3}$ can be given a description as a quotient of a bounded symmetric domain by a discrete, arithmetically defined modular group. Indeed, the isomorphism (12) is Γ-equivariant and therefore:

$$\Gamma\backslash\Omega \simeq \Gamma\backslash O(2, 18)/SO(2) \times O(18).$$

2.2 The Classical F-Theory Moduli Space

The moduli space $\mathcal{M}_F$ of classical vacua associated to F-theory compactified on a $K3$ surface is obtained by identifying complex conjugated structures in $\mathcal{M}_{K3}$. Using identification (14) one obtains that:

$$\mathcal{M}_F = \hat{\Gamma}\backslash\Omega \tag{15}$$

where the new group $\hat{\Gamma}$ is the semi-direct product $\Gamma \rtimes \mathbb{Z}_2 \subset \mathrm{Aut}\left(\mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}\right)$ with the $\mathbb{Z}_2$ factor generated by complex conjugation.

$$\mathcal{M}_F = \hat{\Gamma}\backslash\Omega \simeq \hat{\Gamma}\backslash O(2, 18)/SO(2) \times O(18) \simeq \Gamma\backslash O(2, 18)/O(2) \times O(18). \tag{16}$$

2.3 Arithmetic of Compactification of $\mathcal{M}_{K3}$

The moduli space $\mathcal{M}_{K3}$ is connected but not compact. There exists various arithmetic techniques aiming at compactifying $\Gamma\backslash\Omega$. The simplest one is the Baily-Borel procedure [3] which we briefly review next. Later, we shall turn our attention to a particular case of Mumford's toroidal compactification [1] which plays a central role in the computation we undertake in this paper.

The Baily-Borel procedure [3] introduces an auxiliary space Ω^* with $\Omega \subset \Omega^* \subset \Omega^\vee$. The topological boundary of $\Omega \subset \Omega^\vee$ decomposes into a disjoint union of closed analytic subsets, called boundary components. There are two types of such components. Some are zero-dimensional and are represented by the points in in the real quadric $\Omega^\vee \cap \mathbb{P}(\mathbb{L}_o \otimes_\mathbb{Z} \mathbb{R})$. The others are copies of $\mathbb{P}^1$ and are generated by the complexified images in $\mathbb{P}(\mathbb{L}_o \otimes_\mathbb{Z} \mathbb{C})$ of the 2-dimensional isotropic subspaces of $\mathbb{L}_o \otimes_\mathbb{Z} \mathbb{R}$. Group theoretically, it can be seen that the stabilizer

$$\text{Stab}(F) = \{ \, g \in O^{++}(2,18) \mid gF = F \, \}$$

of a boundary component F is a proper maximal parabolic subgroup of $O^{++}(2,18)$. A boundary component F is called then rational if its stabilizer $\text{Stab}(F)$ is defined over $\mathbb{Q}$. The assignment $P \to F_P$ with $\text{Stab}(F_P) = P$ determines a bijective correspondence between the set of proper maximal rational parabolic subgroups of $O^{++}(2,18)$ and the set of all rational boundary components. One defines then:

$$\Omega^* \;=\; \Omega \;\cup\; \left(\bigcup_P F_P \right)$$

where the right union is made over all proper maximal rational parabolics. The action of Γ extends naturally to Ω^*. Moreover, one can endow Ω^* with the Satake topology, under which the Γ-action is continuous. The Baily-Borel compactification appears then as:

$$(\Gamma\backslash\Omega)^* \;\overset{\text{def}}{=}\; \Gamma\backslash\Omega^*. \tag{17}$$

The main features of this new quotient space are as follows (see [3] for details). The space $(\Gamma\backslash\Omega)^*$ is Hausdorff, compact, connected and can be given a structure of complex algebraic space. The quotient $\Gamma\backslash\Omega$ is embedded in $(\Gamma\backslash\Omega)^*$ as a Zariski open subset. If $I_i(\mathbb{L}_o)$, $i \in \{1,2\}$ represents the set of primitive isotropic sub-lattices of rank i in $\mathbb{L}_o$ then the complement

$$(\Gamma\backslash\Omega)^* - \Gamma\backslash\Omega$$

consists of $|\Gamma\backslash I_1(\mathbb{L}_o)|$ points and $|\Gamma\backslash I_2(\mathbb{L}_o)|$ copies of $\text{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$. Let us note that the complex conjugation involution on Ω extends to Ω^*. On boundary, it preserves the points and induces complex conjugation on the one-dimensional $\mathbb{P}^1$'s. The procedure provides therefore a compactification for the classical F-theory moduli space $\mathcal{M}_F$:

$$\hat{\Gamma}\backslash\Omega \;\subset\; \left(\hat{\Gamma}\backslash\Omega \right)^*$$

with boundary strata given by points and copies of $(\text{PSL}(2,\mathbb{Z}) \rtimes \mathbb{Z}_2) \backslash \mathbb{H}$ with $\mathbb{Z}_2$ generated by $\tau \to -\bar{\tau}$.

It is known that Baily-Borel construction gives the minimal geometrically meaningful compactification of $\Gamma\backslash\Omega$ in the sense that it is dominated by any other geometric compactification. However, the disadvantage of the method is that the boundary has large codimension (it consists of only points and curves) and contains only partial geometrical information. One avoids these inconveniences by using a blow-up of the Baily-Borel construction, the toroidal compactification of Mumford [1]. This compactification, although not canonical in general, gives divisors as boundary components and carries significantly more geometrical information. The main arguments describing the construction, as presented in [11] and [9], are as follows.

The Mumford boundary components associated to $\Gamma\backslash\Omega$ involve again the maximal rational parabolic subgroups of $O(2,18)$. These are stabilizers of non-trivial isotropic subspaces $V_\mathbb{Q} \subset \mathbb{L}_o \otimes_\mathbb{Z} \mathbb{Q}$. The lattice $\mathbb{L}_o$ has signature $(2,18)$, and hence, if $V_\mathbb{Q}$ is isotropic then its dimension is either 2 or 1. If $\dim(V_\mathbb{Q}) = 1$, then the associated Baily-Borel rational boundary component F is represented by just a point. Such a component is called of Type III. For $\dim(V_\mathbb{Q}) = 2$, the corresponding boundary component F is 1-dimensional. In this case F is said to be of Type II. Each rational Baily-Borel component F will determine a Mumford boundary component $\mathcal{B}(F)$. We shall be concerned here only with describing the components of Type II for which the construction is canonical.

Let $V_\mathbb{Q}$ be a rank-two isotropic lattice and F the associated Baily-Borel component. We denote:

$$P(F) \;=\; \text{Stab}(V_\mathbb{R}) \subset O(2,18)$$

$$W(F) \;=\; \text{the unipotent radical of } P(F)$$
$$U(F) \;=\; \text{the center of } W(F).$$

It turns out that $U(F)$ is 1-dimensional (also definable over $\mathbb{Q}$) and the Lie algebra of its real form can be described as:

$$u(F) \;=\; \{N \in \mathrm{Hom}\,((\mathbb{L}_o)_{\mathbb{R}}, (\mathbb{L}_o)_{\mathbb{R}}) \mid \mathrm{Im}(N) \subset V_{\mathbb{R}} \text{ and } (Na, b) + (a, Nb) = 0,\ \forall\, a, b \in (\mathbb{L}_o)_{\mathbb{R}} \}. \qquad (18)$$

One obtains that any $N \in u(F)$ satisfies $N^2 = 0$, $\mathrm{Im}(N) = V_{\mathbb{R}}$ and $\mathrm{Ker}(N) = V_{\mathbb{R}}^{\perp}$. There is then an associated weight filtration:

$$0 \;\subset\; V_{\mathbb{R}} \;\subset\; V_{\mathbb{R}}^{\perp} \;\subset\; (\mathbb{L}_o)_{\mathbb{R}}. \qquad (19)$$

We pick a primitive integral endomorphism $N \in u(F)$ and consider the groups:

$$\mathrm{U}(N)_{\mathbb{C}} \;=\; \{\exp\,(\lambda N) \mid \lambda \in \mathbb{C} \}$$

$$\mathrm{U}(N)_{\mathbb{Z}} \;=\; \{\exp\,(\lambda N) \mid \lambda \in \mathbb{Z} \} \;=\; \mathrm{U}(N)_{\mathbb{C}} \cap O^{++}(2, 18; \mathbb{Z}).$$

The group $\mathrm{U}(N)_{\mathbb{C}}$ acts upon the extended period domain

$$\Omega^{\vee} \;=\; \{\, [z] \in \mathbb{P}\,(\mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}) \mid (z, z) = 0 \,\}.$$

providing an intermediate filtration $\Omega \subset \Omega(F) \subset \Omega^{\vee}$ where $\Omega(F) = \mathrm{U}(N)_{\mathbb{C}} \cdot \Omega$.

One defines then the Mumford boundary component associated to F as the space of nilpotent orbits:

$$\mathcal{B}(F) \;=\; \Omega(F)/\mathrm{U}(N)_{\mathbb{C}}. \qquad (20)$$

In this setting,

$$\Omega(F)/\mathrm{U}(N)_{\mathbb{Z}} \to \mathcal{B}(F) \qquad (21)$$

is a holomorphic principal bundle with structure group $\mathrm{U}(N)_{\mathbb{C}}/\mathrm{U}(N)_{\mathbb{Z}} \simeq \mathbb{C}^*$. The inclusion

$$\Omega/\mathrm{U}(N)_{\mathbb{Z}} \;\hookrightarrow\; \Omega(F)/\mathrm{U}(N)_{\mathbb{Z}}$$

realizes $\Omega/\mathrm{U}(N)_{\mathbb{Z}}$ as an open subset in the total space of (21). Let then:

$$\overline{\Omega(F)/\mathrm{U}(N)_{\mathbb{Z}}} \;=\; (\Omega(F)/\mathrm{U}(N)_{\mathbb{Z}}) \times_{\mathbb{C}^*} \mathbb{C}. \qquad (22)$$

This amounts to gluing in the zero section in the $\mathbb{C}^*$-fibration (21). One defines then:

$$\overline{\Omega/\mathrm{U}(N)_{\mathbb{Z}}} \;\overset{\text{def}}{=}\; \text{interior of the closure of } \Omega/\mathrm{U}(N)_{\mathbb{Z}} \text{ in } \overline{\Omega(F)/\mathrm{U}(N)_{\mathbb{Z}}}.$$

Set-theoretically, one has:

$$\overline{\Omega/\mathrm{U}(N)_{\mathbb{Z}}} \;=\; \Omega/\mathrm{U}(N)_{\mathbb{Z}} \;\sqcup\; \mathcal{B}(F).$$

Finally:

$$\overline{\Omega} \;\overset{\text{def}}{=}\; \bigcup_{F} \overline{\Omega/\mathrm{U}(N)_{\mathbb{Z}}} \;=\; \Omega \;\sqcup\; \left(\bigsqcup_{F} \mathcal{B}(F)\right) \qquad (23)$$

the union being performed over all rational Baily-Borel boundary components of Type II. This space inherits a topology. The arithmetic action of Γ induces a closed discrete equivalence relation on (23). The quotient space, denoted by $\overline{\Gamma\backslash\Omega}$, enjoys the following properties (see [1], [11] for details):

Theorem 4.
- $\overline{\Gamma\backslash\Omega}$ *is a quasi-projective analytic variety.*
- $\overline{\Gamma\backslash\Omega}$ *contains* $\Gamma\backslash\Omega$ *as a Zariski open dense subset.*

- *The complement $\overline{\Gamma\backslash\Omega} - \Gamma\backslash\Omega$ consists of two irreducible divisors. These divisors are quotients of smooth spaces by finite group actions.*

We shall denote the two divisors by $\mathcal{D}_{E_8 \oplus E_8}$ and $\mathcal{D}_{\Gamma_{16}}$. The reason for this terminology is the following. The two Type II divisors in question correspond to the two distinct orbits in $\Gamma\backslash I_2(\mathbb{L}_o)$ where $I_2(\mathbb{L}_o)$ is the set of primitive isotropic rank-two sub-lattices in $\mathbb{L}_o$. One can identify the orbit to which a certain isotropic sub-lattice belongs using the following recipe. Let $V \in I_2(\mathbb{L}_o)$. The quotient lattice $V^\perp/V$ is even, unimodular, negative-definite and has rank 16. It is known that, up to isomorphism, there exists only two lattices of this type: $-(E_8 \oplus E_8)$ and $-\Gamma_{16}$. The two isomorphism classes perfectly differentiate the two orbits in $\Gamma\backslash I_2(\mathbb{L}_o)$. There are therefore only two distinct Baily-Borel boundary curves in

$$(\Gamma\backslash\Omega)^* - \Gamma\backslash\Omega$$

and, accordingly, there are two Type II components in Mumford's compactification.

In fact, for each isotropic sub-lattice V there is a natural projection:

$$\mathcal{B}(F) \to F \tag{24}$$

defined by assigning to a nilpotent orbit $\{U(N)_\mathbb{C} \cdot \omega\}$ the complex line $\{\omega\}^\perp \cap V_\mathbb{C} \subset V_\mathbb{C}$. We shall see the geometrical significance of (24) in the next section. At this point, we just note that these projections descend to maps from the Type II Mumford divisors to the two Baily-Borel boundary curves under

$$\overline{\Gamma\backslash\Omega} \to (\Gamma\backslash\Omega)^* .$$

As mentioned earlier, the main goal of this paper is to describe explicitly the structure of $\Gamma\backslash\Omega$ in a neighborhood of the two Type II divisors $\mathcal{D}_{E_8 \oplus E_8}$ and $\mathcal{D}_{\Gamma_{16}}$. Our description will go along the following direction. Let F be a Type II Baily-Borel component and denote by $\Gamma_F = P(F) \cap \Gamma$ the stabilizer of the associated isotropic sub-lattice V. As subgroup of Γ, the group Γ_F induces an equivalence relation on $\overline{\Omega}$ dominating the Γ-one. One obtains therefore the following sequence of analytic projections:

$$\overline{\Omega} \to \Gamma_F\backslash\overline{\Omega} \to \overline{\Gamma\backslash\Omega}. \tag{25}$$

Then, as explained in Chapter 5 of [1]:

Lemma 5. *There exists an open subset*

$$\mathcal{U}_F \subset \overline{\Omega/U(N)_\mathbb{Z}} \subset \overline{\Omega},$$

tubular neighborhood of the Mumford boundary component $\mathcal{B}(F) \subset \overline{\Omega}$ such that on $\Gamma_F \cdot \mathcal{U}_F$, the Γ-equivalence reduces to Γ_F-equivalence.

In the light of this lemma, the analytic projection:

$$\Gamma_F\backslash (\Gamma_F \cdot \mathcal{U}_F) \to \Gamma\backslash(\Gamma \cdot \mathcal{U}_F) \tag{26}$$

is an isomorphism. One has therefore an analytic identification between an open neighborhood of the Mumford divisor associated to F in $\overline{\Gamma\backslash\Omega}$ and

$$\mathcal{V}_F \overset{\text{def}}{=} \Gamma_F\backslash (\Gamma_F \cdot \mathcal{U}_F) \subset \Gamma_F\backslash \overline{\Omega/U(N)_\mathbb{Z}} \subset \Gamma_F\backslash \overline{\Omega(F)/U(N)_\mathbb{Z}}.$$

But, as observed earlier,

$$\widetilde{\Theta} \colon \overline{\Omega(F)/U(N)_\mathbb{Z}} \to \mathcal{B}(F) \tag{27}$$

is a holomorphic line bundle. After factoring out the action of Γ_F, one obtains a holomorphic fibration with complex lines:

$$\Theta \colon \Gamma_F\backslash\overline{\Omega(F)/U(N)_\mathbb{Z}} \to \Gamma_F\backslash\mathcal{B}(F). \tag{28}$$

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One can regard then $\mathcal{V}_F$ as an open punctured tubular neighborhood of the zero-section in (28).

Based on the above arguments, one concludes that an open subset of the period domain $\Gamma\backslash\Omega$ which is a neighborhood of one of the two possible Type II divisors can be identified with an open punctured tubular neighborhood of the zero-section in the parabolic fibration (28). Therefore, in order to describe the structure of $\mathcal{M}_{K3}$ in the vicinity of one of the two Type II divisors $\mathcal{D}_{E_8 \oplus E_8}$ and $\mathcal{D}_{\Gamma_{16}}$, it is essential to explicitly describe the holomorphic type of (28). We accomplish this task in section 5.

We finish this section with a note on the behavior of complex conjugation within the framework of the above construction. The complex conjugation on Ω extends naturally to an involution of $\overline{\Omega}$ giving complex conjugation on each Type II Mumford boundary component $\mathcal{B}(F)$. One can perform therefore a similar partial compactification:

$$\mathcal{M}_F = \hat{\Gamma}\backslash\Omega \ \subset \ \overline{\hat{\Gamma}\backslash\Omega}$$

with $\overline{\hat{\Gamma}\backslash\Omega} \ - \ \hat{\Gamma}\backslash\Omega$ consisting of two boundary divisors (obtained as quotients of the two Type II divisors of $\Gamma\backslash\Omega$ by complex conjugation). Open neighborhoods of $\mathcal{M}_F$ near the boundary divisors are still described by open neighborhoods of the zero-section in the total space of the parabolic cover (28).

2.4   Boundary Components and Hodge Structures

One can give a Hodge theoretic interpretation for the boundary component $\mathcal{B}(F)$. A period $\omega \in \Omega$ determines automatically a polarized Hodge structure of weight two on $\mathbb{L}\otimes_{\mathbb{Z}}\mathbb{C}$, corresponding geometrically to a marked elliptic $K3$ surface with section. Taking orthogonals with respect to the fixed hyperbolic sub-lattice $H \subset \mathbb{L}$ (which by construction consist of $(1,1)$-cycles and is therefore orthogonal to the period line), one obtains a polarized Hodge structure of weight two on $\mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}$,

$$0 \ \subset \ \{\omega\} \ \subset \ \{\omega\}^{\perp} \ \subset \ \mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}. \tag{29}$$

Let then $V \subset \mathbb{L}_o$ be the primitive isotropic rank-two sub-lattice corresponding to the Type II Baily-Borel boundary component F. There is an induced weight filtration:

$$0 \ \subset \ \mathrm{V}_{\mathbb{C}} \ \subset \ (\mathrm{V}_{\mathbb{C}})^{\perp} \ \subset \ \mathbb{L}_o \otimes \mathbb{C}. \tag{30}$$

Together, filtrations (29) and (30) yield a mixed Hodge structure on $\mathbb{L}_o \otimes_{\mathbb{Z}} \mathbb{C}$. Taking this point of view, one can regard the domain $\Omega(F) = U(N)_{\mathbb{C}} \cdot \Omega$ as the space of mixed Hodge structures on the weight filtration (30). These structures are acted upon by the group $U(N)_{\mathbb{C}}$. The Type II Mumford boundary component

$$\mathcal{B}(F) = \Omega(F)/U(N)_{\mathbb{C}}$$

appears then as the space of nilpotent orbits of such mixed Hodge structures.

There are three $U(N)_{\mathbb{C}}$-invariant graded pure Hodge structures associated to each nilpotent orbit in $\mathcal{B}(F)$:

$$0 \ \subset \ \{\omega\}^{\perp} \cap V_{\mathbb{C}} \ \subset \ V_{\mathbb{C}} \tag{31}$$

$$0 \ \subset \ \left(\{\omega\} \cap V_{\mathbb{C}}^{\perp} + V_{\mathbb{C}}\right)/V_{\mathbb{C}} \ \subset \ \left(\{\omega\}^{\perp} \cap V_{\mathbb{C}}^{\perp} + V_{\mathbb{C}}\right)/V_{\mathbb{C}} \ \subset \ V_{\mathbb{C}}^{\perp}/V_{\mathbb{C}} \tag{32}$$

$$0 \ \subset \ \left(\{\omega\} + V_{\mathbb{C}}^{\perp}\right)/V_{\mathbb{C}}^{\perp} \ \subset \ \left(\{\omega\}^{\perp} + V_{\mathbb{C}}^{\perp}\right)/V_{\mathbb{C}}^{\perp} \ \subset \ (\mathbb{L}_o)_{\mathbb{C}}/V_{\mathbb{C}}^{\perp} \tag{33}$$

The first one, which we denote by $\mathcal{H}$, is a pure Hodge structure of weight one induced on $V_{\mathbb{C}}$ and is polarized with respect to a certain non-degenerate skew-symmetric bilinear form $(\cdot,\cdot)_1$ on V. Let $(\cdot,\cdot)_3$ be the bilinear form on $\mathbb{L}_o/V^{\perp}$ given by $(x,y)_1 = (x, Ny)$ and $(\cdot,\cdot)_1$ be the form on V under which the isomorphism:

$$N\colon \mathbb{L}_o/V^{\perp} \to V \tag{34}$$

becomes an isometry. One has then $(x,y)_1 = (\widetilde{x}, y)$ for $x,y \in V$, where $\widetilde{x}$ is a lift of x to $\mathbb{L}_o$. The bilinear form $(\cdot,\cdot)_1$ is non-degenerate and skew-symmetric. The space $\mathcal{B}(F)$ of nilpotent $U(N)_{\mathbb{C}}$-orbits has two connected components and one can check that the Hodge structure (31) is polarized with respect to $(\cdot,\cdot)_1$ or $-(\cdot,\cdot)_1$

depending on the component the nilpotent orbit is part of. We agree to denote by $\mathcal{B}^+(F)$ the component for which (31) is polarized with respect to $(\cdot, \cdot)_1$. Then

$$\mathcal{B}(F) = \mathcal{B}^+(F) \sqcup \overline{\mathcal{B}^+(F)}.$$

The second graded Hodge structure, described in (32), has pure weight two and can be seen to be of type $(1, 1)$. Finally, the third Hodge structure (33) has weight three, but one can check that, under the isomorphism (34), filtration (33) is just the $(1, 1)$-shift of Hodge structure (31).

A mixed Hodge structure contains considerably more than the sum of its graded pieces. The first two graded parts are glued together by the extension of mixed Hodge structures:

$$\{0\} \to V \to V^\perp \to V^\perp/V \to \{0\}. \tag{35}$$

In fact, one can check that the Hodge structure (31) together with the extension (35) completely determines the nilpotent orbit of ω. This gives a natural isomorphism between $\mathcal{B}(F)$ and the space of equivalence classes of extensions of type (35). Such extensions of mixed Hodge structures have been studied by Carlson in [5]. They are classified, up to isomorphism, by an abelian group homomorphism:

$$\psi \colon \Lambda \to J^1(\mathcal{H}) \tag{36}$$

where Λ is the lattice $\left(V^\perp/V\right)_{\mathbb{Z}}$ and $J^1(\mathcal{H}) = V_{\mathbb{C}}/\left(\{\omega\}^\perp \cap V_{\mathbb{C}} + V_{\mathbb{Z}}\right)$ is the generalized Jacobian associated to the pure Hodge structure $\mathcal{H}$ described in (31). As mentioned before, Λ has to be unimodular, even, negative-definite and of rank 16.

One obtains then that, for a given Type II Baily-Borel component F, the Mumford boundary points lying in $\mathcal{B}(F)$ can be identified with pairs $(\mathcal{H}, \psi)$ consisting of polarized Hodge structures $\mathcal{H}$ of weight one on $V_{\mathbb{C}}$ together with homomorphisms $\psi \colon \Lambda \to J^1(E)$ where $\Lambda = \left(V^\perp/V\right)_{\mathbb{Z}}$. The projection to the $\mathcal{H}$-component $(\mathcal{H}, \psi) \to \mathcal{H}$ recovers exactly the projection:

$$\mathcal{B}(F) \ \to \ F$$

mentioned in the arithmetic discussion of previous section.

3 Stable $K3$ Surfaces

To this point we have described the partial compactification:

$$\Gamma \backslash \Omega \ \subset \overline{\Gamma \backslash \Omega} \tag{37}$$

from a purely arithmetic point of view. In this section, we claim that the above compactification also has a geometrical interpretation. Namely, under the period map identification $\mathcal{M}_{K3} = \Gamma \backslash \Omega$, (37) amounts to enlarging the moduli space $\mathcal{M}_{K3}$ by allowing certain explicit degenerations of elliptic $K3$ surfaces with section.

Let $\Lambda_1 = (E_8 \oplus E_8)$ and $\Lambda_2 = \Gamma_{16}$ be the two possible equivalence classes of unimodular, even, positive-definite lattices of rank 16. We claim that there is an identification:

$$\left\{ \begin{array}{c} \text{points on the Mumford} \\ \text{boundary divizor } \mathcal{D}_{\Lambda_i} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{elliptic Type II stable} \\ K3 \text{ surfaces with section} \\ \text{in } \Lambda_i\text{-category} \end{array} \right\}$$

and furthermore, the above correspondence can be regarded as a natural extension of the period map to the boundary.

3.1 Definition and Examples

Let us start by reviewing the notion of a Type II stable $K3$ surface (following [10] [11]) and the reason why these objects are natural geometrical candidates to be associated with the arithmetic Type II Mumford boundary points.

Definition 6. *([11]) A* **Type II stable K3 surface** *is a surface with normal crossings*

$$Z_o = X_1 \cup X_2 \tag{38}$$
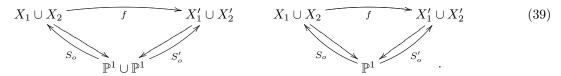
satisfying the properties:

- X_1 *and* X_2 *are smooth rational surfaces.*
- X_1 *and* X_2 *intersect with normal crossings and* $D = X_1 \cap X_2$ *is a smooth elliptic curve.*
- $D \in |-K_{X_i}|$ *for* $i = 1, 2$.
- $N_{D/X_1} \otimes N_{D/X_2} = \mathcal{O}_D$ *(d-semi-stability).*

Let us note that the above conditions imply that $\omega_{Z_o} \simeq \mathcal{O}_{Z_o}$, where ω_{Z_o} is the dualizing sheaf. Specializing the above definition, we say that a Type II stable $K3$ surface is endowed with an **elliptic structure with section** if Z_o is in one of the following categories:

(a) Both smooth rational surfaces X_i are endowed with elliptic fibrations $X_i \to \mathbb{P}^1$ with sections $S_i \subset X_i$. The double curve D is a smooth elliptic fiber on both sides. The two sections S_1 and S_2 meet D at the same point.

(b) Both smooth rational surfaces X_i carry rulings defining maps $X_i \to \mathbb{P}^1$. The two restrictions on the double curve D agree, providing the same branched double-cover $D \to \mathbb{P}^1$. In addition X_1 is endowed with a fixed section of the ruling, denoted S_o, disjoint from D.

In short, a stable surface Z_o is, in the case (a), the total space of an elliptic fibration $X_1 \cup X_2 \to \mathbb{P}^1 \cup \mathbb{P}^1$ with a fixed section given by $S_o = S_1 \cup S_2$. In the case (b), Z_o is the total space of a fibration $X_1 \cup X_2 \to \mathbb{P}^1$ whose generic fiber is a union of two smooth rational curves meeting at two points. The fixed rational curve $S_0 \subset X_1 - D$ is a section for the fibration. For reasons to be clarified shortly, we shall sometime refer to (a) and (b) as $E_8 \oplus E_8$ and Γ_{16} categories, respectively.

Two elliptic Type II stable $K3$ surfaces with section Z_o and Z_o' are said to be isomorphic if there exists an isomorphism of analytic varieties $f : Z_o \to Z_o'$ entering a commutative diagram (depending on the category):

 (39)

The reasons why one considers the configurations in (a)-(b) as elliptic structures with section on a stable $K3$ surface will be explained in section 3.3. Let us next describe explicit examples of such configurations. Our construction pattern is as follows. Let E be a smooth elliptic curve. Consider $p_0, q_0 \in E$ and let

$$E \overset{\varphi_1}{\hookrightarrow} \mathbb{P}^2 \qquad E \overset{\varphi_2}{\hookrightarrow} \mathbb{P}^2,$$

be the projective embeddings determined by the linear systems $|3p_0|$ and $|3q_0|$. Pick 18 more points $p_1, p_2, \cdots p_{18}$ (not necessarily distinct) on E and partition them into two ordered subsets

$$\{p_1, p_2, \cdots p_t\} \cup \{p_{t+1}, p_{t+2}, \cdots p_{18}\}.$$

Blow up the first copy of $\mathbb{P}^2$ at $p_1, p_2, \cdots p_t$ (in the given order) and perform the same blow-up procedure on the second copy of $\mathbb{P}^2$ using the points $p_{t+1}, p_{t+2}, \cdots p_{18}$. Let X_1 and X_2 be the resulting surfaces. A surface Z_o with normal crossings is obtained by gluing X_1 and X_2 together along the proper transforms of $\varphi_1(E)$ and $\varphi_2(E)$ using, as gluing map, the isomorphism $(\varphi_2)^{-1} \circ \varphi_1$.

Definition 7. *A collection* $\{3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\}$, *with* $3p_0$ *and* $3q_0$ *considered as divisor classes in* $\mathrm{Pic}(E)$, *is called a* **special family** *if one of the following sets of conditions holds:*

(a) $t = 9$, $p_9 = p_{18}$ *and*

$$\mathcal{O}_E(p_1 + p_2 + \cdots + p_8 + p_9) = \mathcal{O}_E(9p_0), \quad \mathcal{O}_E(p_{10} + p_{11} + \cdots + p_{18}) = \mathcal{O}_E(9q_0).$$

(b) $2 \leq t \leq 17$, $p_1 = p_2$ *and*

$$\mathcal{O}_E(p_1 + p_2 + \cdots + p_{18}) = \mathcal{O}_E(9p_0 + 9q_0), \quad \mathcal{O}_E(3p_0 - p_1) = \mathcal{O}_E(3q_0 - p_{t+1}).$$

Let $\{3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\}$ be a special family on E. Denote by

$$Z_o\left(E; 3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\right)$$

the surface with normal crossings constructed by the pattern described earlier.

Theorem 8. *The surface:*

$$Z_o\left(E; 3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\right)$$

is an elliptic Type II stable $K3$ *surface with section. Moreover the surface falls in category (a) when the special family satisfies condition (a), and in category (b) when the special family satisfies condition (b).*

Proof. Assume that $\{3p_0; p_1, p_2, \cdots p_9; 3q_0; p_{10} \cdots p_{18}\}$ is a special family of type (a). Then, the double curve D of Z_o is smooth elliptic and satisfies $D \in |-K_{X_i}|$, $D^2 = 0$. A computation involving Riemann-Roch theorem leads to $h^0(X_i, D) = 2$. The linear system $|D|$ is a base-point free pencil on each X_i and induces elliptic fibrations $X_i \to \mathbb{P}^1$. The exceptional curves E_9 and E_{18} corresponding to p_9 and p_{18} are sections in the two fibrations and they meet the double curve D at the same point. The d-stability condition on Z_o is satisfied as both normal bundles N_{D/X_i} are holomorphically trivial. We have therefore an explicit model

$$Z_o\left(E; 3p_0; p_1, p_2, \cdots p_9; 3q_0; p_{10} \cdots p_{18}\right) = X_1 \cup X_2 \to \mathbb{P}^1 \cup \mathbb{P}^1$$

for an elliptic Type II stable $K3$ surface with section in the (a)-category.

We treat now the case when $\{3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\}$ is a special family of type (b). Let H_1, H_2 be the hyper-plane divisors of the two copies of $\mathbb{P}^2$ and denote by E_i the exceptional curve corresponding to p_i. The linear systems $|H_1 - E_1|$ and $|H_2 - E_{t+1}|$ are base-point free pencils inducing rulings $X_i \to \mathbb{P}^1$. The restrictions of the two rulings agree on the double curve D, recovering the branched double cover $E \to \mathbb{P}^1$ associated to the pencil $|3p_0 - p_1| = |3q_0 - p_{t+1}|$. Moreover, if one denotes by S_o the proper transform of E_1 in X_1, then S_o is a smooth rational curve, with self-intersection -2, disjoint from D, and realizing a section of the ruling $X_1 \to \mathbb{P}^1$. The d-semi-stability condition on $Z_o\left(E; 3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\right)$ is satisfied since the line bundle $N_{D/X_1} \otimes N_{D/X_2}$ is represented on E by the principal divisor $9p_0 + 9q_0 - p_1 - p_2 - \cdots - p_{18}$. We obtain therefore an elliptic Type II stable $K3$ surface with section

$$Z_o\left(E; 3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18}\right) = X_1 \cup X_2 \to \mathbb{P}^1$$

in the (b)-category. $\qquad\square$

The surfaces of Theorem 8 represent quite a large set of examples of elliptic Type II stable $K3$ surfaces with section. In fact, one can see that, up to certain explicit transformations, these surfaces actually exhaust all possibilities.

Definition 9. *Let* Z_o *be an elliptic Type II stable* $K3$ *surface with section. A* **blowdown**

$$\rho\colon Z_0 \to \mathbb{P}^2 \cup \mathbb{P}^2$$

consists of two sequences of applications:

$$\rho_1\colon X_1 = X_1^{(n)} \to X_1^{(n-1)} \to \cdots \to X_1^{(1)} \to X_1^{(0)} \tag{40}$$

$$\rho_2\colon X_2 = X_2^{(m)} \to X_2^{(m-1)} \to \cdots \to X_2^{(1)} \to X_2^{(0)} \tag{41}$$

such that:

1. The surfaces $X_1^{(0)}$ and $X_2^{(0)}$ are copies of $\mathbb{P}^2$.

2. Each map $X_i^{(l)} \to X_i^{(l-1)}$ is a contraction of an exceptional curve in $X_i^{(l)}$.

3. If Z_o is of type (a) then S_1, S_2 are the exceptional curves associated to $X_1^{(n)} \to X_1^{(n-1)}$ and $X_2^{(m)} \to X_2^{(m-1)}$.

4. If Z_o is of type (b) then the exceptional curve associated to $X_i^{(l)} \to X_i^{(l-1)}$, $l \geq 2$, is a component of a reducible fiber of the ruling. Moreover, for $i = 1$, $l \geq 3$ this exceptional curve is disjoint from S_o.

Due to their specific construction pattern, the special surfaces $Z_o (E; 3p_0; p_1, p_2, \cdots p_t; 3q_0; p_{t+1} \cdots p_{18})$ carry a canonical blow-down. Furthermore, if a stable surface Z_o admits a blow-down, then Z_o is isomorphic to a special surface of Theorem 8. Indeed, let us assume a choice of blow-down $\rho \colon Z_o \to \mathbb{P}^2 \cup \mathbb{P}^2$. Choose $p_0, q_0 \in D$ such that $3p_0$ and $3q_0$ are hyper-plane section divisors for the embeddings:

$$D \hookrightarrow X_i \overset{\rho_i}{\to} X_i^{(0)}, \quad i = 1, 2.$$

Let $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots y_m$ be the points of intersection between the exceptional curves of $X_i^{(l)} \to X_i^{(l-1)}$ and the double curve D. A cohomology calculation shows that $m + n = 18$. Then, one can see that $\{3p_0; x_1, x_2, \cdots, x_n; 3q_0; y_1, y_2, \cdots y_m\}$ is a special family and there is a canonical isomorphism:

$$Z_o \simeq Z_o(D; 3p_0; x_1, x_2, \cdots, x_n; 3q_0; y_1, y_2, \cdots y_m)$$

restricting to identity over D.

Proposition 10. *For any elliptic Type II stable $K3$ surface with section Z_o in category (a), there exists a blow-down $\rho \colon Z_o \to \mathbb{P}^2 \cup \mathbb{P}^2$.*

Proof. As rational surfaces, both X_1 and X_2 have to dominate one of the geometrically ruled rational surfaces $\mathbb{F}_n$, $n \geq 0$. Since D meets all exceptional curves, the double curve has to be the proper transform of an effective anti-canonical divisor in $\mathbb{F}_n$. Such divisors exist only if $n \leq 2$. But $\mathbb{F}_1$ dominates $\mathbb{P}^2$ and $\mathbb{F}_0$, $\mathbb{F}_2$ also dominate $\mathbb{P}^2$ after blowing up a point on an anti-canonical curve. Therefore, if X_i is neither $\mathbb{F}_o$ nor $\mathbb{F}_2$ (which is the case here since X_1, X_2 are elliptic), one can always find blow-up sequences as in (40) and (41). Since $D^2 = 0$ on each X_i, it has to be that $n = m = 9$. Moreover, one can always choose the section components S_i as the first exceptional curves to be contracted on each side. We have therefore a blow-down $\rho \colon Z_0 \to \mathbb{P}^2 \cup \mathbb{P}^2$ as in definition (9). $\qquad\square$

Not all stable surfaces of category (b) admit blow-downs in the sense of Definition 9. X_2 may be $\mathbb{F}_0$ or $\mathbb{F}_2$ and X_1 may be $\mathbb{F}_2$. None of these surfaces dominate $\mathbb{P}^2$. However, it can be shown that any Z_o of category (b) can be transformed, using certain explicit modifications, to a surface that admits blow-downs.

An **elementary modification** of an elliptic Type II stable $K3$ surface with section Z_o of category (b) consists of the blow-down of an exceptional curve C lying inside a fiber of the ruling $X_i \to \mathbb{P}^1$ and disjoint from S_o, followed by the blow-up of the resulting point on the opposite rational surface. The resulting Z_o' is still an elliptic Type II $K3$ surface with section in category (b).

Proposition 11. *Any elliptic Type II stable $K3$ surface Z_o with section of category (b) can be transformed, using elementary modifications, to a surface which admits a blow-down.*

Proof. We claim that, using elementary modifications, one can transform Z_o to a new stable surface Z_o' such that $X_2' = \mathbb{F}_1$. Indeed, using an argument mentioned during the proof of Proposition 10, X_2 is either $\mathbb{F}_0$, or $\mathbb{F}_2$ or dominates $\mathbb{F}_1$. If there is a blow-down $X_2 \to \mathbb{F}_1$ then perform elementary transformations consisting of flipping successively to X_1 the exceptional curves involved in the blow-down. The new X_2' is clearly $\mathbb{F}_1$. If X_2 is rather a copy of $\mathbb{F}_0$ or $\mathbb{F}_2$ then, choose an exceptional curve C sitting inside a fiber of the ruling on X_1 (X_1 and X_2 cannot be simultaneously geometrically ruled). Let p be the point where C meets the double curve. Perform the elementary transform that takes C to X_2 and flip back to X_1 the proper transform of

the initial rational fiber through p in X_2. The resulting X_1' is then a copy of $\mathbb{F}_1$. Contracting the unique section of negative self-intersection one obtains:

$$X_2 \to X_2^{(0)} \tag{42}$$

with $X_2^{(0)}$ isomorphic to a projective space $\mathbb{P}^2$.

Assuming $X_2 = \mathbb{F}_1$, one has that X_1 is ruled but not geometrically ruled. Let us then describe the blow-down process

$$X_1 = X_1^{(n)} \to X_1^{(n-1)} \to \cdots \to X_1^{(1)} \to X_1^{(0)}. \tag{43}$$

We contract successively exceptional curves inside the reducible fibers of X_1, making sure that the exceptional curves in question do not intersect S_o. One can use this procedure to reduce X_1 to a new ruled surface $X_1^{(2)}$ which has a unique reducible fiber F consisting of a union $C_1 \cup C_2$ of two smooth exceptional curves. Pick the curve, among C_1, C_2, which intersects S_o and contract it. One obtains in this manner a projection $X_1^{(2)} \to X_1^{(1)}$ with $X_1^{(1)}$ geometrically ruled of type $\mathbb{F}_1$. After contracting the image of S_o, we are left with $X_1^{(0)}$ which is a copy of $\mathbb{P}^2$. Sequences (43) and (42) determine a blowdown $\rho\colon Z_o \to \mathbb{P}^2 \cup \mathbb{P}^2$. $\qquad\square$

Summarizing the facts, every elliptic Type II stable $K3$ surface with section of category (a) is isomorphic to a surface $Z_o(D; 3p_0; x_p, p_2, \cdots, p_9; 3q_0; p_{10}, p_{11}, \cdots p_{18})$ with $\{3p_0; x_p, p_2, \cdots, p_9; 3q_0; p_{10}, p_{11}, \cdots p_{18}\}$ a special family on D. Every stable surface of category (b) can be transformed, after elementary modifications to a surface $Z_o(D; 3p_0; p_1, p_2, \cdots, p_{17}; 3q_0; p_{18})$ associated to a special family $\{3p_0; p_1, p_2, \cdots, p_{17}; 3q_0; p_{18}\}$.

3.2 Stable Periods and Torelli Theorem

We are now in position to provide the formal connection between Type II elliptic stable $K3$ surfaces with section and Type II boundary points in the arithmetic partial compactification of $\Gamma\backslash\Omega$. This correspondence will be later justified geometrically as an extended period map, using the theory of $K3$ degenerations.

Theorem 12. *Let Z_o be an elliptic Type II $K3$ surface with section. Denote by D the double curve. One can naturally associate to Z_o a rank-sixteen unimodular even negative-definite lattice Λ_{Z_o} together with an abelian group homomorphism:*

$$\psi_{Z_o}\colon \Lambda_{Z_o} \to \mathrm{Jac}(D).$$

Moreover, Λ_{Z_o} is a lattice of type $-(E_8 \oplus E_8)$, for surfaces Z_o in the (a)-category, and is of type $-\Gamma_{16}$, for Z_o in the (b)-category.

Proof. We shall use a few known facts (see [11] and [12]) concerning the Hodge theory of a Type II stable $K3$ surface. The rank of $H^2(Z_o, \mathbb{Z})$ is 21. The complex cohomology group $H^2(Z_o, \mathbb{C})$ carries a canonical mixed Hodge structure of weight filtration:

$$0 \subset \mathrm{W}_1 \subset \mathrm{W}_2 = H^2(Z_o). \tag{44}$$

The two associated graded Hodge structures involved satisfy:

$$\mathrm{W}_1 \simeq H^1(D) \quad \text{(isomorphism of Hodge structures)},$$

$$\mathrm{W}_2/\mathrm{W}_1 \simeq \mathrm{Ker}\left(H^2(X_1) \oplus H^2(X_2) \to H^2(D)\right).$$

One deduces that $\mathrm{W}_2/\mathrm{W}_1$ has rank 19 and carries a pure Hodge structure of type $(1,1)$.

The mixed structure on $H^2(Z_o)$ produces an extension of mixed Hodge structures:

$$0 \to \mathrm{W}_1 \to \mathrm{W}_2 \to \mathrm{W}_2/\mathrm{W}_1 \to 0 \tag{45}$$

which, according to Carlson [5], is classified by the associated abelian group homomorphism:

$$\widetilde{\psi}_{Z_o}\colon (\mathrm{W}_2/\mathrm{W}_1)_{\mathbb{Z}} \to J^1(\mathrm{W}_1). \tag{46}$$

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Here $J^1(\mathrm{W}_1) = W_1/\, F^1 W^1 + (W_1)_{\mathbb{Z}}$ is the generalized Jacobian associated to the Hodge structure on W_1. There is a purely geometrical description for (46). Since the Hodge structure on W_1 is isomorphic to the geometrical weight-one Hodge structure of the double curve D, one has a natural identification:

$$J^1(\mathrm{W}_1) \simeq \mathrm{Jac}(D) = \mathrm{Pic}^o(D).$$

Moreover, since the two surfaces X_1, X_2 are rational, any given cohomology class

$$[L] \in (\mathrm{W}_2/\mathrm{W}_1)_{\mathbb{Z}} = \mathrm{Ker}\left(H^2(X_1,\mathbb{Z}) \oplus H^2(X_2,\mathbb{Z}) \to H^2(D,\mathbb{Z})\right)$$

is uniquely represented by a pair of holomorphic line bundles $L = (L_1, L_2) \in \mathrm{Pic}(X_1) \times \mathrm{Pic}(X_2)$ satisfying $L_1 \cdot D = L_2 \cdot D$. The image of $[L]$ under (46) can be then described as:

$$\widetilde{\psi}_{Z_o}\left([L]\right) \;=\; \mathcal{O}_D(L_1) \otimes \mathcal{O}_D(-L_2) \in \mathrm{Pic}^o(D) = \mathrm{Jac}(D). \tag{47}$$

In particular $\widetilde{\psi}_{Z_o}\left([L]\right) = 0$ for any cohomology class $[L]$ representing a Cartier divisor on Z_o.

The lattice $(\mathrm{W}_2/\mathrm{W}_1)_{\mathbb{Z}}$ has rank 19 and is indefinite. However, the elliptic structure with section on Z_o induces a series of Cartier divisors producing special cohomology classes. Firstly, the section S_o, which in case (a) is represented by two rational curves in X_1 and X_2 meeting D at the same point, while in case (b) is a unique rational curve in X_1 disjoint from D, determines a Cartier divisor $\mathcal{S}_o$ on Z_o. Secondly, the fiber on Z_o, which in case (a) consists of elliptic fibers merging at D, while in case (b) consists of rulings on each X_i agreeing over the double curve, determines a Cartier divisor class $\mathcal{F}_o$. Thirdly, let:

$$\xi_1 = \mathcal{O}_{X_1}(-D) \in \mathrm{Pic}(X_1), \quad \xi_2 = \mathcal{O}_{X_2}(D) \in \mathrm{Pic}(X_2).$$

The d-stability condition assures us that the two line bundles agree over the double curve and therefore they can be seen to determine a line bundle ξ_o over Z_o. The three Cartier divisors $\mathcal{S}_o, \mathcal{F}_o, \xi_o$ on Z_o determine integral cohomology classes:

$$[\mathcal{S}_o], [\mathcal{F}_o], [\xi_o] \in (\mathrm{W}_2/\mathrm{W}_1)_{\mathbb{Z}}$$

satisfying $[\mathcal{S}_o]^2 = -2$, $[\mathcal{F}_o]^2 = 0$, $[\xi_o]^2 = 0$, $[\mathcal{S}_o].[\mathcal{F}_o] = 1$, $[\mathcal{S}_o].[\xi_o] = 0$, $[\mathcal{F}_o].[\xi_o] = 0$.

Denote by $\{[\xi_o]\}^\perp$ the sub-lattice of $(\mathrm{W}_2/\mathrm{W}_1)_{\mathbb{Z}}$ orthogonal to the class $[\xi_o]$. Clearly all three elements $[\xi_o]$ $[\mathcal{S}_o]$ and $[\mathcal{F}_o]$ belong to $\{[\xi_o]\}^\perp$. Then, define:

$$\Lambda_{Z_o} \,\subset \{[\xi_o]\}^\perp / \left(\mathbb{Z} \cdot [\xi_o]\right)$$

as the sub-lattice orthogonal to the equivalence classes induced by $[\mathcal{S}_o]$ and $[\mathcal{F}_o]$. A simple observation involving the Hodge index theorem on X_1 and X_2 allows one to conclude that Λ_{Z_o} is even, unimodular, negative-definite and of rank 16. As mentioned earlier, the extension homomorphism (47) vanishes on cohomology classes representing Cartier divisors. In particular $\widetilde{\psi}_{Z_o}$ vanishes an all $[\mathcal{S}_o], [\mathcal{F}_o]$ and $[\xi_o]$. Therefore, without losing geometrical information, one can descend (47) to an abelian group homomorphism:

$$\psi_{Z_o} \colon \Lambda_{Z_o} \to \mathrm{Jac}(D). \tag{48}$$

The isomorphism type of the lattice Λ_{Z_o} is characterized by the category to which the stable surface Z_o belongs. Assume that Z_o is a surface in the (a)-category. There is then a natural splitting $\Lambda_{Z_o} = \Lambda^1_{Z_o} \oplus \Lambda^2_{Z_o}$ where

$$\Lambda^i_{Z_o} \;=\; \{\gamma \in H^2(X_i,\mathbb{Z}) \mid \gamma \cdot [D] = 0, \; \gamma \cdot [S_i] = 0 \}.$$

Pick a blow-down $\rho\colon Z_o \to \mathbb{P}^2 \cup \mathbb{P}^2$ as in Definition 9 and consider the associated classes:

$$\{H_1, H_2, E_1, \cdots E_{18}\} \subset H^2(X_1,\mathbb{Z}) \oplus H^2(X_1,\mathbb{Z})$$

representing the proper transforms of a hyper-planes in $\mathbb{P}^2$ and the total transforms of the exceptional curves associated to the blow-ups $X_1^{(1)} \to X_1^{(0)}$, $X_1^{(2)} \to X_1^{(1)}$, $\cdots$ $X_1^{(9)} \to X_1^{(8)}$, $X_2^{(1)} \to X_2^{(0)}$, $X_2^{(2)} \to X_2^{(1)}$, $\cdots$ $X_2^{(9)} \to X_2^{(8)}$. Let $\alpha_1, \alpha_2, \cdots \alpha_8, \beta_1, \beta_2, \cdots \beta_8$ be the following sixteen elements in Λ_{Z_o}:

$$\alpha_1 = E_1 - E_2, \; \alpha_2 = E_2 - E_3, \; \cdots, \; \alpha_7 = E_7 - E_8, \; \alpha_8 = H_1 - E_1 - E_2 - E_3 \tag{49}$$

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$$\beta_1 = E_{10} - E_{11}, \ \beta_2 = E_{11} - E_{12}, \ \cdots, \ \beta_7 = E_{16} - E_{17}, \ \beta_8 = H_2 - E_{10} - E_{11} - E_{12}.$$

One verifies that $\{\alpha_1, \alpha_2, \cdots \alpha_8\}$ and $\{\beta_1, \beta_2, \cdots \beta_8\}$ are basis for $\Lambda^1_{Z_o}$ and $\Lambda^2_{Z_o}$. Moreover, analyzing the intersection numbers, one finds out that, after changing the sign of the quadratic pairing, each of the two lines in (49) consists of a set of E_8 simple roots.

The lattice Λ_{Z_o} is therefore isomorphic to $-(E_8 \oplus E_8)$.

One can do a similar analysis in the case when Z_o is in category (b). Note that the isomorphism class of the pair $(\Lambda_{Z_o}, \psi_{Z_o})$ does not change under elementary modifications. Indeed, a modification that flips an exceptional curve C from X_1 to X_2 induces an isometry:

$$
\begin{array}{ccc}
H^2(X_1, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z}) & \longrightarrow & H^2(X_1', \mathbb{Z}) \oplus H^2(X_2', \mathbb{Z}) \\
\simeq \downarrow & & \downarrow \simeq \\
H^2(X_1', \mathbb{Z}) \oplus \mathbb{Z}[C] \oplus H^2(X_2, \mathbb{Z}) & \longrightarrow & H^2(X_1', \mathbb{Z}) \oplus \mathbb{Z}[C] \oplus H^2(X_2, \mathbb{Z}).
\end{array}
\tag{50}
$$

This map sends $[C] \in H^2(X_1, \mathbb{Z})$ to $-[C] \in H^2(X_2', \mathbb{Z})$, $[\mathcal{F}_o]$ to $[\mathcal{F}_o']$, $[\mathcal{S}_o]$ to $[\mathcal{F}_o']$ and $[\xi_o]$ to $[\xi_o']$. There is then an induced lattice isomorphism $\Lambda_{Z_o} \simeq \Lambda_{Z_o'}$ which clearly makes the diagram:

$$
\begin{array}{ccc}
\Lambda_{Z_o} & \xrightarrow{\ \simeq\ } & \Lambda_{Z_o'} \\
\ {\scriptstyle \psi_{Z_o}}\searrow & & \swarrow {\scriptstyle \psi_{Z_o'}} \\
& \mathrm{Jac}(D) &
\end{array}
$$

commutative.

According to Proposition 11, Z_o can be transformed, using elementary modifications, such that the resulting surface Z_o' admits a blow-down $\rho \colon Z_o' \to \mathbb{P}^2 \cup \mathbb{P}^2$ associated to a special family $\{3p_0; p_1, p_2, \cdots p_{17}; 3q_0; p_{18}\}$ on D. In such conditions, we define classes of cycles $\{\gamma_1, \gamma_2, \cdots \gamma_{16}\}$ in $\Lambda_{Z_o'}$ as follows:

$$\gamma_1 = H_1 - E_1 - E_2 - E_3,$$

$$\gamma_2 = E_3 - E_4, \ \gamma_3 = E_4 - E_5, \ \cdots, \gamma_{14} = E_{15} - E_{16} \tag{51}$$

$$\gamma_{15} = E_{16} - E_{17}, \ \gamma_{16} = H_2 - E_{18} + E_{16} + E_{17}.$$

One verifies that, after reversing the sign of the pairing, (51) is a root system of type D_{16}.

They determine therefore a sublattice of type $-D_{16}$ inside $\Lambda_{Z_o'}$. The index of the sub-lattice is two with a generator of the quotient group being given by the class of:

$$2H_1 - E_1 - E_2 - E_{17} + 2H_2 - 3E_{18}. \tag{52}$$

The lattice Λ_{Z_o} is then of type $-D_{16}^+$ and is therefore isomorphic to $-\Gamma_{16}$. $\qquad\square$

We make now the connection with the arithmetic Mumford boundary points. In the notation of 2.3, assume that F is a Type II Bailey-Borel component for $\Gamma\backslash\Omega$, corresponding to the isotropic rank-two sub-lattice $V \subset \mathbb{L}_o$, and $\mathcal{B}(F)$ is the associated Type II Mumford boundary divisor. Let Λ be the rank-sixteen lattice $V^{\perp}/V$. Recall from 2.4 that $\mathcal{B}(F)$ decomposes into two connected components

$$\mathcal{B}^+(F) \sqcup \overline{\mathcal{B}^+(F)}$$

and there is a bijective identification between boundary points in $\mathcal{B}^+(F)$ and pairs $(\mathcal{H}, \psi)$ consisting of weight-one Hodge structures on $V_{\mathbb{C}}$ polarized with respect to the skew-symmetric form $(\cdot, \cdot)_1$ together with abelian group homomorphisms $\psi \colon \Lambda \to J^1(\mathcal{H})$.

Let Z_o be an elliptic Type II stable $K3$ surface with section as in definition 6 (a)-(b). Attach to Z_o a set of markings ϕ_1, ϕ_2 consisting of isometries

$$\phi_1 \colon H^1(D, \mathbb{Z}) \to V, \quad \phi_2 \colon \Lambda_{Z_o} \to \Lambda. \tag{53}$$

The marking ϕ_1 can be used to transport the geometrical weight-one Hodge structure W_1 of D to a formal weight-one polarized Hodge structure $\mathcal{H}$ on V. There is then an induced isomorphism of abelian groups $\operatorname{Jac}(D) \simeq J^1(\mathcal{H})$. This isomorphism, together with the marking ϕ_2, allows one to transport the homomorphism ψ_{Z_o} of Theorem 12 to a formal homomorphism $\psi \colon \Lambda \to J^1(\mathcal{H})$. In the light of the arguments in previous paragraph, this procedure can be regarded as a period correspondence, associating to every marked elliptic Type II stable $K3$ surface with section (Z_o, ϕ_1, ϕ_2) in Λ-category a **marked stable period** in the form of a pair $(\mathcal{H}, \psi) \in \mathcal{B}^+(F)$.

One can further refine this correspondence by removing the markings and considering the pairs $(\mathcal{H}, \psi)$ modulo the isometries of Λ and V. Let us denote by $\operatorname{SL}_2(\mathbb{Z})$ the group of automorphisms of V preserving the skew-symmetric pairing $(\cdot, \cdot)_1$. This group acts naturally on the set of weight-one Hodge structures on V polarized with respect to $(\cdot, \cdot)_1$. Consider $\operatorname{Aut}(\Lambda)$ to be the group of isometries of lattice Λ. The product group $\mathcal{G} = \operatorname{Aut}(\Lambda) \times \operatorname{SL}_2(\mathbb{Z})$ acts then on the set of pairs $(\mathcal{H}, \psi)$ as:

$$(f, \alpha).(\mathcal{H}, \psi) \;=\; \big(\alpha(\mathcal{H}), \widetilde{\alpha} \circ \psi \circ f^{-1}\big)$$

where $\widetilde{\alpha} \colon J^1(\mathcal{H}) \to J^1(\alpha(\mathcal{H}))$ is the natural isomorphism induced by α. It is clear that, given a marked triplet (Z_o, ϕ_1, ϕ_2) inducing a marked pair $(\mathcal{H}, \psi)$, a variation of markings ϕ_1, ϕ_2 or a change of Z_o under an isomorphisms as in (39) leaves $(\mathcal{H}, \psi)$ within the same $\mathcal{G}$-orbit. Therefore, one can associate to any elliptic Type II stable $K3$ surface with section a well-defined **stable period** in $\mathcal{G}\backslash\mathcal{B}^+(F)$.

Definition 13. *Two elliptic Type II stable K3 surfaces with section in the same category:*

$$Z_o = X_1 \cup X_2 \quad \text{and} \quad Z_o' = X_1' \cup X_2'$$

are said to be **equivalent** *if one of the following holds:*

1. *Z_o and Z_o' are isomorphic (as in (39)).*

2. *Z_o and Z_o' are both of category (a) and Z_o is isomorphic to $X_2' \cup X_1'$.*

3. *Z_o and Z_o' are both of category (b) and can be made to be isomorphic by transforming each of them using a finite sequence of elementary modifications.*

Let then $\mathcal{M}_{\Lambda}^{\text{stable}}$, $\Lambda = E_8 \oplus E_8$ or Γ_{16}, be the coarse moduli spaces of equivalence classes in category (a), respective (b). It can be easily seen that the stable period of a surface $Z_o = X_1 \cup X_2$ does not change when Z_o gets replaced by $X_2 \cup X_1$ (if Z_o is of category (a)) or when Z_o gets transformed by an elementary modification. One has therefore a well-defined period map:

$$\operatorname{per}_{\Lambda} \colon \mathcal{M}_{\Lambda}^{\text{stable}} \to \mathcal{G}\backslash\mathcal{B}^+(F). \tag{54}$$

Furthermore, as we shall see from the analysis in section 5.2, if one denotes by Γ^+ the index-two subgroup of Γ consisting of isometries preserving the connected components of Ω and defines $\Gamma_F^+ \colon = P(F) \cap \Gamma^+$ (recall

that $P(F)$ is the rational parabolic subgroup associated to the Baily-Borel boundary component F), then there exists a natural group isomorphism $\mathcal{G} \simeq \Gamma_F^+$. Moreover, under this isomorphism, the action of $\mathcal{G}$ on $\mathcal{B}^+(F)$ reduces to the standard arithmetic action of $P(F) \cap \Gamma^+$. This produces a natural identification:

$$\mathcal{G} \backslash \mathcal{B}^+(F) \simeq \Gamma_F^+ \backslash \mathcal{B}^+(F) = \Gamma_F \backslash \mathcal{B}(F) = \mathcal{D}_\Lambda.$$

One can therefore interpret the stable period of an elliptic Type II stable $K3$ surface with section as a point of the arithmetic Mumford divisor $\mathcal{D}_\Lambda$ and hence regard (54) as a map $\mathrm{per}_\Lambda \colon \mathcal{M}_\Lambda^{\mathrm{stable}} \to \mathcal{D}_\Lambda$.

Theorem 14. *The map* (54) *is an isomorphism.*

We prove this statement in two steps. To begin with, let us show that (54) is injective.

Theorem 15. *Two stable surfaces Z_o and Z_o of category (a), which have the same stable period, are equivalent.*

Proof. This follows from standard results concerning E_8 del Pezzo surfaces (see [14] [7] and [22] for details). If $Z_o = X_1 \cup X_2$ and $Z_o' = X_1' \cup X_2'$ are stable surfaces of category (a), then, after contracting the sections, one obtains four E_8 del Pezzo surfaces $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_1'$, $\widetilde{X}_2'$. Moreover, one has isomorphisms:

$$\Lambda_{Z_o} = \Lambda_{Z_o}^1 \oplus \Lambda_{Z_o}^2 \simeq [K_{\widetilde{X}_1}]^\perp \oplus [K_{\widetilde{X}_2}]^\perp \subset H^2(\widetilde{X}_1, \mathbb{Z}) \oplus H^2(\widetilde{X}_2, \mathbb{Z})$$

$$\Lambda_{Z_o'} = \Lambda_{Z_o'}^1 \oplus \Lambda_{Z_o'}^2 \simeq [K_{\widetilde{X}_1'}]^\perp \oplus [K_{\widetilde{X}_2'}]^\perp \subset H^2(\widetilde{X}_1', \mathbb{Z}) \oplus H^2(\widetilde{X}_2', \mathbb{Z}).$$

It was proved in [14] that the isomorphism class of a pair $(\widetilde{X}, D)$, consisting of an E_8 del Pezzo surface $\widetilde{X}$ with an embedded smooth elliptic curve D, is determined by the map $[K_{\widetilde{X}}]^\perp \to \mathrm{Jac}(D)$ modulo Weyl equivalence. Based on this argument, assuming that Z_o and Z_o' determine the same stable period in $\mathcal{G} \backslash \mathcal{B}^+(F)$, it follows that there is an isomorphism of elliptic curves $D \simeq D'$ which extends to an isomorphism of stable surfaces of either $X_1 \cup X_2 \simeq X_1' \cup X_2'$ form or $X_1 \cup X_2 \simeq X_2' \cup X_1'$ form. $\square$

We use different arguments for justifying the analog of Theorem 15 for stable surfaces of category (b). As shown earlier, given an elliptic Type II stable $K3$ surface with section $Z_o = X_1 \cup X_2$ of category (b), one can always transform Z_o, by performing elementary modifications, to a new stable surface Z_o' such that $X_2' \simeq \mathbb{F}_1$. In this setting, there exists blowdowns $\rho \colon Z_o' \to \mathbb{P}^2 \cup \mathbb{P}^2$ and each choice of such blowdown induces a D_{16} simple root system

$$\{\gamma_1, \ \gamma_2, \ \cdots \gamma_{16}\}$$

for $\Lambda_{Z_o'}$, as described in (51). The model Z_o' and the blowdown ρ are far from being unique. One can further transform Z_o, using sequences of elementary modifications, to new surfaces Z_o'', satisfying $X_2'' \simeq \mathbb{F}_1$, but not isomorphic to Z_o'. However, any modification from Z_o to Z_o' induces a canonical isomorphism $\Upsilon \colon \Lambda_{Z_o} \to \Lambda_{Z_o'}$ (see (50)) entering the commutative diagram:

$$
\begin{array}{ccc}
\Lambda_{Z_o} & \xrightarrow{\ \Upsilon\ } & \Lambda_{Z_o'} \\
& {}_{\psi_{Z_o}} \searrow \quad \swarrow {}_{\psi_{Z_o'}} & \\
& \mathrm{Jac}(D). &
\end{array}
$$

Lemma 16. *Let Z_o be an elliptic Type II stable surface with section, of category (b). For any basis of of simple roots $\mathbb{S} \subset \Lambda_{Z_o}$, there exists a sequence of elementary modifications transforming Z_o to a new stable surface $Z_o' = X_1' \cup X_2'$ with $X_2' \simeq \mathbb{F}_1$ and a blowdown $\rho \colon Z_o' \to \mathbb{P}^2 \cup \mathbb{P}^2$ such that the simple root system associated to ρ is $\Upsilon(\mathbb{S})$.*

Proof. Any two sets of D_{16} simple roots can be transformed one into the other using a Weyl transformation. It suffices then to show that, given Z_o with $X_2 \simeq \mathbb{F}_1$ and fixing a blowdown $\rho_0 \colon Z_o \to \mathbb{P}^2 \cup \mathbb{P}^2$ with associated set of simple roots $\mathbb{S}_0$, for any Weyl transformation $w \in W(\Lambda_{Z_o})$, there exists a sequence of

elementary modifications transforming Z_o to Z'_o with $X'_2 \simeq \mathbb{F}_1$ and a blowdown $\rho\colon Z'_o \to \mathbb{P}^2 \cup \mathbb{P}^2$, such that the simple root set associated to ρ is $\Upsilon(w \cdot \mathbb{S}_0)$.

Let $H_1, E_1, \cdots E_{17}$ and H_2, E_{18} be the hyper-plane sections and the total transforms of the exceptional curves associated to ρ_0. The simple root set $\mathbb{S}_0$ is:

$$\gamma_1 = H_1 - E_1 - E_2 - E_3, \; \gamma_2 = E_3 - E_4, \; \gamma_3 = E_4 - E_5, \; \cdots, \gamma_{14} = E_{15} - E_{16} \qquad (55)$$

$$\gamma_{15} = E_{16} - E_{17}, \; \gamma_{16} = H_2 - E_{18} + E_{16} + E_{17}.$$

We define $\varepsilon_1, \varepsilon_2 \cdots, \varepsilon_{16}$, elements of $\Lambda_{Z_o} \otimes \mathbb{Q}$ given by:

$$\varepsilon_1 \; = \frac{1}{2}(H_2 - E_{18}) + H_1 - E_1 - E_2, \;\; \varepsilon_l = \frac{1}{2}(H_2 - E_{18}) + E_{l+1}, \;\; 2 \le l \le 16.$$

The set $\{\varepsilon_1, \varepsilon_2 \cdots, \varepsilon_{16}\}$ forms an orthonormal basis for $\Lambda^1_{Z_o} \otimes \mathbb{Q}$ (when changing the sign of the quadratic form) and the roots in $\mathbb{S}_0$ appear as:

$$\gamma_l = \varepsilon_l - \varepsilon_{l+1}, \; \text{for } 1 \le l \le 15, \; \text{and } \gamma_{16} = \varepsilon_{15} + \varepsilon_{16}.$$

In this setting, it is known that the Weyl group $W(\Lambda_{Z_o})$ is generated by permutations of $\varepsilon_1, \varepsilon_2 \cdots, \varepsilon_{16}$ and transformations t_{ij}, $1 \le i < j \le 16$ taking $\varepsilon_i, \varepsilon_j$ to $-\varepsilon_i, -\varepsilon_j$ and leaving all other ε_l unchanged. In what follows, we shall indicate the elementary modifications and the change in blowdown sequence generating, at the level of roots, transpositions $(\varepsilon_i, \varepsilon_j)$. One can use a similar technique to treat the transformations t_{ij}.

We shall denote by $X_1^{(l)}$ the surfaces obtained from X_1 during the blowdown ρ_0, by E_l the corresponding contracting curves, and by p_l the intersection points $D \cap E_l$. Start with $X_1 = X_1^{(17)}$ and contract successively $E_{17}, \cdots E_{j+2}$. The resulting surface is $X_1^{(j+1)}$. The total transform of E_{i+1} on $X_1^{(j+1)}$ is a chain $C_1 \cup C_2 \cup \cdots \cup C_k$ of smooth rational curves with self-intersection -2, with the exception of C_1 which is exceptional. One has intersecting numbers $C_l \cdot C_{l+1} = 1$ for $1 \le l \le k-1$ and $C_l \cdot C'_l = 0$ otherwise. Flip $C_1, C_2, \cdots C_{k-1}$, successively, to X_2 and then contract C_k. Then flip $C_{k-1}, C_{k-2}, \cdots C_2$ back. Flip E_{j+1} to the right. Denote the resulting stable surface by $\widetilde{X}_1^j \cup \widetilde{X}_2$. Next, if $i \ge 2$ then contract successively the curves $E_j, E_{j-1}, \cdots E_{i+2}$ on $\widetilde{X}_1^j$. Let the resulting surface be denoted $\widetilde{X}_1^{i+1}$. Flip E_{j+1} back from $\widetilde{X}_2$ and contract it on $\widetilde{X}_1^{i+1}$. The resulting surface is exactly $X_1^{(i)} \cup X_2$. Keep then the rest of the blowdown intact and construct the upper part of the new blowdown ρ by retracing the steps and blowing up successively the points $p_{j+1}, p_{i+2}, p_{i+3}, \cdots, p_j, p_{i+1}, p_{j+2}, \cdots p_{17}$, on $X_1^{(i)}$.

The new stable surface $Z'_o = X'_1 \cup X_2$ is obtained from Z_o through such a sequence of elementary modifications and the simple root system associated to the blowdown ρ is $\Upsilon(w \cdot \mathbb{S}_0)$.

The case i $= 1$ requires a slight modification of the above procedure. After obtaining $\widetilde{X}_1^{j+1}$ continue by contracting the proper transform of the line passing through p_1 with multiplicity two. Flip E_{j+1} to X_2. Denote the resulting stable surface by $\widetilde{X}_1^j \cup \widetilde{X}_2$. Contract successively $E_j, E_{j-1}, \cdots E_3$ on $\widetilde{X}_1^j$. Flip E_{j+1} back to the left. Contract the image of the proper transform of the line passing through p_1 and p_{l+1} and then contract the image of the proper transform of the line passing through p_1 and p_{l+1}. The resulting surface is a copy of $\mathbb{P}^2$. $\qquad \square$

We are then in position to justify the injectivity of the stable period map (54) for category (b) surfaces.

Theorem 17. *Two stable surfaces Z_o and Z'_o of category (b), which have the same stable period, are equivalent.*

Proof. Since elementary modifications do not vary the stable period, we can assume that both X_2 and X'_2 are copies of $\mathbb{F}_1$. Choose a blowdown $\rho\colon Z_o \to \mathbb{P}^2 \cup \mathbb{P}^2$ and denote by $\mathbb{S} \subset \Lambda_{Z_o}$ the associated basis of simple roots.

Let (ϕ_1, ϕ_2), (ϕ'_1, ϕ_2) markings for Z_o, Z'_o as in (53). Denote by $(\mathcal{H}, \psi)$, $(\mathcal{H}', \psi')$ the induced marked periods. Since the stable periods of the two surfaces are identical, there must exist isometries α and f for V and Λ, respectively, such that:

$$\mathcal{H}' = \widetilde{\alpha}(\mathcal{H}), \quad \psi' = \widetilde{\alpha} \circ \psi \circ f^{-1}.$$

Let $\mathbb{S}' = \left((\phi'_2)^{-1} \circ f \circ \phi_2 \right)(\mathbb{S})$. Then $\mathbb{S}'$ is a basis of simple roots in $\Lambda_{Z'_o}$ and, according to Lemma 16, there exists a new stable surface Z''_o, obtained from Z'_o through a sequence of elementary modifications, which admits a blowdown $\rho''\colon Z''_o \to \mathbb{P}^2 \cup \mathbb{P}^2$ such that the simple root basis $\mathbb{S}''$ associated to ρ'' satisfies $\mathbb{S}'' = \Upsilon(\mathbb{S}')$. Let then $(3p_0; p_1, p_2, p_3, \cdots, p_{16}, p_{17}; 3q_0; p_{18})$ and $(3p''_0; p''_1, p''_2, p''_3, \cdots, p''_{16}, p''_{17}; 3q''_0; p''_{18})$ be the two special families on D and D' induced by the blowdowns ρ and ρ'', respectively. Fix base points on D and D' and consider the induced identifications:

$$D \simeq \mathrm{Jac}(D) = J^1(\mathcal{H}), \quad D' \simeq \mathrm{Jac}(D') = J^1(\mathcal{H}').$$

Use these identifications to define an abelian group isomorphism:

$$\widetilde{\eta}\colon\ D \xrightarrow{\ \simeq\ } J^1(\mathcal{H}) \xrightarrow{\ \widetilde{\alpha}\ } J^1(\mathcal{H}') \xrightarrow{\ \simeq\ } D'. \tag{56}$$

and then construct $\eta\colon D \to D'$ with $\eta(p) = \widetilde{\eta}(p) - \widetilde{\eta}(p_1) + p''_1$. It turns out then that the isomorphism η transports the special family $(3p_0; p_1, p_2, p_3, \cdots, p_{16}, p_{17}; 3q_0; p_{18})$ to $(3p''_0; p''_1, p''_2, p''_3, \cdots, p''_{16}, p''_{17}; 3q''_0; p''_{18})$. This implies that Z_o and Z''_o are isomorphic which, in turn, implies that Z_o and Z'_o are equivalent. $\qquad\square$

One concludes from Theorems 15 and 17 that the stable period map:

$$\mathrm{per}_\Lambda\colon \mathcal{M}^{\mathrm{stable}}_\Lambda \to \mathcal{G}\backslash \mathcal{B}^+(F) \tag{57}$$

is injective. Let us then complete the proof of Theorem 14:

Theorem 18. *The period map (57) is surjective.*

Proof. Let $(\mathcal{H}, \psi)$ be a pair in $\mathcal{B}^+(F)$. We show that there exists a marked surface (Z_o, ϕ_1, ϕ_2) with stable marked period $(\mathcal{H}, \psi)$. The Hodge-theoretic Jacobian $J^1(\mathcal{H})$ is itself a pointed elliptic curve endowed with a natural group structure. We agree to call it (E, p_0) and denote by $\phi_1\colon H^1(E, \mathbb{Z}) \simeq V$ a marking that sends the geometrical Hodge structure of E to $\mathcal{H}$. In particular, ϕ_2 induces a group isomorphism

$$E \simeq \mathrm{Jac}(E) \simeq J^1(\mathcal{H}). \tag{58}$$

If $\Lambda = E_8 \oplus E_8$, pick a basis for $\{a_1, \cdots a_8, b_1, \cdots b_8\}$ for Λ such that $\{a_1, \cdots a_8\}$ and $\{b_1, \cdots b_8\}$ are E_8 systems of simple roots. In what follows, we construct 19 points on E, denoted $q_0, x_1, x_2 \cdots x_9, y_1, y_2 \cdots y_9$. Choose $x_1 \in E$ such that:

$$3x_1 = 2\psi(a_1) + \psi(a_2) - \psi(a_8).$$

Then construct $p_2, \cdots p_9$, recursively, by the rule:

$$x_l = x_{l-1} - \psi(a_{l-1}), \text{ for } 2 \leq l \leq 8$$

$$x_9 = -(x_1 + x_2 + \cdots + x_8).$$

Then set $y_9 = x_9$ and:

$$y_1 = y_9 + (7\psi(b_1) + 6\psi(b_2) + 5\psi(b_3) + \cdots + 2\psi(b_6) + \psi(b_7)) - 3(2\psi(b_1) + \psi(b_2) - \psi(b_8)).$$

Construct then recursively $y_l = y_{l-1} + \psi(b_{l-1})$ for $2 \le l \le 8$ and then pick $q_0 \in E$ such that:

$$3q_0 = 2\psi(b_1) + \psi(b_2) - \psi(b_8) + 3y_1.$$

One verifies that $(3p_0; x_1, x_2, \cdots x_9; 3q_0; y_1, y_2, \cdots y_9)$ is a special family of category (a) on E. The stable surface

$$Z_o \;=\; Z_o\,(E; 3p_0; x_1, x_2, \cdots x_9; 3q_0; y_1, y_2, \cdots y_9) \tag{59}$$

is then an elliptic Type II stable $K3$ surface with section, of category (a). Moreover, Z_o comes endowed with a natural blow-down. Let $\{\alpha_1, \alpha_2, \cdots \alpha_8, \beta_1, \beta_2, \cdots \beta_8\} \subset \Lambda_{Z_o}$ be the ordered set of simple roots associated to the respective blow-down. Then, under the isomorphism (58), $\psi_{Z_o}(\alpha_i) = \psi(a_i)$, $\psi_{Z_o}(\beta_i) = \psi(b_i)$ for $1 \le i \le 8$. In other words, if $\phi_2 \colon \Lambda_{Z_o} \to \Lambda$ is the marking sending $\{\alpha_1, \alpha_2, \cdots \alpha_8, \beta_1, \beta_2, \cdots \beta_8\}$ to $\{a_1, \cdots a_8, b_1, \cdots b_8\}$, then the diagram:

$$
\begin{array}{ccc}
\Lambda_{Z_o} & \xrightarrow{\;\psi_{Z_o}\;} & \mathrm{Jac}(E) \\
{\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \simeq} \\
\Lambda & \xrightarrow{\;\;\psi\;\;} & J^1(\mathcal{H})
\end{array}
$$

is commutative. we conclude that the marked stable period of the marked triplet (Z_o, ϕ_1, ϕ_2) coincides with the pair $(\mathcal{H}, \psi)$.

A similar procedure can be used if $\Lambda = \Gamma_{16}$. Fix $\{c_1, c_2, \cdots c_{16}\}$ a basis of D_{16} simple roots in Λ. We shall construct a set of 19 points $q_0, p_1, p_2, \cdots p_{18}$ in E. To begin with pick $p_1 \in E$ such that:

$$3p_1 \;=\; -\left(2\psi(c_1) + 2\psi(c_2) + \cdots + 2\psi(c_{13}) + 2\psi(c_{14}) + \psi(c_{15}) + \psi(c_{16})\right).$$

Define then $p_2 = p_1$ and construct recursively $p_l \;=\; p_{l-1} - \psi(c_{l-2})$ for $3 \le l \le 17$. Pick then $q_0 \in E$ such that:

$$6q_0 \;=\; 2p_1 + p_2 + p_3 + \cdots p_{17}.$$

Finally, set $p_{18} = p_1 + 3q_0$. It follows then that $(3p_0; p_1, p_2, \cdots p_{17}; 3q_0; p_{18})$ is a special family of category (b) on E. Then,

$$Z_o \;=\; Z_o\,(E; 3p_0; p_1, p_2, \cdots p_{17}; 3q_0; p_{18}) \tag{60}$$

is an elliptic Type II stable $K3$ surface with section endowed with a canonical blowdown. If

$$\{\gamma_1, \gamma_2, \cdots \gamma_{16}\} \;\subset\; \Lambda_{Z_o}$$

is the ordered set of simple roots associated to the respective blow-down, then, under the isomorphism (58), one has. $\psi_{Z_o}(\gamma_l) = \psi(c_l)$, $1 \le i \le 16$. Therefore, if one sets a marking $\phi_2 \colon \Lambda_{Z_o} \to \Lambda$ such that the ordered basis $\{\gamma_1, \gamma_2, \cdots \gamma_{16}\}$ is sent to $\{c_1, c_2, \cdots c_{16}\}$, then the marked stable period associated to (Z_o, ϕ_1, ϕ_2) is $(\mathcal{H}, \psi)$. $\qquad\square$

3.3 Stable Surfaces as K3 Degenerations

We have seen that the two Type II Mumford boundary divisors $\mathcal{D}_{E_8 \oplus E_8}$ involved in the partial compactification of $\Gamma \backslash \Omega$ can be regarded as moduli spaces of periods for elliptic Type II stable $K3$ surfaces with section in the $E_8 \oplus E_8$ and Γ_{16} category, respectively. In this section we justify the presence of such surfaces from a geometrical point of view, as they appear naturally as central fibers for certain degenerations of $K3$ surfaces.

Definition 19. *A one-variable degeneration of elliptically fibered $K3$ surfaces with section consists of a commutative diagram of analytic maps:*

$$
\begin{array}{ccc}
Z & \xrightarrow{\;\pi\;} & \Delta \\
\downarrow & \nearrow & \\
S & &
\end{array}
\tag{61}
$$

where Z is a smooth three-fold, S is a smooth surface, Δ is the unit disk. In addition, the structure requires the presence of an analytic section $s\colon S \to Z$ and of a line bundle $\mathcal{F}$ on Z, such that:

- *For every $t \in \Delta^*$, Z_t is a smooth K3 surface, S_t is a smooth rational curve and the projection $Z_t \to S_t$ is an elliptic fibration with section $s_t\colon S_t \to Z_t$.*

- *The restriction of $\mathcal{F}$ on Z_t coincides with the line bundle associated to the elliptic fiber in $Z_t \to S_t$.*

Two degenerations $Z \xrightarrow{\pi} \Delta$ and $Z' \xrightarrow{\pi'} \Delta$ as in (61) are said to be equivalent if one has a birational map $\alpha\colon Z \to Z'$ entering the commutative diagram:

$$
\begin{array}{ccc}
Z & \xRightarrow{\;\;\alpha\;\;} & Z' \\
\end{array}
\tag{62}
$$

and satisfying $\mathcal{F} = \alpha^* \mathcal{F}'$.

One can think of an equivalence class of $K3$ degenerations as a punctured disc embedded in the moduli space $\mathcal{M}_{K3}$. Intuitively, one can then regard the degenerated central fibers Z_o as geometrical representatives for boundary points in a compactification of $\mathcal{M}_{K3}$. A major difficulty appears here due to the fact that equivalent degenerations can have quite different central fibers. One tries to surmount this obstacle by restricting to more distinguished degenerations in the hope of obtaining a canonical model of central fiber for each degenerating equivalence class, which is a requirement for any attempt of geometrical partial compactification. Along this reasoning line (see [9] [27] [11] [20] for details), we restrict ourself to degenerations $Z \xrightarrow{\pi} \Delta$ as in (61) which are semi-stable (meaning that the central fiber Z_o is a surface with normal crossings) and satisfy $K_Z = \mathcal{O}_Z$. We shall call these Kulikov degenerations.

Since $\pi_1(\Delta^*) \simeq \mathbb{Z}$, it is not necessarily possible to attach a consistent set of markings to the surfaces in such a Kulikov family. Attached to each degeneration, there is a monodromy operator:

$$
\mathrm{T} \in \mathrm{Aut}\left(H^2(Z_t, \mathbb{Z})\right)
\tag{63}
$$

which can be described explicitly as the Picard-Lefschetz transformation obtained by transporting cycles around origin $t = 0$ in Δ while preserving the classes representing the elliptic structure and section. The operator T is unipotent, meaning $(T - I)^3 = 0$ which is equivalent to saying that its logarithm

$$
\mathrm{N} \;=\; (T - I) - \frac{1}{2}(T - I)^2
\tag{64}
$$

is a nilpotent endomorphism of $H^2(Z_t, \mathbb{Q})$ satisfying $N^3 = 0$. Complexifying the picture, one obtains a monodromy weight filtration:

$$
\{0\} \;\subset\; \mathrm{Im}\left(N^2\right) \;\subset\; \mathrm{Im}\left(N\right) \cap \mathrm{Ker}\left(N\right) \;\subset\; \mathrm{Im}\left(N\right) + \mathrm{Ker}\left(N\right) \;\subset\; \mathrm{Ker}\left(N^2\right) \;\subset\; H^2(Z_t, \mathbb{C}).
\tag{65}
$$

Moreover, as explained in [29], the degeneration data produces a mixed Hodge structure on $H^2(Z_t, \mathbb{C})$ with weight filtration (65), the limiting mixed Hodge structure. With respect to this structure, the nilpotent endomorphism N becomes a morphism of mixed Hodge structures of type $(-1, -1)$.

Kulikov degenerations fall into three categories, denoted Type I, Type II and Type III, depending on whether $N = 0$, $N^2 = 0$ but $N \neq 0$, or $N^3 = 0$ but $N^2 \neq 0$. We shall restrict our attention here only to Type II families ($N \neq 0$ but $N^2 = 0$) as that will turn to be the case relevant to our prior discussion. In this case N is always an integral endomorphism. The elliptic Type II stable $K3$ surfaces with section appear then naturally as central fibers for Type II degenerations with primitive N.

Proposition 20.

1. Let $Z \xrightarrow{\pi} \Delta$ be a degeneration as in (61) which is Kulikov of Type II with primitive endomorphism N. The central fiber Z_o is then an elliptic Type II stable $K3$ surface with section.

2. For every elliptic Type II stable $K3$ surface with section Z_o, there exists a Type II Kulikov degeneration $Z \xrightarrow{\pi} \Delta$ as in (61) with central fiber Z_o.

Proof. Both statements can be deduced easily from standard results on $K3$ degenerations (see [9] [10] [11] and [20]). Indeed, to prove the first part of the proposition, assume that Z_o is a central fiber of a degeneration:

$$Z \xrightarrow{\pi} \Delta \qquad (66)$$
$$s \Big\updownarrow \quad \nearrow$$
$$S$$

as in (61), which is Kulikov of Type II with N primitive. Recall the following fact from [20]:

Theorem 21. *The central fiber of a Type II Kulikov degeneration of $K3$ surfaces is always a chain of smooth rational surfaces:*

$$Z_0 \;=\; X_1 \cup X_2 \cup \cdots \cup X_{r+1}.$$

The surfaces V_2, V_3, $\cdots$ V_r are smooth elliptic ruled. The chain contains only double curves and all double curves are smooth and elliptic.

The integer r can be detected by from the arithmetic of the degeneration by writing $N = rN_o$ with N_o primitive. Since we expect that N itself is primitive, it has to be that $r = 1$ and therefore the central fiber Z_o is a union $X_1 \cup X_2$ of two rational surfaces glued along an elliptic curve D. Let us analyze then the central fiber configuration:

$$X_1 \cup X_2 \to S_o. \qquad (67)$$

Since $S \to \Delta$ is a degeneration of smooth rational curves, S_o has to be a chain of rational curves. The map (67) is proper and its domain is a union of two irreducible varieties. Therefore, S_o cannot have more than two irreducible components. We divide then our discussion into two cases:

1. S_o is a union $S_1 \cup S_2$ of two copies of $\mathbb{P}^1$ meeting at one point.

2. S_o is a smooth rational curve.

In the first case, S_i represents the image of X_i through (67) and D is the fiber above the common point. We have therefore two elliptic fibrations $X_i \to S_i$ agreeing over the double curve. The section $s_o \colon S_o \to Z_o$ allows us to regard S_1 and S_2 as two smooth rational curves embedded in X_1 and X_2, respectively. The two curves S_1 and S_2 meet D at the same point. This is exactly the configuration required for Z_o to be an elliptic Type II stable $K3$ surface with section of category (a).

In the second case, the section S_o lies entirely inside one of two surfaces V_i. Assume $S_o \subset V_1$. Since S_o corresponds to a Cartier divisor on Z_o, it cannot intersect the double curve D. Therefore $S_o^2 = -2$. The projection (67) restricts to rulings:

$$V_i \to \mathbb{P}^1$$

with the double curve D playing the role of a bi-section on each side. We obtain therefore that Z_o is an elliptic Type II stable $K3$ surface with section of category (b).

In order to prove the second assumption, we recall some facts pertaining to the deformation theory for stable $K3$ surfaces.

Theorem 22. *([10]) Let Z_o be a Type II stable $K3$ surface.*

1. Z_o *is smoothable and appears as central fiber in a Kulikov semi-stable degeneration.*

2. *The space of first-order deformations of Z_o:*

$$\mathbf{T}^1_{Z_o} = \text{Ext}^1\left(\Omega^1_{Z_o}, \mathcal{O}_{Z_o}\right)$$

is 21-dimensional.

3. *The versal deformation space of Z_o looks like $V_1 \cup V_2 \subset \mathbf{T}^1_{Z_o}$ where V_1 and V_2 are two smooth divisors meeting normally. The points of V_1 corresponds to locally trivial deformations of Z_o. The points of $V_2 \backslash V_1$ represent deformations of $K3$ surfaces. $V_1 \cup V_2$ corresponds to locally trivial deformations of Z_o which remain d-semi-stable.*

Therefore, given an elliptic Type II stable $K3$ surface with section Z_o, there are always plenty of smoothings of Z_o. We just have to show that, on some of these deformations, the two Cartier divisors $\mathcal{S}_o$ and $\mathcal{F}_o$ can be extended on the three-fold. The obstruction to extending a Cartier divisor of the central fiber is measured by the Yoneda pairing [33]:

$$< \cdot, \cdot >: \ \text{Ext}^1\left(\Omega^1_{Z_o}, \mathcal{O}_{Z_o}\right) \otimes H^1\left(Z_o, \Omega^1_{Z_o}\right) \ \rightarrow \ \text{Ext}^2\left(\mathcal{O}_{Z_o}, \mathcal{O}_{Z_o}\right) = H^2\left(Z_o, \mathcal{O}_{Z_o}\right) \tag{68}$$

which is non-degenerate for stable $K3$ surfaces. The Zariski tangent space to the smoothing component V_2 is given by the hyper-plane (see [11]):

$$\left\{\sigma \in \mathbf{T}^1_{Z_o} \mid \ < \sigma, [\xi_o] >= 0 \ \right\} \ \subset \ \mathbf{T}^1_{Z_o}$$

where $[\xi_o]$ is the class in $H^1(Z_o, \Omega^1_{Z_o})$ associated to the Cartier divisor ξ_o. The formal Zariski tangent space to the space of smoothings extending the elliptic structure and section is then given by:

$$\left\{\sigma \in \mathbf{T}^1_{Z_o} \mid \ < \sigma, [\xi_o] >=< \sigma, [\mathcal{F}_o] >=< \sigma, [\mathcal{S}_o] >= 0 \ \right\}. \tag{69}$$

Since Yoneda pairing is non-degenerate and $[\xi_o]$, $[\mathcal{F}_o]$ and $[\mathcal{S}_o]$ are independent in $H^1(Z_o, \Omega^1_{Z_o})$, (69) is 18-dimensional. The space of versal deformations extending the elliptic structure and section has then a unique smoothing component V'_2 of dimension 18. The points V'_2 away from the discriminant locus correspond to deformations as in Definition 19. $\qquad\square$

Let us then present the stable period map (54) as a natural extension of the $K3$ period correspondence $\mathcal{M}_{K3} \simeq \Gamma \backslash \Omega$. Assume that $Z \rightarrow \Delta$ is a degeneration of elliptic $K3$ surfaces with section, as in Definition 19, which is Kulikov, Type II semi-stable, and has primitive endomorphism N. There is then a corresponding Griffiths' period map (see [18]):

$$\Phi \colon \Delta^* \ \rightarrow \ \Gamma \backslash \Omega. \tag{70}$$

Following results of Mumford [1] and Schmid [29], one sees that (70) extends to a holomorphic map:

$$\widetilde{\Phi} \colon \Delta \ \rightarrow \ \overline{\Gamma \backslash \Omega}.$$

Recall the construction of the boundary point $\widetilde{\Phi}(0)$. Choose a compatible marking $H^2(Z_t, \mathbb{Z}) \simeq \mathbb{L}$ as in section 2.1. The endomorphism N is integral and vanishes on both cohomology classes $[\mathcal{F}_t], [\mathcal{S}_t] \in H^2(Z_t, \mathbb{Z})$ corresponding to the elliptic fiber and section in $Z_t \rightarrow S_t$. Therefore, it defines an isotropic rank-two sub-lattice $V \subset \mathbb{L}_o$. Define $\Lambda = V^\perp/V$ and let F be the Baily-Borel component associated to V. The monodromy weight filtration (65) associated to the degeneration $Z \rightarrow \Delta$ is just the complexification of:

$$\{0\} \ \subset \ \text{Im}\,(N) \ \subset \ \text{Ker}\,(N) \ \subset \ H^2(Z_t, \mathbb{Z}). \tag{71}$$

Taking the orthogonal part to the fiber and section classes $[\mathcal{F}_t]$ and $[\mathcal{S}_t]$, reduces (71) to:

$$\{0\} \ \subset \ \text{Im}\,(N) \ \subset \ \text{Ker}\,(N) \cap \{[\mathcal{F}_t], [\mathcal{S}_t]\}^\perp \ \subset \ H^2(Z_t, \mathbb{Z}) \cap \{[\mathcal{F}_t], [\mathcal{S}_t]\}^\perp \tag{72}$$

which corresponds under the marking to:

$$\{0\} \subset V \subset V^\perp \subset \mathbb{L}_o. \tag{73}$$

By the classical construction of Schmid [29], the family $Z \to \Delta$ induces a nilpotent orbit of limiting mixed Hodge structures with weight filtration (71). These structures descend, under the marking, to give a nilpotent orbit of polarized mixed Hodge structures on (73). The resulting $U(N)_\mathbb{C}$-orbit consists essentially of the decreasing filtrations:

$$\mathbb{L}_o \otimes \mathbb{C} \supset \{\exp(zN) \cdot \omega_t\}^\perp \supset \{\exp(zN) \cdot \omega_t\} \supset \{0\}, \quad z \in \mathbb{C} \tag{74}$$

where $\{\omega_t\} \subset \mathbb{L}_o \otimes \mathbb{C}$ is the marked period line of Z_t. But, as explained in section 2.4, such a nilpotent orbit of Hodge structures is equivalent to a point on the Type II Mumford component $\mathcal{B}^+(F)$. $\widetilde{\Phi}(0)$ is the class of this point on the quotient boundary divisor $\mathcal{D}_\Lambda \subset \overline{\Gamma \backslash \Omega} - \Gamma \backslash \Omega$.

Let then Z_o be the central fiber of $Z \to \Delta$. According to Proposition 20, Z_o is an elliptic Type II stable $K3$ surface with section.
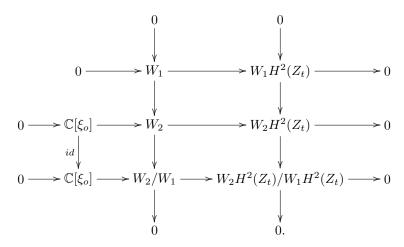
Theorem 23. *The boundary point $\widetilde{\Phi}(0) \in \mathcal{D}_\Lambda$ coincides with the stable period of Z_o.*

Proof. The Clemens-Schmid exact sequence (see [17] for details) allows one to relate the geometric mixed Hodge structure of Z_o with the limiting mixed Hodge structure associated to the degeneration $Z \to \Delta$. Essentially, this argument asserts the exactness of the following sequence of mixed Hodge structures:

$$\{0\} \to H^0(Z_t) \xrightarrow{(-2,-2)} H_4(Z_o) \xrightarrow{(3,3)} H^2(Z_o) \xrightarrow{(0,0)} H^2(Z_t) \xrightarrow{N} H^2(Z_t) \xrightarrow{(-2,-2)} H_2(Z_o) \xrightarrow{(3,3)} H^4(Z_t) \cdots . \tag{75}$$

The sequence also holds as cohomology sequence over $\mathbb{Z}$.

We use the notation from section 3.2. The homology group $H_4(Z_o, \mathbb{Z})$ has rank two and is generated by the two fundamental cycles $[X_i] \in H_4(Z_o, \mathbb{Z})$, $i = 1, 2$. Moreover, the image of $[X_1]$ and $[X_2]$ through the morphism $H_4(Z_o) \to H^2(Z_o)$ is given by $[\xi_o]$ and $-[\xi_o]$. The exactness of (75) allows then one to construct the following commutative diagram of polarized mixed Hodge structures:

This leads to the following isomorphism of extensions of mixed Hodge structures:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \text{(76)} \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & W_1 & \longrightarrow & W_1 H^2(Z_t) & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & W_2/\mathbb{C}[\xi_o] & \longrightarrow & W_2 H^2(Z_t) & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
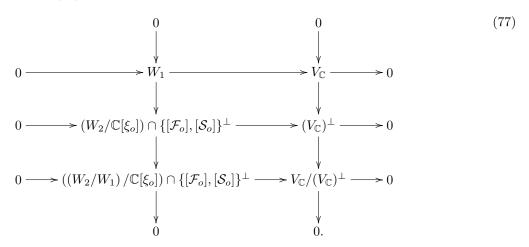0 & \longrightarrow & (W_2/W_1)/\mathbb{C}[\xi_o] & \longrightarrow & W_2 H^2(Z_t)/W_1 H^2(Z_t) & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0. & & & &
\end{array}
$$

Take then the orthogonal subspaces with respect to $\{[\mathcal{F}_o],[\mathcal{S}_o]\}$ and $\{[\mathcal{F}_t],[\mathcal{S}_t]\}$ in $W_2/\mathbb{C}[\xi_o]$ and $W_2 H^2(Z_t)$, respectively. One can make use of the marking to identify

$$
W_1 H^2(Z_t) \simeq V_{\mathbb{C}}, \quad W_2 H^2(Z_t) \cap \{[\mathcal{F}_t],[\mathcal{S}_t]\}^{\perp} = H^2(Z_t) \cap \{[\mathcal{F}_t],[\mathcal{S}_t]\}^{\perp} \simeq (V_{\mathbb{C}})^{\perp}.
$$

In this setting, the diagram (76) becomes:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & & & \text{(77)} \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & W_1 & \longrightarrow & V_{\mathbb{C}} & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (W_2/\mathbb{C}[\xi_o]) \cap \{[\mathcal{F}_o],[\mathcal{S}_o]\}^{\perp} & \longrightarrow & (V_{\mathbb{C}})^{\perp} & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & ((W_2/W_1)/\mathbb{C}[\xi_o]) \cap \{[\mathcal{F}_o],[\mathcal{S}_o]\}^{\perp} & \longrightarrow & V_{\mathbb{C}}/(V_{\mathbb{C}})^{\perp} & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & & & \\
& & 0 & & 0. & & & &
\end{array}
$$

Note that $J^1(W_1) = \mathrm{Jac}(D)$,
$$
\Lambda_{Z_o} = ((W_2/W_1)_{\mathbb{Z}}/\mathbb{Z}[\xi_o]) \cap \{[\mathcal{F}_o],[\mathcal{S}_o]\}^{\perp}
$$
and the classifying homomorphism for the left hand-side extension of mixed Hodge structure is the abelian group homomorphism $\psi_{Z_o}\colon \Lambda_{Z_o} \to \mathrm{Jac}(D)$ associated to Z_o by Theorem 12. As for the right hand-side extension, we have $V^{\perp}/V = \Lambda$ and the weight-one Hodge structure on $V_{\mathbb{C}}$, denoted $\mathcal{H}$, is given by the decreasing filtration
$$
V_{\mathbb{C}} \supset \{\omega_t\}^{\perp} \cap V_{\mathbb{C}} \supset \{0\}.
$$
If $\psi\colon \Lambda \to J^1(\mathcal{H})$ is the classifying homomorphism, then the pair $(\mathcal{H},\psi)$ represents $\widetilde{\Phi}(0)$ in $\mathcal{B}^+(F)$ (see the discussion in 2.4).

The diagram (77) produces isometries

$$
\varphi_1\colon H^1(D,\mathbb{Z}) = (W_1)_{\mathbb{Z}} \to V, \quad \varphi_2\colon \Lambda_{Z_o} \to V^{\perp}/V = \Lambda
$$

which together define a marking for Z_o. Moreover, the marking (φ_1, φ_2) induces an isomorphism of elliptic curves $\mathrm{Jac}(D) \approx J^1(\mathcal{H})$ which enters a commutative diagram

$$
\begin{array}{ccc}
\Lambda_{Z_o} & \xrightarrow{\ \varphi_2\ } & \Lambda \\
{\scriptstyle \psi_{Z_o}} \downarrow & & \downarrow {\scriptstyle \psi} \\
\mathrm{Jac}(D) & \xrightarrow{\ \approx\ } & J^1(\mathcal{H}).
\end{array}
\tag{78}
$$

This allows one to conclude that $\widetilde{\Phi}(0)$ in $\mathcal{D}_\Lambda$ is exactly the stable period of Z_o. $\qquad\square$

4 Boundary Components and Flat Bundles

There exists a second geometric interpretation, more relevant from the point of view of heterotic/F-theory duality, for the boundary points on the two Type II Mumford divisors $\mathcal{D}_\Lambda$ with $\Lambda = E_8 \oplus E_8$ or $\Lambda = \Gamma_{16}$. Recall that, given a Baily-Borel component F, $\mathcal{D}_\Lambda = \Gamma_F^+ \backslash \mathcal{B}^+(F)$ and the points in $\mathcal{B}^+(F)$ are in one-to-one correspondence to pairs $(\mathcal{H}, \psi)$ of polarized weight-one mixed Hodge structures $\mathcal{H}$ on V together with abelian group homomorphisms $\psi \colon \Lambda = V^\perp/V \to J^1(\mathcal{H})$. Such a pair is known to determine a flat G-connection over the elliptic curve $E = J^1(\mathcal{H})$. The Lie group G is $(E_8 \times E_8) \rtimes \mathbb{Z}_2$ if $\Lambda = E_8 \oplus E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ if $\Lambda = \Gamma_{16}$.

Let us briefly review the connection. For explicit details, see [15]. It is a standard fact that, given a compact Lie group G and a smooth two-torus E, there is a bijective correspondence between the equivalence classes of flat G-connections on E and their associated holonomy morphisms $\pi_1(E) \to G$, up to conjugation. One can therefore formally identify a flat connection with a commuting pair of elements in G, up to simultaneous conjugation. Fix a maximal torus $T \subset G$. If G is simply connected, (in particular for $G = (E_8 \times E_8) \rtimes \mathbb{Z}_2$), it was shown that any given pair of commuting elements in G can be simultaneously conjugated in T. The same statement is true for $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, providing that one considers only connections which can be lifted to $\mathrm{Spin}(32)$-connections. In this way, a flat G-connection on E can be formally understood as an element in:

$$\mathrm{Hom}\,(\pi_1(E), T)\,/W$$

where W is the Weyl group of G. The lattice Λ plays the role of the lattice of the maximal torus T. In this framework:

$$\mathrm{Hom}\,(\pi_1(E), T)\ \simeq\ \mathrm{Hom}\,(\pi_1(E), U(1) \otimes \Lambda)\ \simeq\ \mathrm{Hom}\,(\pi_1(E), U(1)) \otimes \Lambda.$$

The first factor of the last term above represents the set of gauge equivalence classes of flat hermitian line bundles over E. In the presence of a complex structure on E, one can identify $\mathrm{Hom}\,(\pi_1(E), U(1))$ to $\mathrm{Pic}^o(E)$ which, in turn, is a complex torus isomorphic to E. There exists then a bijective correspondence between flat G-connections and points of the analytic quotient:

$$E \otimes \Lambda/W \tag{79}$$

which one can see as the moduli space of flat G-bundles over E.

Due to the unimodularity $\Lambda^* \simeq \Lambda$, each element in (79) can be regarded as a class of a morphism $\Lambda \to E$. Along this idea, one can associate to any Type II boundary point of $\mathcal{B}^+(F)$, a smooth elliptic curve $E = J^1(\mathcal{H})$ and a flat G-bundle. In section 5.2 we shall show that all possible flat G-bundles are realized[1] and that two points in $\mathcal{B}^+(F)$ determine equivalent pairs of elliptic curves and flat bundles exactly when they belong to the same Γ_F^+-orbit. This will lead to a holomorphic identification between the boundary divisors $\mathcal{D}_{E_8 \oplus E_8}$, $\mathcal{D}_{\Gamma_{16}}$ and the moduli spaces $\mathcal{M}_{E, E_8 \times E_8 \rtimes \mathbb{Z}_2}$ and $\mathcal{M}_{E, \mathrm{Spin}(32)/\mathbb{Z}_2}$ of equivalence classes of pairs of elliptic curves and flat bundles, respectively.

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[1] For $G = \mathrm{Spin}(32)/\mathbb{Z}_2$, we only look at flat bundles liftable to $\mathrm{Spin}(32)$.

5 Explicit Description of the Parabolic Cover

In this section we give an explicit description of the two Type II boundary divisors $\mathcal{D}_\Lambda = \Gamma_F \backslash \mathcal{B}(F)$ and identify precisely the holomorphic type of the parabolic fibrations given in (28):

$$\Theta \colon \Gamma_F \backslash \overline{\Omega(F)/U(N)_{\mathbb{Z}}} \;\to\; \Gamma_F \backslash \mathcal{B}(F). \tag{80}$$

This leads, following 2.3, to a description of the structure of $\mathcal{M}_{K3}$ in a neighborhood of $\mathcal{D}_\Lambda$.

5.1 Fixing the Parabolic Group

Let F be a fixed Type II Baily-Borel boundary component for $\Gamma \backslash \Omega$. Denote by V the associated primitive isotropic rank-two sub-lattice of $\mathbb{L}_o$ and set, as in 2.3:

$$\Lambda = -\left(V^\perp / V\right)$$

$$\mathrm{P(F)} \;=\; \mathrm{Stab}\,(V_{\mathbb{R}}) \subset O^{++}(2,18)$$

$$\mathrm{W}(F) \;=\; \text{the unipotent radical of P(F)}$$

$$\mathrm{U}(F) \;=\; \text{the center of W(F)}.$$

It follows then that $\mathrm{U}(F)$ is a one-dimensional Lie group with Lie algebra:

$$\mathrm{u}(F) \;=\; \{\mathrm{N} \in \mathrm{End}_{\mathbb{R}}\,(\mathbb{L}_o \otimes \mathbb{R}) \mid \mathrm{Im}(N) = V_{\mathbb{R}} \text{ and } (Nx,y) + (x,Ny) = 0\}.$$

Lemma 24. *There exists a basis $\{A, B\}$ for V such that the endomorphism $N \colon \mathbb{L}_o \to \mathbb{L}_o$ defined by:*

$$N(x) = (x,B)A - (x,A)B \tag{81}$$

is primitive, integral and belongs to $\mathrm{u}(F)$.

Proof. Let A be a primitive element of V. Due to unimodularity, there exists $A' \in \mathbb{L}_o$ satisfying $(A, A') = 1$. Pick $B \in V$, primitive, such that $(B, A') = 0$. It follows that $\{A, B\}$ forms a basis for V.

Let then N be the endomorphism defined in (81). Pick $C \in \mathbb{L}_o$ such that $(B,C) = 1$ and define:

$$B' \;=\; C - (C,A')A - (C,A)A' + (C,A)(A,A')A \;\in V.$$

One has verifies that $N(A') = -B$ and $N(B') = A$. Therefore, N is primitive and $\mathrm{Im}(N) = V$. Moreover, since:

$$(Nx,y) \;=\; (x,B)(y,A) - (x,A)(y,B) \;=\; -(x,Ny)$$

for any $x, y \in \mathbb{L}_o$, the endomorphism N belongs to $\mathrm{u}(F)$. $\qquad\square$

In order to facilitate future computations, we shall introduce a special coordinate system on $\mathbb{L}_o$. The linearly independent family $\{A', B', A, B\}$, can be seen to provide a decomposition:

$$\mathbb{L}_o \;\simeq\; \underbrace{(\mathbb{Z} \cdot A' \oplus \mathbb{Z} \cdot B')}_{\mathbb{Z}^2} \;\oplus\; \underbrace{(\mathbb{Z} \cdot A \oplus \mathbb{Z} \cdot B)}_{\mathbb{Z}^2} \;\oplus\; \Lambda. \tag{82}$$

In this light, any element $\mathbb{L}_o$ (or $\mathbb{L}_o \otimes \mathbb{C}$) can be written uniquely as:

$$x_1 A' + x_2 B' + y_1 A + y_2 B + z.$$

We convene therefore to regard the elements of $\mathbb{L}_o$ as a triplets (x,y,z) with $x = (x_1, x_2) \in \mathbb{Z}^2$, $y = (y_1, y_2) \in \mathbb{Z}^2$ and $z \in \Lambda$. The quadratic pairing on $\mathbb{L}_o$ is recovered as:

$$((x,y,z),\ (x',y',z')) = x.y' + x'.y - (z,z')$$

where the first two dot-pairings on the left represent the standard Euclidean pairing on $\mathbb{Z}^2$ and $(\cdot, \cdot)$ is the pairing of Λ. Under this rule, the isotropic lattice V corresponds to the space of triplets $(0, y, 0)$ and the integral endomorphism N is given by $N(x, y, z) = (0, Tx, 0)$ with $T: \mathbb{Z}^2 \to \mathbb{Z}^2$ is the standard skew-adjoint endomorphism $T(x_1, x_2) = (x_2, -x_1)$.

As in 2.3, we define the groups:

$$\mathrm{U}(N)_{\mathbb{C}} \; := \; \{\exp(\lambda N) \mid \lambda \in \mathbb{C}\}$$

$$\mathrm{U}(N)_{\mathbb{Z}} \; := \; \{\exp(\lambda N) \mid \lambda \in \mathbb{Z}\}$$

leading to the sequence of inclusions:

$$\Omega \; \subset \; \Omega(F) = \mathrm{U}(N)_{\mathbb{C}} \cdot \Omega \; \subset \; \Omega^{\vee}.$$

We shall use the newly introduced coordinate system to analyze these inclusions. Let $r: \mathbb{L}_o \otimes \mathbb{C} \to \mathbb{R}$ be the function defined by $r(\omega) = -i(N\omega, \bar{\omega})$. This function is invariant under the action of $U(N)_{\mathbb{C}}$. In fact, if $\omega = (x, y, z)$ then $r(\omega) = 2\mathrm{Im}(x_1 \overline{x_2})$. Let

$$\Omega^{\vee} = \Omega^{-}(F) \; \cup \; \Omega^{0}(F) \; \cup \; \Omega^{+}(F)$$

be the decomposition of $\Omega^{\vee}$ in subsets for which $r(\omega)$ is strictly negative, zero and strictly positive, respectively.

Proposition 25. *The following statements hold:*

1. $\Omega(F) = \Omega^{-}(F) \; \cup \; \Omega^{+}(F)$.
2. *If* $[\omega] = [a, b, c] \in \Omega^{+}(F)$, *then:*

$$-\frac{a_1}{a_2} \tag{83}$$

   is a well-defined element of the upper-half plane.

Proof. Let $[\omega] \in \Omega(F)$. We show that $r(\omega) \neq 0$ by proving that the opposite statement leads to a contradiction. Indeed, assume that $r(\omega) = 0$. Since $\omega \in \Omega(F)$, there exists $\omega_o \in \Omega$ such that $\omega = \exp(zN).\omega_o = \omega_o + zN\omega_o$ for some $z \in \mathbb{C}$. Then:

$$(\omega, \bar{\omega}) = (\omega_o, \bar{\omega_o}) - 2\,\mathrm{Im}(z) \cdot r(\omega_o) = (\omega_o, \bar{\omega_o}) > 0. \tag{84}$$

But $r(\omega) = 0$ also implies that $\{\omega, N\omega, N\bar{\omega}\}$ span an isotropic subspace of $\mathbb{L}_o \otimes \mathbb{C}$. Clearly, ω and $N\omega$ are independent (otherwise $\omega \in V_{\mathbb{C}}$, contradicting (84)). Since the largest isotropic subspace in $\mathbb{L}_o \otimes \mathbb{C}$ is two-dimensional, it has to happen that $N\bar{\omega}$ is generated by ω and $N\omega$. But that also implies $\omega \in V_{\mathbb{C}}$, leading to a contradiction.

This shows that:

$$\Omega(F) \subset \Omega^{-}(F) \; \cup \; \Omega^{+}(F).$$

The reverse inclusion is straightforward.

Turning to the second statement, one has $r(\omega) = 2\mathrm{Im}(a_1 \overline{a_2}) > 0$. The denominator of (83) is therefore non-zero. Moreover, the same formula leads to:

$$\mathrm{Im}\left(-\frac{a_1}{a_2}\right) = \frac{r(\omega)}{2|a_2|^2}$$

which assures us that (83) is an element of the upper half-plane. $\qquad\square$

We are now in position to write explicit formulas for the geometric assignment, described in 2.4, that associates to a nilpotent orbit in
$$\mathcal{B}(F) \;=\; \Omega(F)/\mathrm{U}(N)_{\mathbb{C}},$$
a pair $(\mathcal{H}, \psi)$ consisting of a weight-one Hodge structure $\mathcal{H}$ on V and a homomorphism $\psi \colon \Lambda \to J^1(\mathcal{H})$.

Under the identification $V \simeq \mathbb{Z}^2$, provided by the basis $\{A, B\}$, the skew-symmetric bilinear form $(\cdot, \cdot)_1$ is transported to $(x, y)_1 = x.Ty = x_1 y_2 - x_2 y_1$. The Hodge structures of weight one on V which are polarized with respect to $(\cdot, \cdot)_1$ are then indexed by purely imaginary complex numbers τ belonging to the upper half-plane $\mathbb{H}$. Every such τ induces the polarized weight-one Hodge structure:
$$0 \;\subset\; \{A + \tau B\} \;\subset\; V_{\mathbb{C}} \tag{85}$$
and the correspondence is one-to-one.

Let then $[\omega] = [a, b, c]$ be an element in $\Omega(F)$. As described in 2.4, the Hodge structure $\mathcal{H}$ associated to the nilpotent orbit of $[\omega]$ in
$$\mathcal{B}(F) \;=\; \Omega(F)/\, \mathrm{U}(N)_{\mathbb{C}}$$
is given by the filtration:
$$0 \;\subset\; \{[\omega]\}^{\perp} \,\cap\, V_{\mathbb{C}} \;\subset\; V_{\mathbb{C}}. \tag{86}$$
Using the coordinate framework, the middle space in (86) is:
$$\{[\omega]\}^{\perp} \,\cap\, V_{\mathbb{C}} \;=\; \{(0, y, 0) \in V_{\mathbb{C}} \mid a.y = 0 \}.$$
An identification of the two filtrations (85) and (86) leads one to:
$$\tau \;=\; -\frac{a_1}{a_2}. \tag{87}$$

Connecting (87) to Proposition 25 we see that the decomposition
$$\mathcal{B}(F) = \Omega(F)/U(N)_{\mathbb{C}} = \Omega^+(F)/U(N)_{\mathbb{C}} \,\cup\, \Omega^-(F)/U(N)_{\mathbb{C}}$$
corresponds to the decomposition $\mathcal{B}(F) = \mathcal{B}^+(F) \cup \overline{\mathcal{B}(F)}$ of section 2.4. The Hodge structure $\mathcal{H}$ is polarized with respect to $(\cdot, \cdot)_1$ if and only if $[\omega] \in \Omega^+(F)$.

The coordinate framework can also be used to give a straightforward procedure constructing the extension homomorphism:
$$\psi \colon \Lambda \to J^1(\mathcal{H}) = V_{\mathbb{C}}/\left(\{\omega\}^{\perp} \cap\, V_{\mathbb{C}}\right) + V \tag{88}$$
associated to $[\omega] = [a, b, c]$. If $\gamma \in \Lambda$, choose a lifting $\widetilde{\gamma} = (0, \beta, \gamma) \in V_{\mathbb{C}}^{\perp}$ such that $(\widetilde{\gamma}, \omega) = 0$. This amounts to choosing $\beta \in V_{\mathbb{C}}$ with $\beta.a = \gamma.c$. Clearly, such a β is not unique but two different choices always differ by an element in $\{[\omega]\}^{\perp} \cap\, V_{\mathbb{C}}$. Moreover, if one denotes:
$$z \;=\; \frac{c}{a_2} \;=\; z_2 - \tau z_1, \quad z_1, z_2 \in \Lambda_{\mathbb{R}} \tag{89}$$
then the homomorphism ψ can be described as assigning:
$$\gamma \;\longmapsto\; ((\gamma, z_1),\ (\gamma, z_2)) \in J^1(\mathcal{H}).$$
The element $z \in \Lambda_{\mathbb{C}}$, defined as in (89), totally controls the homomorphism ψ.

We have reached therefore the following conclusion:

Theorem 26. *The geometric correspondence of section* 2.4 *which associates to boundary points in* $\mathcal{B}^+(F)$ *pairs* $(\mathcal{H}, \psi)$ *of polarized weight-one Hodge structures on* $V_{\mathbb{C}}$ *and extension homomorphisms* $\psi \colon \Lambda \to J^1(\mathcal{H})$ *induces an identification:*
$$\mathcal{B}^+(F) \;=\; \Omega^+(F)/\mathrm{U}(N)_{\mathbb{C}} \;\simeq\; \mathbb{H} \times \Lambda_{\mathbb{C}}. \tag{90}$$
Under this identification, the holomorphic $\mathbb{C}$*-fibration of* (27) *is described by the the map:*
$$\widetilde{\Theta} \colon \Omega^+(F) \to \mathbb{H} \times \Lambda_{\mathbb{C}}, \quad \widetilde{\Theta}\,([a, b, c]) \;=\; \left(-\frac{a_1}{a_2},\ \frac{c}{a_2}\right). \tag{91}$$

One immediately verifies in (91) the main features of (27), namely :

- $\widetilde{\Theta}$ is an onto holomorphic map.
- $\widetilde{\Theta}$ is invariant under the action of $U(N)_{\mathbb{C}}$ on $\Omega^+(F)$ and the fibers of $\widetilde{\Theta}$ coincide with the orbits of the $U(N)_{\mathbb{C}}$-action. Θ is therefore a holomorphic $U(N)_{\mathbb{C}}$-principal bundle.

At this point, recall that one obtains the parabolic cover (80) by further taking the quotient with respect to the action of the parabolic group of integral isometries $\Gamma_F^+ = P(F) \cap \Gamma^+$ (Γ^+ consists of those isometries preserving the connected components of Ω). It is important therefore to understand the group Γ_F^+ and its action on $\Omega^+(F)/U(N)_{\mathbb{Z}}$ and $\mathbb{H} \times \Lambda_{\mathbb{C}}$.

5.2 Description of Γ_F^+ and its action on $\mathcal{B}^+(F)$

The integral isometries Γ_F^+ can be given a matrix description using the coordinate framework (82).

Lemma 27. *A transformation in Γ, stabilizing the isotropic sub-lattice V is of the form:*

$$g(m, Q, R, F) = \begin{pmatrix} m & 0 & 0 \\ R & \widetilde{m} & Qf \\ Q^t m & 0 & f \end{pmatrix} \tag{92}$$

where:

1. $m \in \mathrm{GL}_2(\mathbb{Z})$.
2. $\widetilde{m} = (m^t)^{-1}$.
3. $Q \in \mathrm{Hom}(\Lambda, \mathbb{Z}^2)$, $R \in \mathrm{End}(\mathbb{Z}^2)$ *satisfying* $R^t m + m^t R = m^t Q Q^t m$.
4. f *is an isometry of Λ.*

Moreover $g(m, Q, R, f) \in \Gamma_F^+$ if and only if $m \in \mathrm{SL2}(\mathbb{Z})$.

This gives a matrix characterization for Γ_F^+. The group multiplication law goes as follows:

$$g(m_1, Q_1, R_1, d_1) \cdot g(m_2, Q_2, R_2, d_2) = \tag{93}$$

$$= g(m_1 m_2, \ Q_1 + \widetilde{m_1} Q_2 f_1^t, \ R_1 m_2 + \widetilde{m_1} R_2 + Q_1 f_1 Q_2^t m_2, \ f_1 f_2).$$

In particular:

$$g(m, Q, R, f)^{-1} = g(m^{-1}, \ -m^t Q f, \ R^t, \ f^{-1}).$$

We single out the following special subgroups of Γ_F^+ :

1. $U(N)_{\mathbb{Z}} = \{g(I, 0, R, I) \mid R + R^t = 0\} \simeq \mathbb{Z}$.
2. $\mathcal{S} = \{g(m, 0, 0, I) \mid m \in \mathrm{SL}_2(\mathbb{Z})\} \simeq \mathrm{SL}_2(\mathbb{Z})$.
3. $\mathcal{W} = \{g(I, 0, 0, f) \mid f \in \mathcal{O}(\Lambda)\} \simeq \mathcal{O}(\Lambda)$.
4. $\mathcal{T} = \{g(I, Q, R, I) \mid R + R^t = QQ^t\}$.

It can be verified that:

- $U(N)_{\mathbb{Z}} \subset Z(\Gamma_F^+)$
- $\mathcal{T} \lhd \Gamma_F^+$.
- $\mathcal{S} \cap \mathcal{W} = \{\pm I\}$.
- Γ_F^+ decomposes as a semi-direct product $\mathcal{T} \rtimes (\mathcal{W} \times_{\{\pm I\}} \mathcal{S})$.

The parabolic subgroup Γ_F^+ acts on the total space $\Omega^+(F)$ of the holomorphic $\mathbb{C}^*$-bundle:

$$\widetilde{\Theta}\colon \Omega^+(F) \;\to\; \mathbb{H} \times \Lambda_{\mathbb{C}}, \quad \widetilde{\Theta}\left([a,b,c]\right) \;=\; \left(-\frac{a_1}{a_2},\; \frac{c}{a_2}\right). \tag{94}$$

There is a compatible action on $\mathbb{H} \times \Lambda_{\mathbb{C}}$ which carries an important geometric significance.

Recall that a pair $(\tau, z) \in \mathbb{H} \times \Lambda_{\mathbb{C}}$ determines a polarized mixed Hodge structure on $V_{\mathbb{C}}$ together with a homomorphism $\psi\colon \Lambda \to J^1(\mathcal{H})$ given essentially by $\psi(\gamma) \;=\; ((\gamma, z_1), (\gamma, z_2))$ where $z = z_2 - \tau z_1$. As mentioned earlier, the Jacobian $J^1(\mathcal{H})$ can be regarded as an elliptic curve $E_\tau \;=\; \mathbb{C}/\mathbb{Z} \oplus \tau \mathbb{Z}$ and, in this setting, the morphism ψ determines a flat G-connection over E_τ (the Lie group G is $E_8 \times E_8 \rtimes \mathbb{Z}_2$ if $\Lambda = E_8 \oplus E_8$ lattice and $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ if $\Lambda = \Gamma_{16}$).

Denote by $\pi\colon \mathbb{H} \times \Lambda_{\mathbb{C}} \to \mathbb{H}$ the projection on the first coordinate. Taking then:

$$\mathcal{L}_\tau\colon \; = \{\tau\} \times \Lambda \otimes (\mathbb{Z} \oplus \tau \mathbb{Z}) \subset \pi^{-1}(\tau)$$

one obtains a family of 32-dimensional lattices, parameterized by τ, moving in the fibration π.

Definition 28. *Let Π be the group of holomorphic automorphisms of the fibration π which preserve the lattice family $\mathcal{L}$ and cover $\mathrm{PSL}(2, \mathbb{Z})$ transformations on $\mathbb{H}$.*

It turns out that two elements (τ, z) and (τ', z') of $\mathbb{H} \times \Lambda_{\mathbb{C}}$ determine isomorphic pairs of elliptic curves and flat G-connections if and only if they can be transformed one into another through an isomorphism in Π. In this sense, the analytic space:

$$\mathcal{M}_{E,G} \;=\; \Pi \setminus (\mathbb{H} \times \Lambda_{\mathbb{C}}) \tag{95}$$

can be seen as the moduli space of pairs of elliptic curves and flat G-bundles[2].

Theorem 29. *There is a short exact sequence of groups:*

$$\{1\} \;\to\; \mathrm{U}(N)_{\mathbb{Z}} \;\to\; \Gamma_F^+ \;\overset{\alpha}{\to}\; \Pi \;\to\; \{1\}. \tag{96}$$

with respect to which the analytic fibration $\widetilde{\Theta}$ of (94) is α-equivariant. This induces a holomorphic identification:

$$\mathcal{D}_\Lambda \;=\; \Gamma_F^+ \backslash \mathcal{B}^+(F) \;\simeq\; \Pi \backslash (\mathbb{H} \times \Lambda_{\mathbb{C}}) \;=\; \mathcal{M}_{E,G} \tag{97}$$

between the Type II boundary divisor $\mathcal{D}_\Lambda$ corresponding to F and the moduli space of pairs of elliptic curves and flat G-bundles. Moreover, under (97), the quotient map:

$$\Theta\colon \Gamma_F^+ \backslash \Omega^+(F) \;\to\; \Pi \backslash (\mathbb{H} \times \Lambda_{\mathbb{C}}). \tag{98}$$

is exactly the parabolic Seifert fibration (28) of section 2.3.

Proof. For any $\varphi \in \Gamma_F^+$, one can construct a well-defined automorphism of $\mathbb{H} \times \Lambda_{\mathbb{C}}$ by taking

$$(\tau, z) \;\longmapsto\; \Theta\left(\varphi(\omega)\right)$$

where $[\omega]$ is a lift (under $\widetilde{\Theta}$) of (τ, z) in $\Omega^+(F)$. We claim that all such transformations are elements of Π.

Let $(\tau, z) \in \mathbb{H} \times \Lambda_{\mathbb{C}}$ and $g(m, Q, R, f) \in \Gamma_F^+$ defined as in (92). Choose $[\omega] = [x, y, z] \in \widetilde{\Theta}^{-1}(\tau, z)$. It can be assumed that $x = (-\tau, 1)$ and $z = z_2 - z_1 \tau$ with $z_1, z_2 \in \Lambda_{\mathbb{R}}$.

If $m \in \mathrm{SL}_2(\mathbb{Z})$ is has the matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\widetilde{m} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

---

[2]Again, in the case $\Lambda = \Gamma_{16}$, one considers only $\mathrm{Spin}(32)$-liftable connections

and the action of $g(m, Q, R, f)$ is just:

$$g(m, Q, R, f).[\omega] \;=\; \left[m \cdot x, \; R \cdot x + \widetilde{m} \cdot y + Q \cdot f \cdot z, \; Q^t \cdot m \cdot x + f \cdot z \right].$$

An easy calculation shows that $\widetilde{\Theta}\left(g(m, Q, R, f).[\omega]\right) = (\tau', z')$ with

$$\tau' = \frac{a\tau - b}{-c\tau + d}, \quad z' = Q^t(-\tau', 1) + (df(z_2) - bf(z_1)) + (-cf(z_2) + af(z_1))\,\tau'.$$

It is clear then that the transformation:

$$(\tau, z) \;\rightarrow\; \widetilde{\Theta}\left(g(m, Q, R, f).[\omega]\right) = (\tau', z') \tag{99}$$

covers a $\mathrm{PSL}(2, \mathbb{Z})$ transformation on the first factor of $\mathbb{H} \times \Lambda_{\mathbb{C}}$ corresponding to the matrix action of

$$\widetilde{m} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

In addition, one notes that the transformation (99) preserves the lattice family $\mathcal{L}$. It is therefore with a well-defined transformation in Π.

The above assignment induces a group homomorphism $\alpha \colon \Gamma_F^+ \to \Pi$. One can see that $\alpha\left(g(m, Q, R, f)\right) = 1$ requires $m = I$, $f = I$ and $Q = 0$. This implies that $\mathrm{Ker}(\alpha) = \mathrm{U}(N)_{\mathbb{Z}}$.

Let us check that α is an onto morphism. For this purpose, we single out the following special subgroups of Π.

- $\mathcal{S}_\Pi \;=\; \left\{ \psi \in \pi \;\middle|\; \psi(\tau, z \otimes \lambda) \;=\; \left(\frac{a\tau + b}{c\tau + d}, \; \frac{z}{c\tau + d} \otimes \lambda \right) \right\}$

- $\mathcal{T}_\Pi \;=\; \left\{ \psi \in \Pi \;\middle|\; \psi(\tau, z \otimes \lambda) = (\tau, z \otimes \lambda + 1 \otimes q_1 + \tau \otimes q_2) \;\; \text{where} \;\; (q_1, q_2) \in \Lambda \oplus \Lambda \right\}$

- $\mathcal{W}_\Pi = \left\{ \psi = \mathrm{id} \oplus (\mathrm{id} \otimes f) \in \Pi \;\middle|\; f \in O(\Lambda) \right\}$.

The three subgroups $\mathcal{S}_\Pi$, $\mathcal{T}_\Pi$ and $\mathcal{W}_\Pi$ generate the entire Π. In addition, note that:

$$\mathcal{S}_\Pi \cap \mathcal{W}_\Pi = \left\{ \psi \in \Pi \;\middle|\; \psi(\tau, z \otimes \lambda) \;=\; (\tau, \pm z \otimes \lambda) \right\} = \{\pm I\}$$

and if $p \colon \Pi \to \mathrm{PSL}(2, \mathbb{Z})$ is the projection to $\mathrm{PSL}(2, \mathbb{Z})$ then $\mathrm{Ker}(p)$ is generated by $\mathcal{W}_\Pi$ and $\mathcal{T}_\Pi$. One concludes from these facts that Π is a semi-direct product:

$$\Pi \;=\; \mathcal{T}_\Pi \rtimes \left(\mathcal{W}_\Pi \times_{\{\pm 1\}} \mathcal{S}_\Pi \right). \tag{100}$$

The above three subgroups are naturally related through the homomorphism α to the three particular subgroups of Γ_F^+ described earlier.

When restricted to $\mathcal{S} \subset \Gamma_F^+$, the morphism α produces an isomorphism $\mathcal{S} \simeq \mathcal{S}_\Pi$ sending $g(m, 0, 0, I)$ with

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to the automorphism in $\mathcal{S}_\Pi$ associated to the matrix:

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

When restricted to $\mathcal{W} \subset \Gamma_F^+$ the morphism α produces an isomorphism $\mathcal{W} \simeq \mathcal{W}_\Pi$, which sends $g(I, 0, 0, f)$ to the automorphism induced by f in $\mathcal{W}_\Pi$.

Finally, when restricted to $\mathcal{T} \subset \Gamma_F^+$, the morphism α produces a surjective morphism $\mathcal{T} \simeq \mathcal{T}_\Pi$ with kernel $\mathrm{U}(N)_\mathbb{Z}$. If $Q \colon \Lambda \to \mathbb{Z}^2$ is given by

$$Q(\gamma) = ((\gamma, q_1),\ (\gamma, q_2)),\quad q_1, q_2 \in \Lambda$$

then α represents the assignment $g(I, Q, R, I) \to (q_2, -q_1)$.

All three subgroups, $\mathcal{S}_\Pi$, $\mathcal{T}_\Pi$ and $\mathcal{W}_\Pi$ are therefore entirely covered by the image of α. Since they generate Π, the morphism α is surjective.

One verifies immediately that the map:

$$\widetilde{\Theta} \colon \Omega^+(F) \to \mathbb{H} \times \Lambda_\mathbb{C},\quad \widetilde{\Theta}\left([a, b, c]\right) = \left(-\frac{a_1}{a_2}, \frac{c}{a_2}\right) \tag{101}$$

is equivariant with respect to α. This leads to the Seifert fibration:

$$\Theta \colon \Gamma_F^+ \backslash \Omega^+(F) \ \to\ \Pi \backslash (\mathbb{H} \times \Lambda_\mathbb{C}) = \mathcal{M}_{E,G} \tag{102}$$

whose fibers are isomorphic to $\mathrm{U}(N)_\mathbb{C}/\mathrm{U}(N)_\mathbb{Z}$ and therefore are copies of $\mathbb{C}^*$.

The identification (97) follows from the arguments above. $\qquad\square$

5.3 Automorphy Factors for the Parabolic Cover

Let us remark that, based on the above arguments, one obtains a canonical isomorphism

$$\mathcal{D}_\Lambda = \Gamma_F^+ \backslash \mathcal{B}^+(F) = \Gamma_F^+ \backslash \Omega^+(F)/U(N)_\mathbb{C} \ \simeq\ \Pi \backslash (\mathbb{H} \times \Lambda_\mathbb{C}) = \mathcal{M}_{E,G} \tag{103}$$

identifying the Type II Mumford divisor $\mathcal{D}_\Lambda$ with the moduli space $\mathcal{M}_{E,G}$ of pairs of elliptic curves and flat G-bundles. Under this isomorphism, the parabolic cover (80) becomes the induced holomorphic Seifert $\mathbb{C}^*$-fibration:

$$\Gamma_F^+ \backslash \Omega^+(F) \ \to\ \Pi \backslash (\mathbb{H} \times \Lambda_\mathbb{C}). \tag{104}$$

Our task in this section is to analyze the holomorphic type of (104).

We use the following strategy. The base space of (104) is a complex orbifold $\Pi \backslash V$ where $V = \mathbb{H} \times \Lambda_\mathbb{C}$. One can describe holomorphic $\mathbb{C}^*$-fibrations over such a space in terms of equivariant line bundles over the cover V. These equivariant objects are line bundles $\mathcal{L} \to V$ where the action of the group Π on the base is given a lift to the fibers. All holomorphic line bundles over V are trivial and, choosing a trivializing section, one obtains a lift of the action to fibers through a set of automorphy factors $(\varphi_a)_{a \in \Pi}$ with $\varphi_a \in H^0(V, \mathcal{O}_V^*)$ satisfying:

$$\varphi_{ab}(x) = \varphi_a(b \cdot x)\varphi_b(x).$$

Such a set determines a 1-cocycle φ in $Z^1(\Pi, H^0(V_G, \mathcal{O}_V^*))$. Two automorphy factors provide isomorphic fibrations on $\mathcal{M}_{E,\,G}$ if and only if they determine the same group cohomology class in $H^1(\Pi, H^0(V, \mathcal{O}_V^*))$. To state this rigorously, there is a canonical map ϕ entering the following exact sequence:

$$\{1\} \to H^1(\Pi, H^0(V, \mathcal{O}_V^*)) \xrightarrow{\phi} H^1(\mathcal{M}_{E,\,G},\ \mathcal{O}_{\mathcal{M}_{E,\,G}}^*) \xrightarrow{p^*} H^1(V,\ \mathcal{O}_V^*) \simeq \{1\}. \tag{105}$$

We are going to write down explicitly a set of automorphy factors for fibration (104). Since the modular group Π is generated by the three subgroups $\mathcal{S}_\Pi$, $\mathcal{W}_\Pi$ and $\mathcal{T}_\Pi$ it will suffice to describe automorphy factors for elements in those subgroups.

The first step towards computing the automorphy factors of (104) is defining a holomorphic trivialization of the covering $\mathbb{C}$-bundle:

$$\widetilde{\Theta} \colon \Omega^+(F) \to \mathbb{H} \times \Lambda_\mathbb{C},\quad \widetilde{\Theta}\left([a, b, c]\right) = \left(-\frac{a_1}{a_2}, \frac{c}{a_2}\right). \tag{106}$$

Recall that this map provides the arithmetic recipe through which one can obtain out of a given $K3$ period an elliptic curve E_τ and a morphism $\psi\colon \Lambda \to E_\tau$ which carries the holonomy information of a flat G-connection. Building a trivializing section for (106) amounts then intuitively to finding a way to recover a $K3$ period out of geometric data given by an elliptic curve endowed with a flat G-connection.

Surprisingly, such a method arises in string theory, precisely in the Narain construction (see [24] [25] [16]) of the lattice of momenta related with toroidal compactification of heterotic strings. This construction leads one to consider the following map (see Appendix 6 for details):

$$\sigma_n\colon \mathbb{H} \times \Lambda_\mathbb{C} \times \mathbb{C} \;\to\; \Omega^+(F) \tag{107}$$

$$\sigma_n(\tau, z, u) \;=\; \exp(u \cdot N) . \left[\, (-\tau, 1),\; \frac{1}{2}\left(\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau},\; \frac{\bar\tau(z,z) - \tau(z,\bar z)}{\bar\tau - \tau} \right),\; z \,\right]. \tag{108}$$

A brief analysis of the above formula reveals the following:

Remark 30.

1. *The image of σ_n indeed lies in the indicated space since for any triplet (τ, z, u),*

$$(\sigma_n(\tau, z, u),\; \sigma_n(\tau, z, u)) = 0 \;\; \text{and} \;\; -i(\, N\sigma_n(\tau, z, u),\; \overline{\sigma_n(\tau, z, u)}\,) = \operatorname{Im}\tau > 0.$$

2. *One has:*

$$\left(\sigma_n(\tau, z, u),\; \overline{\sigma_n(\tau, z, u)}\right) = \; \operatorname{Im}(u)\,\operatorname{Im}(\tau)$$

and therefore, $\sigma_n(\tau, z, u)$ is a $K3$ period for any $u \in \mathbb{C}$ with strictly positive imaginary part.

3. *The map*

$$(\tau, z) \to \sigma_n(\tau, z, 0) \tag{109}$$

makes a smooth section for the line bundle (106).

4. *When one factors out the action of $\mathrm{U}(N)_\mathbb{Z}$, application (108) provides a smooth trivialization for the induced $\mathbb{C}^*$-bundle:*

$$\widetilde{\Theta}\colon \Omega^+(F)/\mathrm{U}(N)_\mathbb{Z} \;\to\; \mathbb{H} \times \Lambda_\mathbb{C}. \tag{110}$$

The above Narain trivialization has a major drawback ! It is not holomorphic. Nevertheless, one can get around this problem and obtain a holomorphic trivialization by perturbing slightly the map (108).

Note that the middle term in expression (108) can be rewritten:

$$\frac{1}{2}\left(\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau},\; \frac{\bar\tau(z,z) - \tau(z,\bar z)}{\bar\tau - \tau} \right) \;=\; \frac{1}{2}\left(\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau},\; (z,z) + \tau\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau} \right) \;=$$

$$= \frac{1}{2}\,(0, (z,z)) \;+\; \frac{1}{2}\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau}\,(1, \tau) \;\; = \;\; \frac{1}{2}\,(0, (z,z)) \;+\; \frac{1}{2}\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau}\,T\,(-\tau, 1)$$

Following the above equality, one can see the Narain section (108) as:

$$\sigma_n(\tau, z, 0) \;=\; \exp\left(\frac{1}{2}\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau}N \right) . \left[\, (-\tau, 1),\; \frac{1}{2}\,(0, (z,z)),\; z \,\right] \;\in \Omega^+(F).$$

The second factor in the right-hand side term is holomorphic. This suggest the following perturbation:

$$\sigma\colon \mathbb{H} \times \Lambda_\mathbb{C} \;\to\; \Omega^+(F) \tag{111}$$

$$\sigma(\tau, z) \;=\; \exp\left(-\frac{1}{2}\frac{(z,z) - (z,\bar z)}{\bar\tau - \tau}N \right) . \sigma_n(\tau, z) \;=\; \left[\, (-\tau, 1),\; \frac{1}{2}\,(0, (z,z)),\; z \,\right].$$

We call this the **perturbed Narain map**. One can immediately check that:

Theorem 31. *The perturbed Narain map* (111) *is a holomorphic section for the line bundle:*

$$\widetilde{\Theta} \colon \Omega^+(F) \to \mathbb{H} \times \Lambda_{\mathbb{C}}, \quad \widetilde{\Theta}\left([a, b, c]\right) = \left(-\frac{a_1}{a_2}, \frac{c}{a_2}\right). \tag{112}$$

It descends to a holomorphic section for the $\mathbb{C}^*$*-fibration* (110), *providing therefore a holomorphic trivialization:*

$$\mathbb{H} \times \Lambda_{\mathbb{C}} \times \mathbb{C}^* \simeq \Omega^+(F)/\mathrm{U}(N)_{\mathbb{Z}}, \quad (\tau, z, u) \to u \cdot \sigma(\tau, z).$$

The $\mathbb{C}^*$*-action in the right-hand side expression represents the action of* $\mathrm{U}(N)_{\mathbb{C}}/\mathrm{U}(N)_{\mathbb{Z}}$ *upon* $\Omega^+(F)/\mathrm{U}(N)_{\mathbb{Z}}$.

We are now in position to compute the automorphy factors of the parabolic cover map:

$$\Theta \colon \Gamma_F^+ \backslash \Omega^+(F) \to \Pi \backslash (\mathbb{H} \times \Lambda_{\mathbb{C}}). \tag{113}$$

In order to obtain a set of factors, one needs to analyze the variation of the perturbed Narain map σ under the action of the modular group Π.

Lemma 32. *Assume* $(q_1, q_2) \in \mathcal{T}_{\Pi}$. *Then:*

$$\sigma(\tau, z + q_1 + \tau q_2) = e^{\pi i (2(q_2, z) + \tau(q_2, q_2))} \cdot g(I, Q, R, I).\sigma(\tau, z) \tag{114}$$

where $Q \in \mathrm{Hom}_{\mathbb{Z}}\left(\Lambda, \mathbb{Z}^2\right)$ *is given by* $Q(\gamma) = (-(\gamma, q_2), (\gamma, q_1))$ *and* $R \in \mathrm{End}(\mathbb{Z}^2)$ *with* $R + R^t = QQ^t$

Proof. Computing in $\Omega^+(F)$, we obtains that $\sigma(\tau, z + q_1 + \tau q_2)$ amounts to:

$$\left[(-\tau, 1), \ \frac{1}{2}\left(0, (z, z) + (q_1, q_1) + \tau^2(q_2, q_2) + 2(z, q_2) + 2\tau(z, q_2) + 2\tau(q_1, q_2)\right), \ z + q_1 + \tau q_2\right].$$

On the other hand,

$$g(I, Q, R, I).\sigma(\tau, z) = \left[(-\tau, 1), \ R(-\tau, 1) + \frac{1}{2}(0, (z, z)) + Qz, \ z + Q^t(-\tau, 1)\right].$$

But $Qz = (-zq_2, zq_1)$ and $Q^t(-\tau, 1) = q_1 + \tau q_2$. Moreover, one can see that:

$$R(-\tau, 1) = \left(-(q_1, q_2) - \frac{1}{2}\tau(q_2, q_2), \ \frac{1}{2}(q_1, q_1)\right) + A(-\tau, 1)$$

where $A \in \mathrm{End}(\mathbb{Z}^2)$ skew-symmetric. One obtains then the following equality in $\Omega^+(F)$:

$$\sigma(\tau, z + q_1 + \tau q_2) = \exp\left(\left((q_2, z) + \frac{1}{2}\tau(q_2, q_2) + \alpha\right) N\right) g(I, Q, R, I).\sigma(\tau, z) \tag{115}$$

with $\alpha \in \mathbb{Z}$. After factoring out the $\mathrm{U}(N)_{\mathbb{Z}}$-action, one is led to (114). $\qquad \square$

Lemma 33. *Assume* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. *Then:*

$$\sigma\left(\frac{a\tau + b}{c\tau + d}, \ \frac{z}{c\tau + d}\right) = e^{\left(-\frac{\pi i c(z, z)}{c\tau + d}\right)} \cdot g(m, 0, 0, I).\sigma(\tau, z), \quad \text{where } m = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}. \tag{116}$$

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Proof. As in the previous lemma, we write the calculations in $\Omega^+(F)$. One has:

$$\sigma\left(\frac{a\tau+b}{c\tau+d},\ \frac{z}{c\tau+d}\right)\ =\ \left[\left(-\frac{a\tau+b}{c\tau+d},\ 1\right),\ \frac{1}{2}\left(0,\ \frac{(z,z)}{(c\tau+d)^2}\right),\ \frac{z}{c\tau+d}\right]. \tag{117}$$

In the same time:

$$g(m,0,0,I).\sigma(\tau,z)\ =\ \left[m(-\tau,1),\ \widetilde{m}\left(\frac{1}{2}(0,(z,z))\right),\ z\right]\ = \tag{118}$$

$$=\ \left[(-(a\tau+b),\ c\tau+d),\ \frac{1}{2}\left(c(z,z),d(z,z)\right),\ z\right]\ =\ \left[\left(-\frac{a\tau+b}{c\tau+d},\ 1\right),\ \frac{1}{2}\left(\frac{c(z,z)}{c\tau+d},\ \frac{d(z,z)}{c\tau+d}\right),\ \frac{z}{c\tau+d}\right].$$

Comparing the two formulas we get the following identity in $\Omega^+(F)$:

$$\sigma\left(\frac{a\tau+b}{c\tau+d},\ \frac{z}{c\tau+d}\right)\ =\ \exp\left(\left(-\frac{1}{2}\frac{c(z,z)}{c\tau+d}\right)N\right)\ g(m,0,0,I).\sigma(\tau,z). \tag{119}$$

Factoring out the $U(N)_{\mathbb{Z}}$-action, one obtains (116). $\qquad\square$

We can state then:

Theorem 34. *Let* $(\varphi_g)_{g\in K}$ *be the automorphy factors of parabolic cover* $\mathbb{C}^*$*-fibration:*

$$\Gamma_F^+ \setminus \Omega^+(F)\ \to\ \Pi \setminus (\mathbb{H} \times \Lambda_{\mathbb{C}}) \tag{120}$$

associated to the trivialization generated by σ. *Then:*

1. $\varphi_g(\tau,z)\ =\ e^{-\pi i(2(q_2,z)+\tau(q_2,q_2))}$ *for* $g=(q_1,q_2)\in \mathcal{T}_\Pi$.
2. $\varphi_g(\tau,z)\ =\ e^{\left(\frac{\pi i c(z,z)}{c\tau+d}\right)}$ *for* $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathcal{S}_\Pi$.
3. $\varphi_g(\tau,z)\ =\ 1$ *for* $g\in \mathcal{W}_\Pi$.

Proof. The first two expressions are direct consequences of Lemmas 32 and 33. The fact that $\varphi_g(\tau,z)\ =\ 1$ for $g\in \mathcal{W}_\Pi$. follows from:

$$\sigma(\tau,f(z))\ =\ \left[(-\tau,1),\ \frac{1}{2}(0,(f(z),f(z))),\ f(z)\right]\ =$$

$$=\ \left[(-\tau,1),\ \frac{1}{2}(0,(z,z)),\ f(z)\right]\ =\ g(I,0,0,f).\sigma(\tau,z)$$

for any $f\in O(\Lambda)$. $\qquad\square$

The three subgroups $\mathcal{T}_\Pi$, $\mathcal{S}_\Pi$ and $\mathcal{W}_\Pi$ generate the entire modular group Π. Therefore, the above automorphy factors are enough to characterize completely the holomorphic type of fibration (120).

5.4 Theta Function Interpretation and Relation to Heterotic String Theory

Given the particular automorphy factor expressions computed in the previous section, one can provide for the parabolic cover $\mathbb{C}^*$-fibration (120) a theta function interpretation.

Let $\mathbb{H} \times \Lambda_{\mathbb{C}}$ be the orbifold cover of $\mathcal{M}_{E,G}$. Since Λ is positive definite, unimodular and even, there is an associated holomorphic theta function (see [19] [23] for details) :

$$\Theta_\Lambda:\ \mathbb{H} \times \Lambda_{\mathbb{C}} \to \mathbb{C},\qquad \Theta_\Lambda(\tau,\ z)=\sum_{\gamma\in\Lambda} e^{\pi i(2(z,\gamma)+\tau(\gamma,\gamma))}. \tag{121}$$

The pairing appearing above represents the bilinear complexification of the integral pairing on Λ. The Λ-character function can be written then as a quotient of Θ_Λ:

$$B_\Lambda \colon \mathbb{H} \times \Lambda_\mathbb{C} \to \mathbb{C}, \quad B_\Lambda(\tau,\ z) = \frac{\Theta_\Lambda(\tau,\ z)}{\eta(\tau)^{16}}. \tag{122}$$

Here, η is Dedekind's eta function:

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{m=1}^{\infty} \left(1 - e^{2\pi i m \tau}\right),$$

which is an automorphic form of weight $1/2$ and multiplier system given by a group homomorphism $\chi \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Z}/24\mathbb{Z}$, in the sense that [32]:

$$\eta(\gamma \cdot \tau) \ = \ \chi(\gamma)\ \sqrt{c\tau + d}\ \eta(\tau) \ \text{ for } \ \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

The character terminology for (122) is justified by its role in the representation theory of infinite-dimensional Lie algebras. The function B_Λ is the zero-character of the level $l = 1$ basic highest weight representation of the Kac-Moody algebra associated to G (see [19] for details).

According to [19], the character function B_Λ obeys the following transformation properties:

Proposition 35. *Under the action of the modular group Π, the character function (122) transforms as :*

$$B_\Lambda \left(g \cdot (\tau, z)\right) \ = \ \varphi_g^{\mathrm{ch}}(\tau, z) \cdot B_\Lambda(\tau, z).$$

The factors $\varphi_g^{\mathrm{ch}}, g \in \Pi$ can be described as:

- $\varphi_{(q_1, q_2)}^{\mathrm{ch}}(\tau, z) = e^{\pi i(-2(q_2 z) - \tau(q_2 q_2))}$ *for $(q_1, q_2) \in \mathcal{T}_\Pi$.*

- $\varphi_m^{\mathrm{ch}} = e^{\frac{\pi i c(zz)}{c\tau + d}}$ *for $m = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathcal{S}_\Pi$.*

- $\varphi_w^{\mathrm{ch}} = 1$ *for $w \in \mathcal{W}_\Pi$.*

The holomorphic function B_Λ descends therefore to a section of a $\mathbb{C}$-fibration:

$$\mathcal{Z} \to \Pi \backslash (\mathbb{H} \times \Lambda_\mathbb{C}) = \mathcal{M}_{E,\ G} \tag{123}$$

with automorphy factors φ_g^{ch} described above. We call this the character fibration.

One can compare then the character fibration (123) with the parabolic cover $\mathbb{C}^*$-fibration:

$$\Theta \colon \Gamma_F^+ \backslash \Omega^+(F) \ \to \ \Pi \backslash (\mathbb{H} \times \Lambda_\mathbb{C}) = \mathcal{M}_{E,\ G} \tag{124}$$

analyzed in the previous section. A look at Theorem 34 and Proposition 35 is enough to convince us that the two holomorphic fibrations are defined through identical automorphy factors. Therefore:

Theorem 36. *The parabolic cover fibration*

$$\Theta \colon \Gamma_F^+ \backslash \Omega^+(F) \ \to \ \Pi \backslash (\mathbb{H} \times \Lambda_\mathbb{C}) = \mathcal{M}_{E,\ G} \tag{125}$$

is holomorphically isomorphic to the character fibration (123) with the zero section removed.

We conclude the section by placing the outcome of Theorem 36 in connection with the parabolic compactification construction presented in section 2, and comparing the resulted structure to the classical moduli spaces of eight dimensional heterotic string theory.

Recall that, up to isomorphism, there exist only two even, positive-definite and unimodular lattices Λ of rank 16. To each choice of Λ one can associate a corresponding Lie group G. For $\Lambda_1 = E_8 \times E_8$ one sets $G_1 = (E_8 \times E_8) \rtimes \mathbb{Z}_2$. If $\Lambda_2 = \Gamma_{16}$ then $G_2 = \mathrm{Spin}(32)/\mathbb{Z}_2$. The moduli space $\mathcal{M}_{K3}$ of $K3$ surfaces with section admits a partial compactification $\overline{\mathcal{M}_{K3}}$ obtained by adding two distinct divisors at infinity $\mathcal{D}_{\Lambda_i}$. Each point on $\mathcal{D}_{\Lambda_i}$ can be identified with an equivalence class of elliptic Type II stable $K3$ surfaces with section, in Λ_i-category, and with an isomorphism class of a pair (E, A) consisting of an elliptic curve E and a flat G_i-connection A. The correspondence gives a natural holomorphic isomorphism:

$$\mathcal{D}_{\Lambda_i} \simeq \mathcal{M}_{E, G_i} \tag{126}$$

where $\mathcal{M}_{E, G_i}$ is the 17-dimensional moduli space of pairs of elliptic curves and flat G_i-bundles[3].

As explained in 2.3, in each of the two cases, one has the parabolic cover

$$\mathcal{P}_{\Lambda_i} \xrightarrow{p} \mathcal{M}_{K3} \tag{127}$$

modeling the projection $\Gamma_F \backslash \Omega \to \Gamma \backslash \Omega$ where Γ_F is the stabilizer in Γ of a rank two isotropic sub-lattice of $\mathbb{L}_o$ determining Λ_i. Moreover, the space $\mathcal{P}_{\Lambda_i}$ fibers holomorphically over the corresponding divisor:

$$\mathcal{P}_{\Lambda_i} \to \mathcal{D}_{\Lambda_i} \tag{128}$$

with all fibers being copies of $\mathbb{C}^*$. Theorem 36 shows that, under identification (126), the above fibration is the character fibration of Λ_i with the zero-section removed. That allows one to holomrphically identify $\mathcal{P}_{\Lambda_i}$ with the total space of the character $\mathbb{C}^*$-fibration.

Turning our attention to the heterotic side of the duality, it was shown in [6] (see Theorems 1 and 2 in section 1) that the moduli space $\mathcal{M}_{\mathrm{het}}^{G_i}$ of classical vacua for heterotic string theory compactified over the torus is holomorphically isomorphic to the same total space of the character $\mathbb{C}^*$-fibration corresponding to the lattice Λ_i. Corroborating these facts to Theorem 36, one obtains a holomorphic isomorphism of $\mathbb{C}^*$-fibrations:

$$
\begin{array}{ccc}
\mathcal{P}_{\Lambda_i} & \xrightarrow{\simeq} & \mathcal{M}_{\mathrm{het}}^{G_i} \\
\downarrow & & \downarrow \\
\mathcal{D}_{\Lambda_i} & \xrightarrow{\simeq} & \mathcal{M}_{E, G_i}
\end{array}
\tag{129}
$$

which can be seen as an identification between the parabolic cover space $\mathcal{P}_{\Lambda_i}$ and the classical moduli space of eight-dimensional heterotic string theory with group G_i.

But, as described in section 2.3, there exists an open subset $\mathcal{V}_i \subset \mathcal{P}_i$, which can be seen as a punctured tubular neighborhood of the zero-section in the comlex line fibration associated to (128), such that the restriction:

$$p|_{\mathcal{V}_i} : \mathcal{V}_i \to p(\mathcal{V}_i) \tag{130}$$

is an isomorphism and $p(\mathcal{V}_i)$ is an open punctured tubular neighborhood of the divisor $\mathcal{D}_{\Lambda_i}$ in $\mathcal{M}_{K3}$. This fact allows us to conclude that:

Theorem 37. *(F-Theory/Heterotic String Duality in Eight Dimensions)*
There exists a holomorphic isomorphism between an open neighborhood of $\mathcal{M}_{K3}$ near the divisor $\mathcal{D}_{\Lambda_i}$ and an open neighborhood of $\mathcal{M}_{\mathrm{het}}^{G_i}$ near the zero-section of the complex line fibration associated to the right fibration in (129).

The open neighborhood of $\mathcal{M}_{\mathrm{het}}^{G_i}$ in the above statement corresponds to large volumes of the elliptic curve. Hence, the two regions identified by Theorem 37 are exactly the sectors that physics predicts should closely resemble each other.

---

[3]Again, if $G = \mathrm{Spin}(32)/\mathbb{Z}_2$ only connection liftable to $\mathrm{Spin}(32)$-connection are considered

6 Appendix: Narain Construction

The parameter fields for 8-dimensional heterotic string theory are, after Narain [16] [24], triplets (A, g, B) consisting of a flat G-connection, a flat metric and a constant anti-symmetric 2-tensor B, all defined over a two-torus E. The Lie group G involved is either $E_8 \times E_8 \rtimes \mathbb{Z}_2$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$.

One usually describes a flat torus as a quotient:

$$E = \mathbb{R}^2/U$$

of the Euclidean space $\mathbb{R}^2$ through a rank two lattice $U = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. In this way, E inherits a flat metric, which in turn generates a volume $v \in \mathbb{R}_+^*$ and a complex structure parameterized by $\tau \in \mathbb{H}$. These parameters are obtained as:

$$v = \sqrt{g_{11}g_{22} - g_{12}^2}$$

$$\tau = \frac{g_{12} + v \cdot i}{g_{11}}.$$

where $g_{ij} = e_i \cdot e_j$.

A flat G-connection on E is, formally, a morphism

$$A : U \rightarrow \Lambda_{\mathbb{R}}.$$

The lattice Λ is the coroot lattice of G if $G = E_8 \times E_8 \rtimes \mathbb{Z}_2$ and the lattice of a maximal torus of G if $G = \mathrm{Spin}(32)/\mathbb{Z}_2$. As is the standard procedure, one parameterizes holomorphically these flat connections by taking:

$$z = z_2 - \tau z_1 \in \Lambda_{\mathbb{C}}, \quad \text{where} \quad z_i := A(e_i) \in \Lambda_{\mathbb{R}}.$$

The last ingredient, the B-field is seen as a two-form $B = b\, (e_1^*) \wedge (e_2^*)$ with $b \in \mathbb{R}$. The B-field holonomy along E is given by

$$\exp\left(i \int_E B\right) = \exp(ibv).$$

One considers then the space:

$$\mathbb{R}^{2,18} = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \Lambda_{\mathbb{R}}$$

endowed with the inner product:

$$(x, y, z).(x', y', z') = x.x' - y.y' - z.z'.$$

The lattice of momenta [16], denoted $\mathcal{L}_{(A,g,B)}$, associated to a heterotic triplet (A, g, B) is obtained as the image of the map:

$$\varphi_{(A,g,B)} : U \oplus U^* \oplus \Lambda \rightarrow \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \Lambda_{\mathbb{R}} \tag{131}$$

$$\varphi_{(A,g,B)}(w, p, l) =$$

$$= \left(\frac{1}{2}p - bTw - \frac{1}{2}A^t l - \frac{1}{4}A^t Aw - w,\ \frac{1}{2}p - bTw - \frac{1}{2}A^t l - \frac{1}{4}A^t Aw + w,\ Aw + l\right).$$

Here $T \colon \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the anti-self adjoint morphism $T(x_1, x_2) = (x_2, -x_1)$. One checks that, in the above formulation, the image

$$\mathcal{L}_{(A,g,B)} := \mathrm{Im}\left(\varphi_{(A,g,B)}\right)$$

with the induced inner product forms a lattice of rank 20 embedded in the ambient space $\mathbb{R}^{2,18}$. The lattice $\mathcal{L}_{(A,g,b)}$ is isomorphic to $H \oplus H \oplus (-\Lambda)$. A basis underlying this decomposition is given by:

$$F_i := \varphi_{(A,g,B)}(-e_i, 0, 0) = \left(bTe_i + \frac{1}{4}A^t Ae_i + e_i,\ bTe_i + \frac{1}{4}A^t Ae_i - e_i,\ -Ae_i\right) \tag{132}$$

$$F_i^* \; : \; = \; \varphi_{(A,g,B)}(0, e_i^*, 0) \; = \; \left(\frac{1}{2} e_i^*, \; \frac{1}{2} e_i^*, \; 0 \right) \tag{133}$$

$$L \; : \; = \; \varphi_{(A,g,B)}(0, 0, l) \; = \; \left(-\frac{1}{2} A^t l, \; -\frac{1}{2} A^t l, \; l \right). \tag{134}$$

It satisfies:

$$F_i.F_j = 0, \quad F_i^*.F_j^* = 0, \quad F_i.F_j^* = \delta_{ij}$$

$$F_i.L = F_j^*.L = 0, \quad L.L' = -\, l.l'.$$

One is interested in the behavior of the oriented positive 2-plane $\mathbb{R}^2 \subset \mathbb{R}^{2,18}$ with respect to the lattice $\mathcal{L}_{(A,g,B)}$. Let us imagine that the lattice $\mathcal{L}$ remains fixed and the oriented $\mathbb{R}^2$ is varying inside $\mathcal{L} \otimes \mathbb{R}$ parameterized by the heterotic variables. This provides an assignment:

$$\{\text{heterotic parameters } (A, g, B)\} \;\rightarrow\; O(2,18)/SO(2) \times O(18). \tag{135}$$

Moreover, the target space in (135) has a natural holomorphic structure. One can equivalently regard positive, oriented, two-planes in $\mathcal{L} \otimes \mathbb{R}$ as complex lines $\omega \subset \mathcal{L} \otimes \mathbb{C}$ satisfying $\omega.\omega = 0$ and $\omega.\overline{\omega} > 0$. There is then a bijective correspondence:

$$O(2,18)/SO(2) \times O(18) \;\simeq\; \{\, \omega \in \mathbb{P}\mathcal{L}_{\mathbb{C}} \mid \omega.\omega = 0, \; \omega.\overline{\omega} > 0 \,\}$$

and the map (135) can be interpreted as sending triplets of heterotic parameters to the 18-dimensional complex period domain Ω of section 2.1.

One can describe explicitly this map. Let (A, g, B) be a heterotic triplet determining:

$$(\tau, z, v, b) \;\in\; \mathbb{H} \times \Lambda_{\mathbb{C}} \times \mathbb{R}_+^* \times \mathbb{R}.$$

Then, the complex line ω is generated by:

$$\omega \;=\; \sum \alpha_i F_i \;+\; \sum \beta_j F_j^* \;+\; \gamma \tag{136}$$

with

$$\alpha_1 \;=\; -\tau, \quad \alpha_2 \;=\; 1, \quad \gamma = z$$

$$\beta_1 \;=\; -2(bv + iv) \;+\; \frac{(z,z) - (z,\bar{z})}{2(\bar{\tau} - \tau)}$$

$$\beta_2 \;=\; -2\tau(bv + iv) \;+\; \frac{\bar{\tau}(z,z) - \tau(z,\bar{z})}{2(\tau - \bar{\tau})}.$$

Take the decomposition $\mathcal{L} = H \oplus H \oplus (-\Lambda)$ with a basis for $H \oplus H$ given by $\{F_1, F_2, F_1^*, F_2^*\}$. The inner product on $\mathcal{L}_{\mathbb{C}}$ appears as:

$$(a, b, c).(a', b', c') \;=\; (a, b') + (b, a') - (c, c').$$

Let $N \in \mathrm{End}(\mathcal{L})$ be the nilpotent anti-self adjoint endomorphism

$$N(a, b, c) \;=\; (0, Ta, 0)$$

and let $\exp(tN) = I + tN$ be its exponential. The Narain correspondence between heterotic parameters and period complex lines in Ω appears then as:

$$\sigma_n \; : \; \mathbb{H} \times \Lambda_{\mathbb{C}} \times \mathbb{H} \;\rightarrow\; \Omega \tag{137}$$

$$\sigma_n(\tau, z, u) \;=\; \exp(-2u \cdot N) \left[(-\tau, 1), \; \frac{1}{2} \left(\frac{(z,z) - (z,\bar{z})}{\bar{\tau} - \tau}, \; \frac{\bar{\tau}(z,z) - \tau(z,\bar{z})}{\bar{\tau} - \tau} \right), \; z \right].$$

The complex variable u represents $bv + iv \in \mathbb{C}$. It is clear that (137) is not holomorphic. However one can move the non-holomorphic part of (137) to the exponential. Indeed:

$$\frac{\bar{\tau}(z,z) - \tau(z,\bar{z})}{\bar{\tau} - \tau} = \tau \, \frac{(z,z) - (z,\bar{z})}{\bar{\tau} - \tau} + (z.z)$$

and therefore, one can rewrite:

$$\sigma_n(\tau, z, u) = \exp\left(\left(-2u + \frac{(z,z) - (z,\bar{z})}{2(\bar{\tau} - \tau)} \right) \cdot N \right) \left[\, (-\tau, 1), \, \frac{1}{2}\left(0, \, (z,z)\right), \, z \, \right] .$$

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