# Modular Invariants for Lattice Polarized K3 Surfaces 

Adrian Clingher * Charles F. Doran ${ }^{\dagger}$

## 1 Introduction

Let X be an algebraic K 3 surface over the field of complex numbers. The $\mathbb{Z}$-module obtained as the image of the first Chern class map:

$$
c_{1}: \mathrm{H}^{1}\left(\mathrm{X}, \mathcal{O}_{\mathrm{X}}^{*}\right) \rightarrow \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})
$$

when endowed with the bilinear pairing induced by the intersection form on $\mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})$, forms an even lattice. By the Lefschetz Theorem on $(1,1)$ classes, this is precisely the Néron-Severi lattice NS $(\mathrm{X})$ of the surface X , namely the group of isomorphism classes of divisors modulo homological equivalence. Furthermore, according to the Hodge Index Theorem, $\mathrm{NS}(\mathrm{X})$ is an indefinite lattice of rank $1 \leq \mathrm{p}_{\mathrm{X}} \leq 20$ and signature of type ( $1, \mathrm{p}_{\mathrm{X}}-1$ ).

In [11], Dolgachev formulated the notion of a lattice polarization of a K 3 surface. If M is an even lattice of signature $(1, r)$ with $r \geq 0$, then an M -polarization on X is, by definition, a primitive lattice embedding:

$$
\begin{equation*}
i: \mathrm{M} \hookrightarrow \operatorname{NS}(X) \tag{1}
\end{equation*}
$$

such that the image $i(\mathrm{M})$ contains a pseudo-ample class. A coarse moduli space $\mathcal{M}_{\mathrm{M}}$ can be defined for equivalence classes of pairs ( $\mathrm{X}, i$ ) of M-polarized K 3 surfaces and an appropriate version of the Global Torelli Theorem holds.

The focus of this paper is on K 3 surfaces which admit a polarization by the unique even unimodular lattice of signature $(1,17)$. This particular lattice can be realized effectively as the orthogonal direct sum

$$
\mathrm{M}=\mathrm{H} \oplus \mathrm{E}_{8} \oplus \mathrm{E}_{8}
$$

where H is the standard rank-two hyperbolic lattice and $\mathrm{E}_{8}$ is the unique even, negative-definite and unimodular lattice of rank eight. Note that not all algebraic K3 surfaces admit such an M-polarization. In fact, the presence of such a structure imposes severe constraints on the geometry of X. In particular, the Picard rank $p_{\mathrm{X}}$ has to be 18,19 or 20 .

A standard observation on the Hodge theory of this special class of $K 3$ surfaces is that the polarized Hodge structure of an M-polarized K 3 surface ( $\mathrm{X}, i$ ) is identical with the polarized Hodge structure of an abelian surface $A=E_{1} \times E_{2}$ realized as a cartesian product of two elliptic curves. Since both types of surfaces involved admit appropriate versions of the Torelli theorem, Hodge theory implies a well-defined correspondence:

$$
\begin{equation*}
(\mathrm{X}, i) \leftrightarrow \mathrm{E}_{1} \times \mathrm{E}_{2} \tag{2}
\end{equation*}
$$

giving rise to a canonical analytic isomorphism between the corresponding moduli spaces on the two sides. By employing a modern point of view from the frontier of algebraic geometry with string theory, one can regard (2) as a Hodge-theoretic duality map, a correspondence that relates two seemingly different types

[^0]of surfaces sharing similar Hodge-theoretic information ${ }^{1}$. Our point in this work is that the resemblance of the two Hodge structures involved in the duality correspondence (2) is not fortuitous, but rather is merely a consequence of a quite interesting geometric relationship.
Theorem 1.1. Let ( $\mathrm{X}, i$ ) be an M-polarized K3 surface.
(a) The surface X possesses a canonical involution $\beta$ defining a Shioda-Inose structure.
(b) The minimal resolution of $\mathrm{X} / \beta$ is a new K3 surface Y endowed with a canonical Kummer structure. This structure realizes Y as the Kummer surface $\mathrm{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ associated to an abelian surface A canonically represented as a cartesian product of two elliptic curves. The elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ are unique, up to permutation.
(c) The construction induces a canonical Hodge isomorphism between the M-polarized Hodge structure of X and the natural H-polarized Hodge structure of the abelian surface $\mathrm{A}=\mathrm{E}_{1} \times \mathrm{E}_{2}$.

Section 3 of the paper is devoted entirely to proving the above theorem.
In the second part of the paper we describe an application of the geometric transform outlined above. One important feature of the special class of M-polarized K3 surfaces is that such polarized pairs ( $\mathrm{X}, i$ ) turn out to be completely classified by two modular invariants $\pi, \sigma \in \mathbb{C}$, much in the same as way elliptic curves over the field of complex numbers are classified by the J-invariant. However, the two modular invariants $\pi$ and $\sigma$ are not geometric in origin. They are defined Hodge-theoretically, and the result leading to the classification is a consequence of the appropriate version of the Global Torelli Theorem for lattice polarized K3 surfaces. However, in the context of the duality map (2), the two invariants can be seen as the standard symmetric functions on the J-invariants of the dual elliptic curves:

$$
\begin{equation*}
\sigma=\mathrm{J}\left(\mathrm{E}_{1}\right)+\mathrm{J}\left(\mathrm{E}_{2}\right), \quad \pi=\mathrm{J}\left(\mathrm{E}_{1}\right) \cdot \mathrm{J}\left(\mathrm{E}_{2}\right) . \tag{3}
\end{equation*}
$$

This interpretation suggests that the modular invariants of an M-polarized K3 surface can be computed by determining the two elliptic curves that appear on the right-side of (2).

Explicit M-polarized K3 surfaces can be constructed by various geometrical procedures. One such method, introduced in 1977 by Inose [17], constructs a two-parameter family $\mathrm{X}(a, b)$ of M-polarized K3 surfaces ${ }^{2}$ by taking minimal resolutions of the projective quartics in $\mathbb{P}^{3}$ associated with the special equations:

$$
\begin{equation*}
y^{2} z w-4 x^{3} z+3 a x z w^{2}-\frac{1}{2}\left(z^{2} w^{2}+w^{4}\right)+b z w^{3}=0, \quad a, b \in \mathbb{C} . \tag{4}
\end{equation*}
$$

In fact, as we will see shortly, this family covers all possibilities. Every M-polarized K3 surface can be realized as $\mathrm{X}(a, b)$ for some $a, b \in \mathbb{C}$. One can regard the Inose quartic (4) as a normal form of an Mpolarized K3 surface.

In the second part of the paper, we use the geometric transform of Theorem 1.1 to explicitly describe the J-invariants of the two elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ associated to the Inose surface $\mathrm{X}(a, b)$.

Theorem 1.2. The J-invariants $\mathrm{J}\left(\mathrm{E}_{1}\right)$ and $\mathrm{J}\left(\mathrm{E}_{2}\right)$ of the two elliptic curves associated to $\mathrm{X}(a, b)$ by the transform of Theorem 1.1 are the two solutions of the quadratic equation:

$$
\mathrm{x}^{2}-\left(a^{3}-b^{2}+1\right) \mathrm{x}+a^{3}=0
$$

As pointed out earlier, as a consequence of the above theorem, one obtains explicit formulas for the two modular invariants of the Inose surface $\mathrm{X}(a, b)$.

[^1]Corollary 1.3. The modular invariants of the Inose surface $\mathrm{X}(a, b)$ are given by:

$$
\begin{equation*}
\pi=a^{3}, \quad \sigma=a^{3}-b^{2}+1 \tag{5}
\end{equation*}
$$

The power of the geometric transformation underlying the duality map (2) is fully revealed by the proof of Theorem 1.2. In the absence of such a geometric argument, in order to prove that statement one would be forced to undertake long and very complex computations of the periods of the quartic $(4)^{3}$.

The method of proof of Theorem 1.2 also provides an explicit description of the geometric cycle in $\mathrm{X} \times \mathrm{A}$ underlying the Hodge isometry between the two surfaces $A$ and $X$, as predicted by the Hodge conjecture. At the suggestion of Johan de Jong, we have included a detailed discussion of this in Section 4.9.

It seems that the geometric transformation described by Theorem 1.1 is a particular case of a more general phenomenon. Evidence for this is provided by an analysis of the slightly more general case of K3 surfaces polarized by the rank 17 lattice $\mathrm{H} \oplus \mathrm{E}_{8} \oplus \mathrm{E}_{7}$. Surfaces in this class still admit a canonical ShiodaInose structure. This leads to a correspondence between these special K3 surfaces and jacobians of smooth genus-two curves. These results will be described in a forthcoming paper.

A very interesting alternative arithmetic approach to the above questions, by Noam Elkies and Abhinav Kumar, has been communicated to the authors [22].

## 2 Hodge Structures for M-polarized K3 Surfaces

Let (X, $i$ ) be an M-polarized K3 surface. Denote by $\omega \in \mathrm{H}^{2}(\mathrm{X}, \mathbb{C})$ the class of a non-zero holomorphic twoform on X . This class is unique, up to multiplication by a non-zero scalar. The Hodge structure of X is then essentially given by the decomposition:

$$
\mathrm{H}^{2}(\mathrm{X}, \mathbb{C})=\mathrm{H}^{2,0}(\mathrm{X}) \oplus \mathrm{H}^{1,1}(\mathrm{X}) \oplus \mathrm{H}^{0,2}(\mathrm{X})
$$

where $\mathrm{H}^{2,0}(\mathrm{X})=\mathbb{C} \cdot \omega, \mathrm{H}^{0,2}(\mathrm{X})=\mathbb{C} \cdot \bar{\omega}$ and $\mathrm{H}^{1,1}(\mathrm{X})=\{\omega, \bar{\omega}\}^{\perp}$. Since the lattice $i(\mathrm{M})$ is generated by classes associated to algebraic cycles, it follows that one has an embedding:

$$
i(\mathrm{M}) \subset \mathrm{H}^{1,1}(\mathrm{X}) \cap \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})
$$

By standard lattice theory (see, for example, the exposition in [27]), the orthogonal complement N of $i(\mathrm{M})$ in $\mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})$ is an even, unimodular sublattice of signature $(2,2)$. Hence the lattice N is isometric to the orthogonal direct sum $\mathrm{H} \oplus \mathrm{H}$ of two rank-two hyperbolic lattices. One can therefore choose a basis

$$
\mathcal{B}=\left\{x_{1}, x_{2}, y_{1}, y_{2}\right\}
$$

of N with intersection matrix:

$$
\left(\begin{array}{llll}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right)
$$

It follows that $\omega$ belongs to $\mathrm{N} \otimes \mathbb{C}$. Moreover, since the elements of the basis $\mathcal{B}$ are isotropic, the class $\omega$ has non-zero intersection with any one of them. The class $\omega$ is therefore uniquely defined as soon as one imposes the normalization condition $\left(\omega, y_{2}\right)=1$.

Let us also note that the basis $\mathcal{B}$ can be chosen such that the isomorphism of real vector spaces:

$$
\begin{equation*}
(\omega, \cdot):\left\langle x_{1}, x_{2}\right\rangle \otimes \mathbb{R} \rightarrow \mathbb{C} \tag{6}
\end{equation*}
$$

is orientation reversing. Then, as discussed in [9], the Hodge-Riemann bilinear relations imply that the normalized period class can be written:

$$
\begin{equation*}
\omega=\tau x_{1}+x_{2}+u y_{1}+(-\tau u) y_{2} \tag{7}
\end{equation*}
$$

[^2]where $\tau, u$ are uniquely defined (but depending on the choice of basis $\mathcal{B}$ ) elements of the complex upper half-plane $\mathbb{H}$.

Definition 2.1. The modular invariants of the M-polarized K3 surface ( $\mathrm{X}, i$ ) are, by definition:

$$
\begin{gather*}
\sigma(\mathrm{X}, i):=\mathrm{J}(\tau)+\mathrm{J}(u)  \tag{8}\\
\pi(\mathrm{X}, i):=\mathrm{J}(\tau) \cdot \mathrm{J}(u)
\end{gather*}
$$

where J is the classical elliptic modular function ${ }^{4}$.
Let us make two observations justifying the importance of the numbers defined above. Firstly, the numbers $\sigma(\mathrm{X}, i)$ and $\pi(\mathrm{X}, i)$ do not depend on the choice of basis $\mathcal{B}$. That is because any new choice of basis $\mathcal{B}^{\prime}$ can be related to $\mathcal{B}$ by an integral isometry $\varphi$ of the lattice N that preserves the spinor norm. That is $\mathcal{B}^{\prime}=\varphi(\mathcal{B})$. But then, as shown for example in Section 2 of [15], the group $O^{+}(\mathrm{N})$ of such integral isometries is naturally isomorphic to the semi-direct product:

$$
(\operatorname{PSL}(2, \mathbb{Z}) \times \operatorname{PSL}(2, \mathbb{Z})) \rtimes \mathbb{Z} / 2 \mathbb{Z}
$$

with the generator of $\mathbb{Z} / 2 \mathbb{Z}$ acting on $\operatorname{PSL}(2, \mathbb{Z}) \times \operatorname{PSL}(2, \mathbb{Z})$ by exchanging the two sides. This clearly proves that such a modification does not affect $\sigma(\mathrm{X}, i)$ and $\pi(\mathrm{X}, i)$.

Secondly, thanks to a lattice polarized version of the Global Torelli Theorem for K3 surfaces [11], the numbers $\sigma(\mathrm{X}, i)$ and $\pi(\mathrm{X}, i)$ fully classify the polarized pairs (X,i) up to isomorphism. Simply put, this result says that there exists a two-dimensional complex analytic space $\mathcal{M}_{M}$ realizing a coarse moduli space for M-polarized K3 surfaces and that the period map to the classifying space of polarized Hodge structures is an isomorphism of analytic spaces.

$$
\begin{equation*}
\mathcal{M}_{\mathrm{M}} \longrightarrow(\operatorname{PSL}(2, \mathbb{Z}) \times \operatorname{PSL}(2, \mathbb{Z})) \rtimes \mathbb{Z} / 2 \mathbb{Z} \backslash(\mathbb{H} \times \mathbb{H}) \tag{9}
\end{equation*}
$$

From this point of view, the modular invariants $\sigma$ and $\pi$ can be regarded as natural coordinates on the moduli space $\mathcal{M}_{\mathrm{M}}$.

Note then that the right-hand side space in (9) also classifies unordered pairs ( $\mathrm{E}_{1}, \mathrm{E}_{2}$ ) of curves of genus one. The two geometric structures, M-polarized K3 surfaces and unordered pairs of elliptic curves, have the same classifying moduli space. Moreover, there is an obvious Hodge-theoretic bijective correspondence relating these structures:

$$
\begin{equation*}
(\mathrm{X}, i) \longleftrightarrow\left(\mathrm{E}_{1}, \mathrm{E}_{2}\right) \tag{10}
\end{equation*}
$$

such that $\sigma(\mathrm{X}, i)=\mathrm{J}\left(\mathrm{E}_{1}\right)+\mathrm{J}\left(\mathrm{E}_{2}\right)$ and $\pi(\mathrm{X}, i)=\mathrm{J}\left(\mathrm{E}_{1}\right) \cdot \mathrm{J}\left(\mathrm{E}_{2}\right)$.

## 3 A Geometric Transformation Underlying the Duality Map

As mentioned in the introduction, our goal is to place the Hodge theoretic correspondence (10) into a geometric setting. In what follows, we provide the details needed for both the statement of Theorem 1.3 as well as its proof. Setting-up the geometric transformation requires a few technical ingredients concerning Shioda-Inose structures, Kummer surfaces and elliptic fibrations on a K3 surface. We shall therefore begin our exposition by presenting some basic facts.

### 3.1 Shioda-Inose Structures

The notion of a Shioda-Inose structure originates in the works [18] of Shioda and Inose and [27] of Nikulin. Their ideas were later refined and generalized by Morrison [24]. The above three papers are the main references for the assertions we review here.

[^3]Definition 3.1. Let X be a K3 surface. An involution $\varphi \in \operatorname{Aut}(\mathrm{X})$ is called a Nikulin involution if $\varphi^{*} \omega=\omega$ for any holomorphic two-form $\omega$.

If a Nikulin involution $\varphi$ exists on X , then $\varphi$ has exactly eight fixed points. In such a case, the quotient space

$$
\mathrm{X} /\left\{\mathrm{id}_{\mathrm{X}}, \varphi\right\}
$$

is a surface with eight rational double point singularities of type $A_{1}$. The minimal resolution of this singular space is a new K 3 surface which we denote by Y . The two K 3 surfaces X and Y are related by a (generically) two-to-one rational map $\pi: \mathrm{X} \rightarrow \mathrm{Y}$.

Denote by $\mathrm{H}_{\mathrm{Y}}^{2}$ the orthogonal complement in $\mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z})$ of the eight exceptional curves. One has then a natural push-forward map (see § 3 of [24] or § 3 of [18]):

$$
\pi_{*}: \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z}) \rightarrow \mathrm{H}_{\mathrm{Y}}^{2}
$$

which restricts to a morphism of $\mathbb{Z}$-modules:

$$
\begin{equation*}
\pi_{*}: \mathrm{T}_{\mathrm{X}} \rightarrow \mathrm{~T}_{\mathrm{Y}} \tag{11}
\end{equation*}
$$

between the transcendental lattices of the two K3 surfaces.
Remark 3.2. The complexification of the morphism (11) is a morphism of Hodge structures, but, in general, (11) does not preserve the lattice pairings. In fact, one can check that:

$$
\left\langle\pi_{*}\left(t_{1}\right), \pi_{*}\left(t_{2}\right)\right\rangle_{\mathrm{Y}}=\left\langle t_{1}, t_{2}\right\rangle_{\mathrm{X}}+\left\langle t_{1}, \varphi^{*}\left(t_{2}\right)\right\rangle_{\mathrm{x}}
$$

Definition 3.3. A Nikulin involution $\varphi$ defines a Shioda-Inose structure on X if Y is a Kummer surface and the morphism (11) is a Hodge isometry $\mathrm{T}_{\mathrm{X}}(2) \simeq \mathrm{T}_{\mathrm{Y}}$.

The notation $\mathrm{T}_{\mathrm{X}}(2)$ means that the bilinear pairing on the transcendental lattice $\mathrm{T}_{\mathrm{X}}$ is multiplied by 2. We refer the reader to section 3.4 for an explanation of the significance of the last condition in the above definition, as well as for a short overview of the basics of Kummer surfaces.

Not every K3 surface admits a Nikulin involution, much less a Shioda-Inose structure. A very effective lattice-theoretic criterion which provides a necessary and sufficient condition for the existence of a ShiodaInose structure on a K3 surface X has been given by Morrison.

Theorem 3.4. (Morrison [24], Theorem 5.7) Let X be an algebraic K3 surface. There exists a Shioda-Inose structure on X if and only if the lattice $\mathrm{E}_{8} \oplus \mathrm{E}_{8}$ can be primitively embedded into the Néron-Severi lattice $\mathrm{NS}(\mathrm{X})$.
The proof of the above statement is based on a result of Nikulin ([27], Theorem 4.3). A primitive embedding $\mathrm{E}_{8} \oplus \mathrm{E}_{8} \hookrightarrow \mathrm{NS}(\mathrm{X})$ allows one to define a special lattice isometry of $\mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})$ which interchanges the two copies of $\mathrm{E}_{8}$ given by the embedding and acts trivially on their orthogonal complement. In this context, Nikulin's theorem asserts that, possibly after conjugation by a reflection in an algebraic class of square -2 , this lattice isometry can be associated to an involution of the K3 surface X. Morrison shows then that this involution defines in fact a Shioda-Inose structure on X.

Closer to the purpose of this paper, note that Theorem 3.4 implies that an M-polarized K3 surface ( $\mathrm{X}, i$ ) admits a Shioda-Inose structure. In fact, there exists a well-defined Shioda-Inose structure $\beta$ on X canonically associated with the M-polarization.

This canonical Shioda-Inose structure associated to an M-polarized K3 surface plays a central role in our construction. However, in this paper we shall take a different point of view towards defining the Nikulin involution $\beta$ underlying the Shioda-Inose structure. Instead of using Theorem 3.4, we shall introduce this involution in a more explicit and geometric manner. The canonical involution $\beta$ appears naturally in the context of a special jacobian fibration on X .

### 3.2 Jacobian Fibrations on K3 Surfaces

During the course of this section we shall assume that X is an algebraic K 3 surface.
Definition 3.5. A jacobian fibration (or elliptic fibration with section) on X is a pair $(\varphi, S)$ consisting of a proper map of analytic spaces $\varphi: \mathrm{X} \rightarrow \mathbb{P}^{1}$ whose generic fiber is a smooth curve of genus one, and a section $S$ in the elliptic fibration $\varphi$.

If $S^{\prime}$ is another section of the jacobian fibration $(\varphi, S)$, then there exists ${ }^{5}$ an automorphism of X preserving $\varphi$ and mapping $S$ to $S^{\prime}$. One can therefore realize an identification between the set of sections of $\varphi$ and the group of automorphisms of X preserving $\varphi$. This is the Mordell-Weil group MW $(\varphi, S)$ of the jacobian fibration.

Note also that a jacobian fibration $(\varphi, S)$ on X induces a sublattice:

$$
\mathcal{H}_{(\varphi, S)} \subset \mathrm{NS}(\mathrm{X})
$$

constructed as the span of the two cohomology classes associated with the elliptic fiber and the section, respectively. The lattice $\mathcal{H}_{(\varphi, S)}$ is isomorphic to the standard rank-two hyperbolic lattice H .

The sublattice $\mathcal{H}_{(\varphi, S)}$ determines uniquely the jacobian fibration $(\varphi, S)$. In other words, there cannot be two distinct jacobian fibrations on X determining the same hyperbolic sublattice in NS(X). However, it is not true that any lattice embedding of H into $\mathrm{NS}(\mathrm{X})$ corresponds to a jacobian fibration. Nevertheless, the following assertions hold:
Lemma 3.6. A lattice embedding $\mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{X})$ can be associated with a jacobian fibration $(\varphi, S)$ if and only if its image in $\mathrm{NS}(\mathrm{X})$ contains a pseudo-ample class.

Lemma 3.7. Let $\Gamma_{\mathrm{X}}$ be the group of isometries of $\mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})$ preserving the Hodge decomposition. For any lattice embedding

$$
e: \mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{X}),
$$

there exists $\alpha \in \Gamma_{\mathrm{X}}$ such that $\operatorname{Im}(\alpha \circ e)$ contains a pseudo-ample class.
Lemma 3.8. One has the following bijective correspondence:

$$
\left\{\begin{array}{c}
\text { isomorphism classes of }  \tag{12}\\
\text { jacobian fibrations on } X
\end{array}\right\} \longleftrightarrow\left\{\begin{array}{c}
\text { cattice embeddings } \\
\mathrm{H} \hookrightarrow \mathrm{NS}(X)
\end{array}\right\} / \Gamma_{X} \text {. }
$$

These are standard well-known results. For proofs, we refer the reader to [30], [21] and [9].
Next, let us consider

$$
\mathcal{W}_{(\varphi, S)} \subset \mathrm{NS}(\mathrm{X})
$$

to be the orthogonal complement of $\mathcal{H}_{(\varphi, S)}$ in the Néron-Severi lattice of X. It follows that $\mathcal{W}_{(\varphi, S)}$ itself is a negative-definite lattice of rank $p_{X}-2$. Moreover, the Néron-Severi lattice decomposes as an orthogonal direct sum:

$$
\mathrm{NS}(\mathrm{X})=\mathcal{H}_{(\varphi, S)} \oplus \mathcal{W}_{(\varphi, S)}
$$

Let $\Sigma \subset \mathbb{P}^{1}$ be the set of points on the base of the elliptic fibration $\varphi$ which correspond to singular fibers. For each $v \in \Sigma$, denote by $\mathrm{T}_{v}$ the sublattice of $\mathcal{W}_{(\varphi, S)}$ spanned by the classes of the irreducible components of the singular fiber over $v$ which are disjoint from $S$. One has then the following result.

## Lemma 3.9.

(a) For each $v \in \Sigma, \mathrm{~T}_{v}$ is a negative-definite lattice of ADE type.

[^4](b) Let $\mathcal{W}_{(\varphi, S)}^{\mathrm{rooot}}$ be the lattice spanned by the $\operatorname{roots}^{6}$ of $\mathcal{W}_{(\varphi, S)}$. Then:
\[

$$
\begin{equation*}
\mathcal{W}_{(\varphi, S)}^{\mathrm{root}}=\bigoplus_{v \in \Sigma} \mathrm{~T}_{v} . \tag{13}
\end{equation*}
$$

\]

The decomposition (13) is unique, up to a permutation of the factors.
(c) There exists a canonical group isomorphism:

$$
\begin{equation*}
\mathcal{W}_{(\varphi, S)} / \mathcal{W}_{(\varphi, S)}^{\text {root }} \xrightarrow{\simeq} \operatorname{MW}(\varphi) . \tag{14}
\end{equation*}
$$

The first two statements of the above lemma are standard facts from Kodaira's classification of singular fibers of elliptic fibrations (see, for example, [20]). The last statement is due to Shioda [33].

Let us briefly indicate the construction of the correspondence in (14). Given $\gamma \in \mathcal{W}_{(\varphi, S)}$, denote by L the unique holomorphic line bundle over X such that $c_{1}(\mathrm{~L})=\gamma$. Let $x \in \mathrm{X}$ be a point belonging to a smooth fiber $\mathrm{E}_{\varphi(x)}$. Then, the restriction of L to $\mathrm{E}_{\varphi(x)}$ is a holomorphic line bundle of degree zero and, therefore, there exists a unique $y \in E_{\varphi(x)}$ such that:

$$
\left.\mathrm{L}\right|_{\mathrm{E}_{\varphi(x)}} \simeq \mathcal{O}_{\mathrm{E}_{\varphi(x)}}(x-y) .
$$

The assignment $x \mapsto y$ extends by continuity to an automorphism of the $K 3$ surface and hence to an element in MW $(\varphi)$.

### 3.3 A Canonical Involution

We shall apply now the general theory presented in the previous section in the context of an M-polarized K3 surface (X, i).

By standard lattice theory (see [27]), there exist exactly two distinct ways (up to an overall isometry) in which one can embed the standard rank-two hyperbolic lattice H isometrically into M . The two possibilities are distinguished by the isomorphism type of the orthogonal complement of the image of the embedding. The orthogonal complement has rank 16 and it is also unimodular, even, and negative-definite. As is well-known, up to isomorphism there exist only two such lattices. One is $\mathrm{E}_{8} \oplus \mathrm{E}_{8}$. The other is $\mathrm{D}_{16}^{+}$, the unimodular index-two overlattice of the negative definite lattice associated with the root system $\mathrm{D}_{16}$.

In the presence of an M-polarization on X , the two distinct isometric embeddings of the rank-two hyperbolic lattice H into M determine two distinct classes of embeddings of H into the Néron-Severi lattice $\mathrm{NS}(X)$. According to Lemma 3.8, one obtains therefore two special jacobian fibrations ( $\left.\Theta_{1}, S_{1}\right)$ and $\left(\Theta_{2}, S_{2}\right)$ on X .

$$
\Theta_{1}, \Theta_{2}: X \rightarrow \mathbb{P}^{1}
$$

We shall use the term standard fibration for $\Theta_{1}$ (associated to the rank-sixteen lattice $\mathrm{E}_{8} \oplus \mathrm{E}_{8}$ ) and alternate fibration for $\Theta_{2}$ (associated to the rank-sixteen lattice $\mathrm{D}_{16}^{+}$).

Proposition 3.10. Let ( $\mathrm{X}, i$ ) be an M-polarized K3 surface.
(a) The standard fibration $\left(\Theta_{1}, S_{1}\right)$ has two singular fibers of Kodaira type $\mathrm{II}^{*}$. The section $S_{1}$ is the unique section of $\Theta_{1}$ whose cohomology class belongs to $i(\mathrm{M})$.
(b) The alternate fibration $\left(\Theta_{2}, S_{2}\right)$ has a singular fiber of type $\mathrm{I}_{12}^{*}$. There are precisely two sections $S_{2}$ and $S_{2}^{\prime}$ of $\Theta_{2}$ with cohomology classes represented in $i(\mathrm{M}) . S_{2}$ and $S_{2}^{\prime}$ are disjoint. Moreover, the Mordell-Weil group $\operatorname{MW}\left(\Theta_{2}\right)$ contains a canonical involution $\beta \in \operatorname{Aut}(X)$ which exchanges $S_{2}$ and $S_{2}^{\prime}$.

[^5]Proof. The above assertions are consequences of the general principles reviewed in Section 3.2. In the case of the standard fibration $\left(\Theta_{1}, S_{1}\right)$, one has an orthogonal decomposition:

$$
\mathcal{W}_{\left(\Theta_{1}, S_{1}\right)}^{\text {root }}=\mathrm{E}_{8} \oplus \mathrm{E}_{8} \oplus \mathcal{U}^{\text {root }}
$$

where $\mathcal{U}$ is the orthogonal complement of $i(\mathrm{M})$ in $\mathrm{NS}(\mathrm{X})$ and $\mathcal{U}^{\text {root }}$ is the root lattice of $\mathcal{U}$. The above decomposition, combined with assertion (b) of Lemma 3.9, proves the existence of two singular fibers of Kodaira type II* in the elliptic fibration $\Theta_{1}$. It also follows immediately that $S_{1}$ is the unique section of $\Theta_{1}$ with associated cohomology class in $i(\mathrm{M})$. One can represent the rational curves obtained as irreducible components of the two $\mathrm{II}^{*}$ fibers of $\Theta_{1}$ as well as the section $S_{1}$ in the following dual diagram.


The case of the alternate fibration $\left(\Theta_{2}, S_{2}\right)$ can be handled similarly. In this situation, one obtains an orthogonal decomposition:

$$
\mathcal{W}_{\left(\Theta_{2}, S_{2}\right)}=\mathrm{D}_{16}^{+} \oplus \mathcal{U}
$$

Since any given root of the above lattice has to lie in one of the two factors, one obtains:

$$
\mathcal{W}_{\left(\Theta_{2}, S\right)}^{\text {root }}=\mathrm{D}_{16} \oplus \mathcal{U}^{\text {root }}
$$

Once more, the assertion (b) of Lemma 3.9 tells one that $\Theta_{2}$ has a singular fiber of type $\mathrm{I}_{12}^{*}$. Next, note that, by assertion (c) of Lemma 3.9, one has an isomorphism of groups:

$$
\begin{equation*}
\operatorname{MW}\left(\Theta_{2}\right) \simeq \mathbb{Z} / 2 \mathbb{Z} \oplus \mathcal{V} / \mathcal{V}^{\mathrm{root}} \tag{16}
\end{equation*}
$$

The image $\beta \in \operatorname{MW}\left(\Theta_{2}\right)$ of the generator of the $\mathbb{Z} / 2 \mathbb{Z}$ factor above determines naturally a non-trivial canonical involution of the K3 surface X. In particular, the jacobian fibration $\left(\Theta_{2}, S_{2}\right)$ has an extra section $S_{2}^{\prime}$, the image of $S_{2}$ through $\beta$. One can easily see from (16) that $S_{2}$ and $S_{2}^{\prime}$ are the only sections of the elliptic fibration $\Theta_{2}$ with cohomology classes represented in the polarizing lattice $i(\mathrm{M})$.

In fact, one can clearly see the special $\mathrm{I}_{12}^{*}$ singular fiber together with two sections in the dual diagram (15). This special singular fiber of $\Theta_{2}$ is given by the divisor:

$$
\begin{equation*}
C_{2}+C_{4}+2\left(C_{3}+C_{5}+\cdots C_{9}+S_{1}+D_{9}+D_{8}+\cdots D_{3}\right)+D_{4}+D_{2} \tag{17}
\end{equation*}
$$

whereas the two sections $S_{2}$ and $S_{2}^{\prime}$ are represented by the two extremal curves $C_{1}$ and $D_{1}$.

Remark 3.11. The effect of the involution $\beta$ on the diagram (15) amounts to a right-left flip which sends the $C$-curves to the corresponding symmetric $D$-curves and vice-versa. In particular, the restriction of $\beta$ to the middle rational curve $S$ is a non-trivial involution of $S$ with two distinct fixed points.

Remark 3.12. The induced morphism $\beta^{*}: \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z}) \rightarrow \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})$ restricts to the identity on the orthogonal complement of $i(\mathrm{M})$. In particular, $\beta^{*}$ acts trivially on the transcendental lattice $\mathrm{T}_{\mathrm{X}}$.

One may guess now that it is the canonical involution $\beta$ of Proposition 3.10 that gives rise to the canonical Shioda-Inose structure on X we mentioned at the end of Section 3.1.

We are now in position to formulate the main result of the paper:
Theorem 3.13. Let (X, i) be an M-polarized K3 surface.
(a) The involution $\beta$ introduced above defines a Shioda-Inose structure on X .
(b) The minimal resolution Y of the quotient $\mathrm{X} / \beta$ is a K3 surface with a canonical Kummer structure. This structure realizes Y as the Kummer surface of an abelian surface $\mathrm{A}=\mathrm{E}_{1} \times \mathrm{E}_{2}$ canonically represented as a cartesian product of two elliptic curves. The two elliptic curves are unique, up to permutation.
(c) The above geometric transformation induces a canonical Hodge isomorphism between the M-polarized Hodge structure of X and the natural H -polarized Hodge structure of A .

Before embarking on the proof of the above theorem, let us comment briefly on the two special jacobian fibrations $\Theta_{1}$ and $\Theta_{2}$ which we have uncovered in this section. These two jacobian fibrations ${ }^{7}$ are canonically associated to an M-polarization on a K3 surface X. However, so far, it is the standard fibration $\Theta_{1}$ that has received the lion's share of attention in the literature ${ }^{8}$. An analysis of $\Theta_{1}$ appears in the original work of Inose [17] and, over the last ten years, $\Theta_{1}$ has been extensively studied in the string theory literature due to its connection with the $\mathrm{E}_{8} \oplus \mathrm{E}_{8}$ heterotic string theory in eight dimensions. The alternate fibration $\Theta_{2}$, however, has been largely overlooked. Nevertheless, it is $\Theta_{2}$, with its non-trivial Mordell-Weil group, that gives rise to a canonical Shioda-Inose structure on the M-polarized K3 surface X and leads one to a geometric explanation for the Hodge-theoretic duality map (10). The alternate fibration $\Theta_{2}$ will play a central role in the remainder of this paper.

### 3.4 Kummer Surfaces

In order to give a proof of Theorem 3.13, we shall need a few classical results concerning the geometry of Kummer surfaces. For detailed proofs of the facts mentioned in this brief review we refer the reader to [26], [30] and [24].

Let A be a two-dimensional complex torus. Such a surface is naturally endowed with an abelian group structure. One can consider therefore on A the special involution given by -id. The fixed locus of -id consists of sixteen distinct points. Therefore the quotient:

$$
\begin{equation*}
\mathrm{A} /\{ \pm \mathrm{id}\} \tag{18}
\end{equation*}
$$

is a singular surface with sixteen rational double point singularities of type $\mathrm{A}_{1}$. It is well-known that the minimal resolution of (18) is a special K 3 surface $\mathrm{Km}(\mathrm{A})$ called the Kummer surface of A.

As a first important feature of Kummer surfaces, we note that the Hodge structures of A and Km(A) are closely related. Indeed, denote by $p: \mathrm{A} \rightarrow \mathrm{Km}(\mathrm{A})$ the rational map induced by the quotienting and resolution procedure described above. Then, as explained for example in [24], one has a natural morphism

$$
p_{*}: \mathrm{H}^{2}(\mathrm{~A}, \mathbb{Z}) \rightarrow \mathrm{H}_{\mathrm{Km}(\mathrm{~A})}^{2}
$$

where $H_{K m(A)}^{2}$ is the sublattice of $H^{2}(\operatorname{Km}(A), \mathbb{Z})$ of classes orthogonal to all the sixteen exceptional curves. The complexification of $p_{*}$ sends the class of a holomorphic two-form on A to a class representing a holomorphic two-form on $\mathrm{Km}(\mathrm{A})$ and, as an immediate consequence of Proposition 3.2 in [24], one obtains:

Proposition 3.14. The map $p_{*}$ is an isomorphism and it induces a canonical Hodge isometry

$$
\begin{equation*}
\mathrm{H}^{2}(\mathrm{~A}, \mathbb{Z})(2) \stackrel{p_{*}}{\approx} \mathrm{H}_{\mathrm{Km}(\mathrm{~A})}^{2} . \tag{19}
\end{equation*}
$$

[^6]Moreover, $p_{*}\left(\mathrm{~T}_{\mathrm{A}}\right)=\mathrm{T}_{\mathrm{Km}(\mathrm{A})}$ and the above identification leads to a Hodge isometry at the level of transcendental lattices:

$$
\begin{equation*}
\mathrm{T}_{\mathrm{A}}(2) \stackrel{p_{*}}{\sim} \mathrm{~T}_{\mathrm{Km}(\mathrm{~A})} . \tag{20}
\end{equation*}
$$

Kummer surfaces represent a large class of K3 surfaces. In fact, it is known (see for example [30]) that they form a dense subset in the moduli space of K3 surfaces. One would therefore like to have a criterion for determining whether a given K3 surface is Kummer. A very effective lattice-theoretic criterion for answering this question has been introduced by Nikulin in [26].

Definition 3.15. ${ }^{9}$ Let V be the four-dimensional vector space over the field $\mathbb{F}_{2}$. Consider the rank-sixteen even lattice

$$
\begin{equation*}
\mathcal{R}=\bigoplus_{v \in \mathrm{~V}} \mathbb{Z} x_{v} \tag{21}
\end{equation*}
$$

whose bilinear form is induced by $\left(x_{v}, x_{v^{\prime}}\right)=-2 \delta_{v v^{\prime}}$. By definition, the Kummer lattice K is the lattice in $\mathcal{R} \otimes \mathbb{Q}$ spanned (over $\mathbb{Z}$ ) by the following elements:

$$
\left\{x_{v} \mid v \in \mathrm{~V}\right\} \cup\left\{\left.\frac{1}{2} \sum_{v \in \mathcal{W}} x_{v} \right\rvert\, \mathcal{W} \subset \mathrm{V} \text { affine hyperplane }\right\}
$$

The Kummer lattice K has rank sixteen, is even and negative-definite, and has the same discriminant group and discriminant form as the orthogonal sum:

$$
\mathrm{H}(2) \oplus \mathrm{H}(2) \oplus \mathrm{H}(2)
$$

where H is the standard rank-two hyperbolic lattice.
The connection with Kummer surfaces is established by the fact that, given a Kummer surface $\mathrm{Km}(\mathrm{A})$, one has a natural primitive lattice embedding:

$$
\mathrm{K} \hookrightarrow \mathrm{NS}(\operatorname{Km}(\mathrm{~A}))
$$

whose image is the minimal primitive sublattice of $\mathrm{NS}(\mathrm{Km}(\mathrm{A}))$ containing the classes of the sixteen exceptional curves.

Nikulin's criterion asserts that the converse of the above statement is also true.
Theorem 3.16. (Nikulin [26])
(a) A K3 surface Y is a Kummer surface if and only if there exists a primitive lattice embedding $\mathrm{K} \hookrightarrow \mathrm{NS}(\mathrm{Y})$.
(b) For every primitive lattice embedding $e: \mathrm{K} \hookrightarrow \mathrm{NS}(\mathrm{Y})$, there exists a unique and canonically defined twodimensional complex torus A and a Hodge isometry $\alpha$ of $\mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z})$ such that $\mathrm{Y}=\mathrm{Km}(\mathrm{A})$ and $\operatorname{Im}(\alpha \circ e)$ is the minimal primitive sublattice of $\mathrm{NS}(\mathrm{Y})$ containing the sixteen exceptional curves arising during the Kummer construction process.

Note that it is possible for a K3 surface Y to have multiple non-equivalent Kummer structures, i.e. there exist non-isomorphic complex tori A and $\mathrm{A}^{\prime}$ such that

$$
\mathrm{Km}(\mathrm{~A}) \simeq \mathrm{Y} \simeq \operatorname{Km}\left(\mathrm{~A}^{\prime}\right)
$$

However, as the last part of Theorem 3.16 illustrates, once a primitive lattice embedding of the Kummer lattice K into $\mathrm{NS}(\mathrm{Y})$ is fixed, there exists a unique complex torus A compatible with the embedding of K . For a detailed treatment of the classification problem for Kummer structures on a K3 surface we refer the reader to the paper [16] of Hosono, Lian, Oguiso and Yau.

[^7]For the remainder of this section we shall restrict our attention to Kummer surfaces $\mathrm{Km}(\mathrm{A})$ associated to abelian surfaces $\mathrm{A}=\mathrm{E}_{1} \times \mathrm{E}_{2}$ realized as a cartesian product of two elliptic curves.

Let us first introduce the basic criterion for an abelian surface A to have the above property. According to the Hodge index theorem, the Néron-Severi lattice of A, denoted by NS(A), is an even lattice of signature $(1, r)$ with $0 \leq r \leq 3$. If A splits as a cartesian product $\mathrm{E}_{1} \times \mathrm{E}_{2}$ of two elliptic curves, then the cohomology classes of the two curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ span a rank-two hyperbolic sublattice of $\mathrm{NS}(\mathrm{A})$. The converse of this statement also holds.

Proposition 3.17. Let A be an abelian surface.
(a) The surface A can be realized as a product of two elliptic curves if and only if there exists a primitive lattice embedding $\mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{A})$.
(b) For every primitive lattice embedding $e: \mathrm{H} \hookrightarrow \mathrm{NS}(\mathrm{A})$, there exist two elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ (unique, up to permutation) and an analytic isomorphism $\mathrm{A} \simeq \mathrm{E}_{1} \times \mathrm{E}_{2}$ such that $\operatorname{Im}(e)$ is spanned by the cohomology classes of $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$.
Proof. This is a lattice-theoretic version of Ruppert's criterion for an abelian surface to be isomorphic to a cartesian product of two elliptic curves. See [31] or Chapter $10 \S 6$ of [4] for proofs.

Note that it is possible for an abelian surface A to be represented as a cartesian product of two elliptic curves in two or more non-equivalent ways. One can see that this phenomenon happens only when the Picard rank of $A$ is maximal $\left(p_{A}=4\right)$. In such a case, the number of non-equivalent representations $A=E_{1} \times \mathrm{E}_{2}$ has an interesting interpretation in the context of the class group theory of imaginary quadratic fields [15].

Let us assume now that a splitting $\mathrm{A}=\mathrm{E}_{1} \times \mathrm{E}_{2}$ has been fixed. In this context, the cartesian product structure of A gives rise to a special configuration of twenty-four curves on the Kummer surface Km(A). In order to introduce this curve configuration, let $\left\{x_{0}, x_{1}, x_{2}, x_{3}\right\}$ and $\left\{y_{0}, y_{1}, y_{2}, y_{3}\right\}$ be the two sets of points of order two on $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$. Denote by $\mathrm{H}_{i}, \mathrm{G}_{j}(0 \leq i, j \leq 3)$ the rational curves on $\mathrm{Km}(\mathrm{A})$ obtained as the proper transforms of the images, under the quotient map, of $\mathrm{E}_{1} \times\left\{y_{i}\right\}$ and $\left\{x_{j}\right\} \times \mathrm{E}_{2}$, respectively. Let also $\mathrm{E}_{i j}$ be the exceptional curve on $\operatorname{Km}(\mathrm{A})$ associated to the double point $\left(x_{i}, y_{j}\right)$ of A . One has then the following intersection numbers:

$$
\begin{gathered}
\mathrm{H}_{i} \cdot \mathrm{G}_{j}=0 \\
\mathrm{H}_{k} \cdot \mathrm{E}_{i j}=\delta_{k i}, \quad \mathrm{G}_{k} \cdot \mathrm{E}_{i j}=\delta_{k j} .
\end{gathered}
$$

Definition 3.18. The configuration of twenty-four rational curves

$$
\begin{equation*}
\left\{\mathrm{H}_{i}, \mathrm{G}_{j}, \mathrm{E}_{i j} \mid 0 \leq i, j \leq 3\right\} \tag{22}
\end{equation*}
$$

is called the double Kummer pencil of $\mathrm{Km}(\mathrm{A})$. The minimal primitive sublattice of $\mathrm{NS}(\mathrm{Km}(\mathrm{A}))$ containing the classes of the curves in (22) is called the double Kummer lattice of $\mathrm{Km}(\mathrm{A})$. We denote the isomorphism class of this lattice by DK.

Remark 3.19. By standard lattice theory (see for example Theorem 1.14 .4 of [27]), up to an overall isometry, the double Kummer lattice DK has a unique primitive embedding into the K3 lattice. The orthogonal complement of any such embedding is isomorphic to

$$
\mathrm{H}(2) \oplus \mathrm{H}(2) .
$$

Let us also note that the double Kummer lattice DK contains a natural finite-index sublattice isomorphic to

$$
\mathrm{K} \oplus \mathrm{H}(2) .
$$

The left-hand side term above is, of course, the minimal primitive sublattice of NS ( $\mathrm{Km}(\mathrm{A}))$ containing the sixteen exceptional curves $\mathrm{E}_{i j}$ whereas the factor on the right-hand side is spanned by the two classes:

$$
\begin{equation*}
2 \mathrm{H}_{i}+\sum_{j=0}^{3} \mathrm{E}_{i j}, \quad 2 \mathrm{G}_{j}+\sum_{i=0}^{3} \mathrm{E}_{i j} . \tag{23}
\end{equation*}
$$

The two classes described above do not depend on the indices $i$ and $j$, respectively. Moreover, one can verify that the two classes of (23) are precisely the images of the cohomology classes in $\mathrm{H}^{2}(\mathrm{~A}, \mathbb{Z})$ associated to $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ under the morphism $\pi_{*}$ of Proposition 3.14.

To summarize, we have seen that every Kummer surface $Z=\operatorname{Km}\left(E_{1} \times E_{2}\right)$ associated to an abelian surface that can be realized as a cartesian product of two genus-one curves comes equipped with a natural primitive lattice embedding $\mathrm{DK} \hookrightarrow \mathrm{NS}(\mathrm{Z})$. In fact, one can see that the existence of such an embedding is a sufficient criterion for a K3 surface Z to be a Kummer surface associated to a product abelian surface.

Proposition 3.20. Let Z be a K3 surface. Assume that a primitive lattice embedding $e$ : $\mathrm{DK} \hookrightarrow \mathrm{NS}(\mathrm{Z})$ has been given. Then there exist two elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ and a Hodge isometry $\alpha$ of $\mathrm{H}^{2}(\mathrm{Z}, \mathbb{Z})$ such that

$$
\mathrm{Z}=\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)
$$

and $\operatorname{Im}(\alpha \circ e)$ is the double Kummer lattice associated to the Kummer construction. The two elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ are unique (up to permutation) and canonically defined.

Proof. The above assertion is a consequence of the results presented earlier in this section. Let

$$
e: \mathrm{DK} \hookrightarrow \mathrm{NS}(\mathrm{Z})
$$

be a primitive lattice embedding. There exists then a primitive embedding of the Kummer lattice K in $e(\mathrm{DK})$. Moreover, by standard lattice theory (see Section 15 of [27]), this embedding is unique, up to an overall Hodge isometry of $\mathrm{H}^{2}(\mathrm{Z}, \mathbb{Z})$. Therefore, by Nikulin's criterion, one has a canonical Kummer structure on Z . In other words $\mathrm{Z}=\mathrm{Km}(\mathrm{A})$ with A uniquely defined. Then, according to Proposition 3.14, one has a Hodge isometry

$$
\begin{equation*}
\mathrm{H}^{2}(\mathrm{~A}, \mathbb{Z})(2) \stackrel{p_{*}}{\sim} \mathrm{H}_{\mathrm{Z}}^{2} . \tag{24}
\end{equation*}
$$

But $\mathrm{H}_{\mathrm{Z}}^{2}$ contains a canonical primitive sublattice of type $\mathrm{H}(2)$, the orthogonal complement of the Kummer lattice in $e(\mathrm{DK})$. The preimage of this lattice in $\mathrm{H}^{2}(\mathrm{~A}, \mathbb{Z})$ is primitively embedded in $\mathrm{NS}(\mathrm{A})$ and is isomorphic to H. Then, by Proposition 3.17, the abelian surface splits canonically as a product of two elliptic curves.

### 3.5 Proof of Theorem 3.13

We are now in position to give detailed proofs for the statements of Theorem 3.13.
Let us begin by observing that $\beta$ is a Nikulin involution. If $\omega$ is a given holomorphic two-form on X , then either $\beta^{*} \omega=\omega$ or $\beta^{*} \omega=-\omega$. But, the latter possibility implies (see, for example, [36]) that either $\beta$ has no fixed locus (case that is ruled out by Remark 3.11) or that the fixed locus of $\beta$ is a union of curves (case that is ruled out by the fact that $\beta$ acts without fixed points on the smooth fibers of $\Theta_{2}$ ). Therefore the only possibility that can occur is $\beta^{*} \omega=\omega$ which, by definition, means that $\beta$ is a Nikulin involution.

Remark 3.21. As is well-known (for a proof of this fact see $\S 5$ of [28]), the fixed locus of a Nikulin involution always consists of eight distinct points. The eight fixed points associated to $\beta$ appear nicely in the context of the alternate fibration $\Theta_{2}$. As noted in Remark 3.11, two of them lie on the smooth rational curve $S_{1}$ (the middle curve of the dual diagram (15), also the section of the standard fibration $\Theta_{1}$ ). The additional six fixed points lie on the singular fibers of $\Theta_{2}$. For instance, in the generic case, the alternate elliptic fibration $\Theta_{2}$ has, in addition to the $I_{12}^{*}$ fiber, another six singular fibers of Kodaira type $\mathrm{I}_{1}$ (each consisting of a reduced rational curve with one node). The extra six fixed points of $\beta$ are precisely the nodes of those fibers.

Let then Y be the K 3 surface obtained as the minimal resolution of the quotient of X through $\beta$. We show now that $Y$ is a Kummer surface. In order to carry out our argument, we denote by $\mathrm{F}_{1}, \mathrm{~F}_{2} \cdots \mathrm{~F}_{8}$ the eight exceptional curves arising after resolving the eight rational singularities. Assume that $F_{1}$ and $F_{2}$ are associated to the two fixed points of $\beta$ that lie on the $I_{12}^{*}$ fiber of $\Theta_{2}$.

Recall that the alternate elliptic fibration $\Theta_{2}$ is left invariant by the involution $\beta$. Therefore, $\Theta_{2}$ induces a new elliptic fibration on Y. We denote this fibration by $\Psi_{2}$.


It is then not hard to see that the $I_{12}^{*}$ fiber of $\Theta_{2}$ becomes a singular fiber of Kodaira type $I_{6}^{*}$ in the fibration $\Psi_{2}$. We represent its irreducible components in the dual diagram below.


The curves $\mathrm{R}_{i}, 1 \leq i \leq 9$ are the images of the curves $\mathrm{C}_{i}$ (and also $\mathrm{D}_{i}$ ) of X (recall diagram (15)). The curve $\widetilde{S}_{1}$ above is the quotient of the rational curve $S_{1}$ of diagram (15) by the involution $\beta$. Note also that $R_{1}$ is a section in $\Psi_{2}$ while the unaccounted for exceptional curves $\mathrm{F}_{3}, \mathrm{~F}_{4} \cdots \mathrm{~F}_{8}$ are disjoint from $R_{1}$ and form irreducible components in the additional singular fibers of $\Psi_{2}$.


Lemma 3.22. Let $\mathcal{L}(\mathrm{Y})$ be the minimal primitive sublattice of $\mathrm{NS}(\mathrm{Y})$ containing the classes associated to the eighteen curves $\mathrm{R}_{i}(1 \leq i \leq 9), \mathrm{F}_{j}(1 \leq j \leq 8)$ and $\widetilde{S}_{1}$. The lattice $\mathcal{L}(\mathrm{Y})$ is isomorphic to the double Kummer lattice DK .
Proof. Let us denote by $\mathcal{N}$ the minimal primitive sublattice of $\mathrm{NS}(\mathrm{Y})$ containing the eight exceptional curves $\mathrm{F}_{i}, 1 \leq i \leq 8$. The lattice $\mathcal{N}$ can also be regarded (see Lemma 5.4 of [24]) as the span of the nine classes

$$
\mathrm{F}_{1}, \mathrm{~F}_{2}, \cdots, \mathrm{~F}_{8}, \frac{1}{2} \sum_{i=1}^{8} \mathrm{~F}_{i}
$$

This is the so-called Nikulin lattice (see $\S 5$ of [24]). It has rank eight and has the same discriminant group and discriminant form as $\mathrm{H}(2) \oplus \mathrm{H}(2) \oplus \mathrm{H}(2)$.

Denote by $H_{Y}^{2}$ the orthogonal complement of $\mathcal{N}$ in $\mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z})$. Then, as we described in Section 3.1, the Shioda-Inose construction induces a natural push-forward morphism:

$$
\pi_{*}: \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z}) \rightarrow \mathrm{H}_{\mathrm{Y}}^{2} \hookrightarrow \mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z})
$$

Adopting the notation of diagram (15), one has that $\pi_{*}\left(S_{1}\right)=2 \widetilde{S}_{1}+F_{1}+F_{2}$ and $\pi_{*}\left(C_{j}\right)=\pi_{*}\left(D_{j}\right)=R_{j}$ for $1 \leq j \leq 9$. In particular, the image under $\pi_{*}$ of the class of the elliptic fiber of the standard fibration $\Theta_{1}$ on X is:

$$
\begin{equation*}
2 R_{1}+4 R_{2}+6 R_{3}+3 R_{4}+5 R_{5}+4 R_{6}+3 R_{7}+2 R_{8}+R_{9} \tag{29}
\end{equation*}
$$

We consider then the following sublattices of $\mathrm{H}_{\mathrm{Y}}^{2}$ :
(a) $\mathcal{E}$ is the span of the curves $\mathrm{R}_{j}, 1 \leq j \leq 8$.
(b) $\mathcal{H}$ is the span of $\pi_{*}\left(S_{1}\right)$ and (29).
(c) $\mathcal{Q}=\pi_{*}\left(i(\mathrm{M})^{\perp}\right)$.

Using Remarks 3.2 and 3.12, we deduce that the three lattices above are orthogonal to each other. Moreover, $\mathcal{E}$ is isomorphic to $\mathrm{E}_{8}$ (hence unimodular), $\mathcal{H}$ is isomorphic to $\mathrm{H}(2)$ and $\mathcal{Q}$ is isomorphic to $\mathrm{H}(2) \oplus \mathrm{H}(2)$. Hence, the discriminant of $\mathcal{E} \oplus \mathcal{H} \oplus \mathcal{Q}$ is $2^{6}$. But the lattice $\mathrm{H}_{\mathrm{Y}}^{2}$ has the same discriminant as its orthogonal complement $\mathcal{N}$ which, in turn, has discriminant $2^{6}$. Since clearly $\mathcal{E} \oplus \mathcal{H} \oplus \mathcal{Q}$ is a sublattice of $\mathrm{H}_{\mathrm{Y}}^{2}$, the equality of the two discriminants allows us to conclude that

$$
\begin{equation*}
\mathrm{H}_{\mathrm{Y}}^{2}=\mathcal{E} \oplus \mathcal{H} \oplus \mathcal{Q} \tag{30}
\end{equation*}
$$

In particular, $\mathcal{Q}$ must be primitively embedded in $\mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z})$.
Now, by standard lattice theory ([27], Theorem 1.14.4), up to an overall isometry there exists a unique primitive lattice embedding of $\mathrm{H}(2) \oplus \mathrm{H}(2)$ into the K3 lattice. By Remark 3.19, the orthogonal complement of such an embedding is isomorphic to the double Kummer lattice DK. We see therefore that $\mathcal{Q}^{\perp}$ is isomorphic to DK.

At this point, let us also note the primitive embedding $\mathcal{L}(\mathrm{Y}) \subset \mathcal{Q}^{\perp}$. In order to show that $\mathcal{L}(\mathrm{Y})=\mathcal{Q}^{\perp}$ all we need to do is verify that the two lattices involved have the same rank. The rank of $\mathcal{Q}^{\perp}$ is 18 , as it is isomorphic to DK. By definition, $\operatorname{rank}(\mathcal{L}(\mathrm{Y})) \leq 18$. But

$$
\mathcal{N} \oplus \mathcal{E} \oplus \mathcal{H} \subset \mathcal{L}(\mathrm{Y})
$$

and therefore $\operatorname{rank}(\mathcal{L}(\mathrm{Y})) \geq 18$. Hence, we have that $\mathcal{L}(\mathrm{Y})=\mathrm{Q}^{\perp}$ and therefore $\mathcal{L}(\mathrm{Y})$ is isomorphic with the double Kummer lattice DK.

The above result shows that, by construction, Y comes endowed with a canonical primitive lattice embedding $\mathrm{DK} \hookrightarrow \mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z})$. This fact, in connection with Proposition 3.20 , implies that there exist two canonically defined elliptic curves $\mathrm{E}_{1}, \mathrm{E}_{2}$ (unique, up to permutation) such that

$$
\begin{equation*}
\mathrm{Y}=\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right) \tag{31}
\end{equation*}
$$

Moreover, Proposition 3.20 together with weak form of the Global Torelli Theorem (Theorem 11.1 of [2] $\S V I I I)$, implies that $\mathcal{L}(\mathrm{Y})$ is precisely the double Kummer lattice associated to the Kummer construction (31).

In order to check the assertion $(c)$ of Theorem 3.13, let us consider the diagram of rational maps:

$$
\begin{equation*}
\mathrm{X} \xrightarrow[\rightarrow]{\pi} \mathrm{Y} \stackrel{p}{\leftrightarrow--} \mathrm{E}_{1} \times \mathrm{E}_{2} \tag{32}
\end{equation*}
$$

where $\pi$ is the map induced by the Shioda-Inose construction and $p$ is the map associated with the Kummer construction. The $K 3$ surface $X$ carries the lattice polarization $i(\mathrm{M}) \subset \mathrm{H}^{2}(\mathrm{X}, \mathbb{Z})$, whereas the abelian surface $\mathrm{E}_{1} \times \mathrm{E}_{2}$ is H-polarized by the sublattice $\mathrm{P} \subset \mathrm{H}^{2}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}, \mathbb{Z}\right)$ spanned by the classes of $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$. In both cases, the orthogonal complement of the polarizing lattice is isomorphic to $\mathrm{H} \oplus \mathrm{H}$. One has then the push-forward morphisms:

$$
\begin{equation*}
\mathrm{H}^{2}(\mathrm{X}, \mathbb{Z}) \xrightarrow{\pi_{*}} \mathrm{H}^{2}(\mathrm{Y}, \mathbb{Z}) \stackrel{p_{*}}{\longleftrightarrow} \mathrm{H}^{2}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}, \mathbb{Z}\right) . \tag{33}
\end{equation*}
$$

From the proof of Lemma 3.22, we have that $\pi_{*}\left(i(\mathrm{M})^{\perp}\right)=\mathcal{Q}=\mathcal{L}(\mathrm{Y})^{\perp}$. By Proposition 3.14, $p_{*}\left(\mathrm{P}^{\perp}\right)=$ $\mathcal{L}(\mathrm{Y})^{\perp}$. Moreover, both the restriction of the $\pi_{*}$ on $i(\mathrm{M})^{\perp}$ and the restriction of $p_{*}$ on $\mathrm{P}^{\perp}$ induce isomorphisms of Hodge structures:

$$
i(\mathrm{M})^{\perp}(2) \stackrel{\pi_{*}}{\sim} \mathcal{L}(\mathrm{Y})^{\perp}, \quad \mathrm{P}^{\perp}(2) \stackrel{p_{*}}{\sim} \mathcal{L}(\mathrm{Y})^{\perp} .
$$

By taking $\left(p_{*}\right)^{-1} \circ \pi_{*}$, one obtains therefore a canonical isomorphism of polarized Hodge structures

$$
i(\mathrm{M})^{\perp} \stackrel{\pi_{*}}{\approx} \mathrm{P}^{\perp}
$$

between the surfaces X and $\mathrm{E}_{1} \times \mathrm{E}_{2}$.

## 4 An Explicit Computation

In the first half of this paper, we have described a geometric correspondence:

$$
(\mathrm{X}, i) \mapsto \mathrm{A}(\mathrm{X})=\mathrm{E}_{1} \times \mathrm{E}_{2}
$$

which associates to any given M-polarized K 3 surface X an abelian surface $\mathrm{A}(\mathrm{X})$ realized as a cartesian product of two elliptic curves. In this second part of the paper we shall make this correspondence explicit. In other words we shall compute the J-invariants of the two elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$.

Note that, by the Hodge theoretic equivalence underlying the correspondence, the modular invariants of an M-polarized K3 surface ( $\mathrm{X}, i$ ) can be written as:

$$
\sigma(\mathrm{X}, i)=\mathrm{J}\left(\mathrm{E}_{1}\right)+\mathrm{J}\left(\mathrm{E}_{2}\right), \quad \pi(\mathrm{X}, i)=\mathrm{J}\left(\mathrm{E}_{1}\right) \cdot \mathrm{J}\left(\mathrm{E}_{2}\right)
$$

Therefore, as an immediate application of the calculation of the two J-invariants of $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$, we shall obtain formulas for the modular invariants associated to an explicitly defined M-polarized K3 surface.

### 4.1 The Inose Form

In his 1977 paper [17], Inose introduced an explicit two-parameter family of K3 surfaces which carry canonical M-polarizations. The surfaces in this family are defined as follows.

Let $a, b \in \mathbb{C}$. Denote by $\mathrm{Q}(a, b)$ the surface in $\mathbb{P}^{3}$ (with homogeneous coordinates $[x, y, z, w]$ ) defined by the quartic equation:

$$
\begin{equation*}
y^{2} z w-4 x^{3} z+3 a x z w^{2}-\frac{1}{2}\left(z^{2} w^{2}+w^{4}\right)+b z w^{3}=0 \tag{34}
\end{equation*}
$$

We shall refer to the polynomial on the left side of the above equation as the Inose form. The surface $\mathrm{Q}(a, b)$ has only rational double point singularities and its minimal resolution, denoted $\mathrm{X}(a, b)$, is a K3 surface. Moreover, by construction, the surface $\mathrm{X}(a, b)$ has a canonical M-polarization. In order to see this, let us note that the intersection of $\mathrm{Q}(a, b)$ with the hyperplane $\{w=0\}$ is a union of two lines $\mathrm{L}_{1} \cup \mathrm{~L}_{2}$ with:

$$
\mathrm{L}_{1}:=\{z=w=0\}, \quad \mathrm{L}_{2}:=\{x=w=0\}
$$

Moreover, by standard singularity theory, the points $[0,1,0,0]$ and $[0,0,1,0]$ are rational double point singularities on $\mathrm{Q}(a, b)$ of types $\mathrm{A}_{11}$ and $\mathrm{E}_{6}$, respectively. As a result, one obtains on the minimal resolution $\mathrm{X}(a, b)$ the following configuration of rational curves:


Note already the similarity with the previously encountered diagram (15). The lattices spanned by:

$$
\begin{gathered}
\left\{a_{1}, a_{2}, L_{1}, a_{3}, a_{4}, a_{5}, a_{6}, a_{7}\right\} \\
\left\{a_{11}, L_{2}, e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{6}\right\} \\
\left\{a_{9}, 2 a_{1}+4 a_{2}+3 L_{1}+6 a_{3}+5 a_{4}+4 a_{5}+3 a_{6}+2 a_{7}+a_{8}\right\}
\end{gathered}
$$

are mutually orthogonal and they are also isomorphic to $\mathrm{E}_{8}, \mathrm{E}_{8}$ and H , respectively. As a consequence, their direct sum provides a canonical primitive lattice embedding $\mathrm{M} \hookrightarrow \mathrm{NS}(\mathrm{X}(a, b))$.

### 4.2 The Main Formula

In the remaining part of the paper, we prove:
Theorem 4.1. Let $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ be the two elliptic curves associated to the M-polarized K 3 surface $\mathrm{X}(a, b)$ by the correspondence of Theorem 3.13. Then $\mathrm{J}\left(\mathrm{E}_{1}\right)$ and $\mathrm{J}\left(\mathrm{E}_{2}\right)$ are the two solutions of the quadratic equation:

$$
\begin{equation*}
x^{2}-\left(a^{3}-b^{2}+1\right) x+a^{3}=0 . \tag{36}
\end{equation*}
$$

As mentioned earlier, as a consequence of the above theorem, one obtains:
Corollary 4.2. The modular invariants of the M-polarized $K 3$ surface $\mathrm{X}(a, b)$ are given by:

$$
\begin{equation*}
\pi=a^{3}, \quad \sigma=a^{3}-b^{2}+1 \tag{37}
\end{equation*}
$$

Corollary 4.3. Every M-polarized K3 surface is isomorphic ${ }^{10}$ to $\mathrm{X}(a, b)$ for some $a, b \in \mathbb{C}$.
Our strategy for proving Theorem 4.1 relies on a detailed analysis of the two basic algebraic invariants associated with the elliptic fibration $\Psi_{2}$ on the Kummer surface Y : the functional and homological invariants.

### 4.3 Invariants Associated to an Elliptic Surface

Let X be a smooth compact complex analytic surface and let $\varphi: \mathrm{X} \rightarrow \mathrm{C}$ be a proper analytic map to a smooth curve such that the generic fiber of $\varphi$ is a smooth elliptic curve. Assume also that $\varphi$ does not have multiple fibers. Kodaira $[19,20]$ associated two fundamental invariants to a such a structure. We present them here following the exposition of Friedman-Morgan in Section 1.3.3 of [12].
(a) The functional invariant is an analytic function $\mathcal{J}_{\varphi}: \mathrm{C} \rightarrow \mathbb{P}^{1}$. It can be defined in the following manner. Let U be the complement in C of the critical values of $\varphi$. Then $\mathcal{J}_{\varphi}$ is the meromorphic continuation of the composite map:

$$
\mathrm{U} \xrightarrow{e} \mathbb{H} / \mathrm{PSL}(2, \mathbb{Z}) \xrightarrow{\mathrm{J}} \mathbb{C}
$$

which takes a smooth elliptic fiber to its associated point in the moduli space of elliptic curves and then evaluates the classical elliptic modular function ${ }^{11}$ at that respective point.
(b) The homological invariant is, by definition, the sheaf $\mathcal{G}_{\varphi}=\mathrm{R}^{1} \varphi_{*} \mathbb{Z}_{\mathrm{X}}$. The restriction of $\mathcal{G}_{\varphi}$ on U is locally constant and oriented and its stalk at every point is isomorphic with $\mathbb{Z} \oplus \mathbb{Z}$. Moreover, since $\varphi$ has no multiple fibers, one has $\mathcal{G}_{\varphi}=i_{*}\left(\left.\mathcal{G}_{\varphi}\right|_{\mathrm{U}}\right)$ where $i: \mathrm{U} \hookrightarrow \mathrm{C}$ and therefore $\mathcal{G}_{\varphi}$ is determined by its restriction on $U$. The latter sheaf is however fully determined by the conjugacy class of its monodromy map:

$$
\begin{equation*}
\rho_{\varphi}: \pi_{1}(\mathrm{U}, t) \longrightarrow \mathrm{SO}\left(\mathrm{H}^{1}\left(\varphi^{-1}(t), \mathbb{Z}\right)\right) \tag{38}
\end{equation*}
$$

One can regard, therefore, the homological invariant of $\varphi$ as an element in $\operatorname{Hom}\left(\pi_{1}(\mathrm{U}), \mathrm{SL}(2, \mathbb{Z})\right)$, modulo conjugation.

The two invariants are not unrelated. Let us assume, for simplicity, that $\mathcal{J}_{\varphi}$ is not constant, as the cases of interest to us will definitely satisfy this condition. Set then $\mathrm{U}_{0} \subset \mathrm{U}$ as the open subset for which $\mathcal{J}_{\varphi} \notin\{0,1\}$ and denote by $\mathbb{H}_{0}$ the set of elements of the upper half-plane $\mathbb{H}$ for which the associated elliptic modular function is neither 0 or 1 . Pick $t \in \mathrm{U}_{0}$. The composition

$$
\mathrm{U}_{0} \stackrel{i}{\hookrightarrow} \mathrm{U} \xrightarrow{e} \mathbb{H} / \operatorname{PSL}(2, \mathbb{Z})
$$

[^8]induces a morphism of fundamental groups:
\[

$$
\begin{equation*}
\pi_{1}\left(\mathrm{U}_{0}, t\right) \xrightarrow{(e \circ i) \neq} \pi_{1}\left(\mathbb{H}_{0} / \operatorname{PSL}(2, \mathbb{Z})\right) \simeq \operatorname{PSL}(2, \mathbb{Z}) \tag{39}
\end{equation*}
$$

\]

The compatibility between the two invariants asserts that the above morphism agrees, modulo conjugation with:

$$
\begin{equation*}
\pi_{1}\left(\mathrm{U}_{0}, t\right) \xrightarrow{i_{\#}} \pi_{1}(\mathrm{U}, t) \xrightarrow{\rho_{\varphi}} \mathrm{SL}(2, \mathbb{Z}) \rightarrow \operatorname{PSL}(2, \mathbb{Z}) . \tag{40}
\end{equation*}
$$

The above compatibility condition can be introduced independent of the actual elliptic fibration over C. Given a (non-constant) meromorphic function $\mathcal{J}: \mathrm{C} \rightarrow \mathbb{C}$ with no poles on U and a morphism of $\mathbb{Z}$ modules $\rho: \pi_{1}(\mathrm{U}) \rightarrow \mathrm{SL}(2, \mathbb{Z})$, the pair $(\mathcal{J}, \rho)$ is said to be compatible if the associated maps (39) and (40) agree modulo conjugation. One has then the following classical theorem of Kodaira.

Theorem 4.4. (Kodaira [20]) For a compatible pair $(\mathcal{J}, \rho)$ as above, there is, up to an isomorphism of elliptic surfaces, exactly one elliptic fibration $\varphi: \mathrm{X} \rightarrow \mathrm{C}$, admitting a section, with functional and homological invariants given by $\mathcal{J}$ and $\rho$.

The above theorem provides one with a very powerful tool for classifying jacobian fibrations. The functional invariant $\mathcal{J}$ describes the smooth part of an elliptic fibration but does not provide enough information for one to establish the topological type of the total surface X. The homological invariant $\rho$, which is essentially a lifting of the local monodromies from $\operatorname{PSL}(2, \mathbb{Z})$ to $\operatorname{SL}(2, \mathbb{Z})$, complements $\mathcal{J}$ and the pair $(\mathcal{J}, \rho)$ fully classifies the elliptic fibration ${ }^{12}$. The effectiveness of Kodaira's criterion is further enhanced by the fact that, given a jacobian fibration as above, the monodromy $\rho(\gamma)$ of a small loop $\gamma$ circling a critical value of $\varphi$ in a manner agreeing with the orientation of C is determined modulo conjugation by the Kodaira type of the associated singular fiber [19]. One has therefore the following very particular consequence of the above discussion.

Corollary 4.5. Let $\varphi$ and $\psi$ be two jacobian fibrations on two K3 surfaces X and $\mathrm{X}^{\prime}$.


The two jacobian fibrations are isomorphic if and only if there exists a a projective automorphism $q$ of $\mathbb{P}^{1}$ such that $q$ maps bijectively the singular locus of $\varphi$ to the singular locus of $\psi, \mathcal{J}_{\varphi}=\mathcal{J}_{\psi} \circ \mathrm{q}$ and, for any $t$ in the singular locus of $\varphi$, the Kodaira type of a singular fiber $\varphi^{-1}(t)$ is the same as the Kodaira type of the singular fiber $\psi^{-1}(\mathrm{q}(t))$.
Our strategy for proving Theorem 4.1 is structured as follows. We first compute the functional invariants and Kodaira types of singular fibers of both the alternate fibration $\Theta_{2}$ on $\mathrm{X}(a, b)$, and the induced jacobian fibration $\Psi_{2}$ on the K 3 surface $\mathrm{Y}(a, b)$. Then, switching our attention to the other side of the correspondence, we show that, for any two elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$, the Kummer surface $\mathrm{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ possesses a canonical jacobian fibration $\Upsilon_{2}$ with the same types of singular fibers as $\Psi_{2}$. Finally, using Corollary 4.5, we prove that the two elliptic fibrations $\Psi_{2}$ and $\Upsilon_{2}$ are equivalent if and only if $J\left(\mathrm{E}_{1}\right)$ and $\mathrm{J}\left(\mathrm{E}_{2}\right)$ are solutions to the quadratic equation (36).

### 4.4 The Alternate Fibration $\Theta_{2}$

It is quite easy to observe the fibration $\Theta_{2}$ on the surface $\mathrm{X}(a, b)$. The alternate fibration is induced by the projection to $[x, w]$ from the quartic $\mathrm{Q}(a, b)$. Indeed, one can easily verify the following facts.

[^9](a) The generic fiber of the projection to $[x, w]$ from $\mathrm{Q}(a, b)$ is an elliptic curve. In fact, the fiber over $[\lambda, 1]$ can be seen as the cubic curve in $\mathbb{P}^{2}(y, z, w)$ given by:
\[

$$
\begin{equation*}
\Theta_{2}^{\lambda}:=\Theta_{2}^{-1}([\lambda, 1])=\left\{2 y^{2} z-\left(8 \lambda^{3}-6 a \lambda-2 b\right) z w^{2}-z^{2} w-w^{3}=0\right\} . \tag{42}
\end{equation*}
$$

\]

This is a smooth cubic as long as $4 \lambda^{3}-3 a \lambda-b \neq \pm 1$.
(b) After resolving the singularities of $\mathrm{Q}(a, b)$, the projection to $[x, w]$ induces an elliptic fibration on the K3 surface $\mathrm{X}(a, b)$.
(c) The singular fiber $\Theta_{2}^{\infty}:=\Theta_{2}^{-1}([1,0])$ is of Kodaira type $I_{12}^{*}$. In the context of the diagram (35), $\Theta_{2}^{\infty}$ appears as the divisor:

$$
a_{2}+L_{1}+2\left(a_{3}+a_{4}+\cdots+a_{11}+L_{2}+e_{1}+e_{2}+e_{3}\right)+e_{4}+e_{5}
$$

(d) The curves $a_{1}$ and $e_{6}$ are sections of $\Theta_{2}$.

Let us then compute the functional invariant of the elliptic fibration $\Theta_{2}$. In order to simplify further calculations, we shall introduce the following polynomial:

$$
\mathrm{P}(X)=4 X^{3}-3 A X-B
$$

With this in place, one can rewrite the cubic equation in (42) in a standard Weierstrass form as:

$$
\begin{equation*}
(\sqrt{2} y z)^{2}=\left(z+\frac{2}{3} \mathrm{P}(\lambda)\right)^{3}+g_{2}(\lambda)\left(z+\frac{2}{3} \mathrm{P}(\lambda)\right)+g_{3}(\lambda) \tag{43}
\end{equation*}
$$

where the terms $g_{2}(\lambda)$ and $g_{2}(\lambda)$ are given by:

$$
g_{2}(\lambda)=1-\frac{4}{3} \mathrm{P}^{2}(\lambda), \quad g_{3}(\lambda)=\frac{16}{27} \mathrm{P}^{3}(\lambda)-\frac{2}{3} \mathrm{P}(\lambda) .
$$

The discriminant of Weierstrass form (43) is then:

$$
\Delta_{\Theta_{2}}(\lambda)=4 g_{2}^{3}(\lambda)+27 g_{3}^{2}(\lambda)=4\left(1-\mathrm{P}^{2}(\lambda)\right) .
$$

In the same manner, the functional invariant of $\Theta_{2}$ can be computed as:

$$
\mathcal{J}_{\Theta_{2}}(\lambda)=\frac{4 g_{2}^{3}(\lambda)}{\Delta_{\Theta_{2}}(\lambda)}=\frac{\left(3-4 \mathrm{P}^{2}(\lambda)\right)^{2}}{9\left(1-\mathrm{P}^{2}(\lambda)\right)}
$$

The explicit formulas for $g_{2}(\lambda), g_{3}(\lambda)$ and $\Delta_{\Theta_{2}}(\lambda)$ allow one to determine not only the location but also the Kodaira type of the singular fibers of $\Theta_{2}$. Using Tate's algorithm [34], one obtains:

Proposition 4.6. The singular fibers of $\Theta_{2}$ are located at $[1,0]$ (the $\mathrm{I}_{12}^{*}$ fiber) and at the points $[\lambda, 1]$ with $\lambda$ belonging to the subset:

$$
\begin{equation*}
\Sigma:=\left\{\lambda \mid \mathrm{P}(\lambda)^{2}=1\right\} . \tag{44}
\end{equation*}
$$

The following five cases can occur:

- $a^{3} \neq(b \pm 1)^{2}$. In this case, both polynomials $\mathrm{P}(X)-1$ and $\mathrm{P}(X)+1$ have three distinct roots. The subset $\Sigma$ consists of six distinct points, each of which corresponds to a singular fiber of type $\mathrm{I}_{1}$ in $\Theta_{2}$.
- $a^{3}=(b+1)^{2}, b \neq 0, a \neq 0$. In this case, $\mathrm{P}(X)+1$ has three distinct roots. However, the polynomial $\mathrm{P}(X)-1$ has a root of order two at $-(b+1) / 2 a$ and a simple root at $(b+1) / a$. The subset $\Sigma$ consists of 5 distinct points.

$$
\Sigma=\left\{\frac{-(b+1)}{2 a}, \frac{b+1}{a}\right\} \cup\{\lambda \mid \mathrm{P}(\lambda)=-1\}
$$

The first value in the above list corresponds to a singular fiber of type $\mathrm{I}_{2}$ in $\Theta_{2}$. The remaining four points correspond to fibers of type $\mathrm{I}_{1}$.

- $a^{3}=(b-1)^{2}, b \neq 0, a \neq 0$. In this case, the polynomial $\mathrm{P}(X)-1$ has three distinct roots. However, $\mathrm{P}(X)+1$ has a root of order two at $-(b-1) / 2 a$ and a simple root at $(b-1) / a$. As in the previous case, the subset $\Sigma$ consists of 5 distinct values.

$$
\Sigma=\left\{\frac{-(b-1)}{2 a}, \frac{b-1}{a}\right\} \cup\{\lambda \mid \mathrm{P}(\lambda)=1\}
$$

The first value in the above list corresponds to a singular fiber of type $\mathrm{I}_{2}$ in $\Theta_{2}$. The remaining four points correspond to fibers of type $\mathrm{I}_{1}$.

- $a=0, b= \pm 1$. Then

$$
\Sigma=\{0\} \cup\left\{\left.\frac{1}{\sqrt[3]{2}} \theta \right\rvert\, \theta^{3}=b\right\}
$$

The value $\lambda=0$ corresponds to a singular fiber of type $\mathrm{I}_{3}$ in $\Theta_{2}$. The remaining three values of $\Sigma$ correspond to fibers of type $\mathrm{I}_{1}$.

- $a^{3}=1, b=0$. In this case one has:

$$
\mathrm{P}(X)-1=\left(2 X-a^{2}\right)^{2}\left(X+a^{2}\right), \mathrm{P}(X)+1=\left(2 X+a^{2}\right)^{2}\left(X-a^{2}\right)
$$

Accordingly,

$$
\Sigma=\left\{\frac{a^{2}}{2},-\frac{a^{2}}{2},-a^{2}, a^{2}\right\}
$$

The first two values in the above list correspond to singular fibers of type $\mathrm{I}_{2}$ while the last two values correspond to fibers of type $\mathrm{I}_{1}$.
Next, we describe explicitly the involution $\beta$ on $\mathrm{X}(a, b)$. Note that, in each of the smooth cubics $\Theta_{2}^{\lambda}$ of (42), the point $[1,0,0]$ is an inflection point. If one chooses this point as the origin of the cubic group law on $\Theta_{2}^{\lambda}$, the point $[0,1,0]$ is a point of order two with respect to this law. Moreover, when regarding $\Theta_{2}^{\lambda}$ as an elliptic fiber in $\mathrm{X}(a, b)$, one has that $[1,0,0]$ and $[0,1,0]$ are the intersections with the two sections $a_{1}$ and $e_{6}$. Therefore, the effect of $\beta$ on $\Theta_{2}^{\lambda}$ can be seen, in the coordinates of (42), as the analytic continuation of:

$$
\begin{gather*}
\Theta_{2}^{\lambda} \backslash\{[1,0,0],[0,1,0]\} \rightarrow \Theta_{2}^{\lambda} \backslash\{[1,0,0],[0,1,0]\}  \tag{45}\\
{[y, z, w] \mapsto\left[-y z, w^{2}, z w\right] .}
\end{gather*}
$$

Finally, the full $\beta$ is induced from the analytic involution:

$$
\begin{gather*}
\beta_{1}: \mathrm{Q}(a, b) \backslash\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2}\right) \rightarrow \mathrm{Q}(a, b) \backslash\left(\mathrm{L}_{1} \cup \mathrm{~L}_{2}\right)  \tag{46}\\
\beta_{1}([x, y, z, w]) \mapsto\left[x z,-y z, w^{2}, z w\right]
\end{gather*}
$$

### 4.5 The Elliptic Fibration $\Psi_{2}$

Let $\mathrm{Y}(a, b)$ be the Kummer surface obtained from $\mathrm{X}(a, b)$ through the Shioda-Inose construction. Recall from section 3.5 that the alternate fibration $\Theta_{2}$ survives on $\mathrm{Y}(a, b)$ in the form of a new elliptic fibration $\Psi_{2}$.

As we already know, the fiber $\Psi_{2}^{\infty}$ has Kodaira type $I_{6}^{*}$. In this section, we describe the location and Kodaira type of the other singular fibers and write an explicit formula for the functional invariant $\mathrm{J}_{\Psi_{2}}$.

Note that the smooth fibers $\Psi_{2}^{\lambda}$ are quotients of the cubics $\Theta_{2}^{\lambda}$ of (42) by the involution (45). By then taking affine coordinates $[y, z, 1]$ on $\Theta_{2}^{\lambda}$ and defining

$$
u=y^{2}-\mathrm{P}(\lambda), v=\frac{1}{2} y\left(z-\frac{1}{z}\right)
$$

one obtains an affine description of $\Psi_{2}^{\lambda}$ as:

$$
\begin{equation*}
v^{2}=(u+\mathrm{P}(\lambda))(u-1)(u+1) \tag{47}
\end{equation*}
$$

This can then be easily transformed to a Weierstrass form:

$$
\begin{equation*}
v^{2}=\left(u+\frac{1}{3} \mathrm{P}(\lambda)\right)^{3}-\left(u+\frac{1}{3} \mathrm{P}(\lambda)\right)\left(\frac{1}{3} \mathrm{P}^{2}(\lambda)+1\right)+\frac{2}{27} \mathrm{P}^{3}(\lambda)-\frac{2}{3} \mathrm{P}(\lambda) \tag{48}
\end{equation*}
$$

which has as discriminant:

$$
\Delta_{\Psi_{2}}(\lambda)=-4\left(\mathrm{P}^{2}(\lambda)-1\right)^{2}
$$

It follows then that the functional invariant of the elliptic fibration $\Psi_{2}$ is:

$$
\begin{equation*}
\mathcal{J}_{\Psi_{2}}(\lambda)=\frac{\left(P^{2}(\lambda)+3\right)^{2}}{9\left(P^{2}(\lambda)-1\right)^{2}} \tag{49}
\end{equation*}
$$

As in the previous section, the above information allows us to also describe the location and Kodaira type of the singular fibers of $\Psi_{2}$.

Proposition 4.7. The singular fibers of the elliptic fibration $\Psi_{2}$ on $\mathrm{Y}(a, b)$ are located at $[1,0]$ (the $\mathrm{I}_{6}^{*}$ fiber) and at the points $[\lambda, 1]$ with $\lambda$ belonging to the subset:

$$
\begin{equation*}
\Sigma:=\left\{\lambda \mid P(\lambda)^{2}=1\right\} \tag{50}
\end{equation*}
$$

The following cases occur:

- $a^{3} \neq(b \pm 1)^{2}$. In this case, both polynomials $\mathrm{P}(X)-1$ and $\mathrm{P}(X)+1$ have three distinct roots. The subset $\Sigma$ consists of six distinct points, each of which corresponds to a singular fiber of type $\mathrm{I}_{2}$ in $\Theta_{2}$.
- $a^{3}=(b+1)^{2}, b \neq 0, a \neq 0$. In this case, $\mathrm{P}(X)+1$ has three distinct roots. However, the polynomial $\mathrm{P}(X)-1$ has a root of order two at $-(b+1) / 2 a$ and a simple root at $(b+1) / a$. The subset $\Sigma$ consists of 5 distinct points.

$$
\Sigma=\left\{\frac{-(b+1)}{2 a}, \frac{b+1}{a}\right\} \cup\{\lambda \mid \mathrm{P}(\lambda)=-1\}
$$

The first value in the above list corresponds to a singular fiber of type $I_{4}$ in $\Theta_{2}$. The remaining four points correspond to fibers of type $\mathrm{I}_{2}$.

- $a^{3}=(b-1)^{2}, b \neq 0, a \neq 0$. In this case, the polynomial $\mathrm{P}(X)-1$ has three distinct roots. However, $\mathrm{P}(X)+1$ has a root of order two at $-(b-1) / 2 a$ and a simple root at $(b-1) / a$. As in the previous case, the subset $\Sigma$ consists of 5 distinct values.

$$
\Sigma=\left\{\frac{-(b-1)}{2 a}, \frac{b-1}{a}\right\} \cup\{\lambda \mid \mathrm{P}(\lambda)=1\}
$$

The first value in the above list corresponds to a singular fiber of type $\mathrm{I}_{4}$ in $\Theta_{2}$. The remaining four points correspond to fibers of type $\mathrm{I}_{2}$.

- $a=0, b= \pm 1$. Then

$$
\Sigma=\{0\} \cup\left\{\left.\frac{1}{\sqrt[3]{2}} \theta \right\rvert\, \theta^{3}=b\right\}
$$

The value $\lambda=0$ corresponds to a singular fiber of type $\mathrm{I}_{6}$ in $\Theta_{2}$. The remaining three values of $\Sigma$ correspond to fibers of type $\mathrm{I}_{2}$.

- $a^{3}=1, b=0$. In this case one has:

$$
\mathrm{P}(X)-1=\left(2 X-a^{2}\right)^{2}\left(X+a^{2}\right), \mathrm{P}(X)+1=\left(2 X+a^{2}\right)^{2}\left(X-a^{2}\right)
$$

Accordingly,

$$
\Sigma=\left\{\frac{a^{2}}{2},-\frac{a^{2}}{2},-a^{2}, a^{2}\right\}
$$

The first two values in the above list correspond to singular fibers of type $\mathrm{I}_{4}$ while the last two values correspond to fibers of type $\mathrm{I}_{2}$.

### 4.6 A Special Elliptic Fibration on $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$

As we already know from Theorem 3.13, the surface $\mathrm{Y}(a, b)$ can be realized in a canonical way as the Kummer surface $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ associated to the product of two elliptic curves. Moreover, in this context, the elliptic fibration $\Psi_{2}$ on $\mathrm{Y}(a, b)$ can be derived directly from the Kummer construction.

Recall from Section 3.4 that the surface $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ has a canonical twenty-four curve configuration $\left\{\mathrm{H}_{i}, \mathrm{G}_{j}, \mathrm{E}_{i j} \mid 0 \leq i, j \leq 3\right\}$ called the double Kummer pencil.

Lemma 4.8. Consider the divisor D on $\mathrm{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ defined as:

$$
\begin{equation*}
\mathrm{D}=\mathrm{E}_{21}+\mathrm{E}_{31}+2\left(\mathrm{G}_{1}+\mathrm{E}_{01}+\mathrm{H}_{0}+\mathrm{E}_{00}+\mathrm{G}_{0}+\mathrm{E}_{10}+\mathrm{H}_{1}\right)+\mathrm{E}_{12}+\mathrm{E}_{13} . \tag{51}
\end{equation*}
$$

Then $\mathrm{D}^{2}=0$ and $|D|$ is a pencil inducing an elliptic fibration $\Upsilon_{2}: \operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right) \rightarrow \mathbb{P}^{1}$. The divisor D is a singular fiber for $\Upsilon_{2}$ and has Kodaira type $\mathrm{I}_{6}^{*}$. The four smooth rational curves $\mathrm{H}_{2}, \mathrm{H}_{3}, \mathrm{G}_{2}$ and $\mathrm{G}_{3}$ form four disjoint sections of $\Upsilon_{2}$.


Proof. The above assertion is a consequence of a classical theorem due to Pjateckii-Š̌apiro and Šafarevič ([30], Chapter 3, Theorem 1).

Remark 4.9. A different selection of the double Kummer pencil curves defining the divisor (51) alters the elliptic fibration $\Upsilon_{2}$ by an analytic automorphism of $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$. The equivalence class of $\Upsilon_{2}$ is therefore well-defined.

Remark 4.10. In [29], Oguiso classified all jacobian fibrations on a Kummer surface associated to a product of two non-isogenous elliptic curves. The elliptic fibration $\Upsilon_{2}$ defined above appears as $\mathcal{J}_{5}$ in Oguiso's classification. It is the only jacobian fibration on such a surface that admits a singular fiber of Kodaira type $\mathrm{I}_{6}^{*}$.

By virtue of the geometric correspondence

$$
\mathrm{X}(a, b) \longrightarrow \mathrm{E}_{1} \times \mathrm{E}_{2}
$$

described in the first part of the paper, one has, as an intermediate step, an isomorphism

$$
\mathrm{Y}(a, b) \simeq \operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)
$$

that maps the jacobian fibration $\Psi_{2}$ on $\mathrm{Y}(a, b)$ to the jacobian fibration $\Upsilon_{2}$ on $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$.

This fact allows one to realize an explicit relation between the Inose parameters $a, b$ of the M-polarized K 3 surface $\mathrm{X}(a, b)$ and the J -invariants of the two resulting elliptic curves $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$. In light of Corollary 4.5, the two jacobian fibrations $\Psi_{2}$ on $\mathrm{Y}(a, b)$ and $\Upsilon_{2}$ on $\operatorname{Km}\left(\mathrm{E}_{2} \times \mathrm{E}_{2}\right)$ are equivalent if and only if their functional invariant and singular locus differ by a projective transformation and the Kodaira types of their singular fibers match.

We have already described in detail the functional invariant and the location and type of the singular fibers of $\Psi_{2}$. In what follows we shall perform a similar analysis for $\Upsilon_{2}$. The comparison between these two pieces of data will then allow us to prove the main statement of Theorem 4.1.

Claim 4.11. The two elliptic fibrations $\Psi_{2}$ and $\Upsilon_{2}$ have equivalent functional and homological invariants if and only if

$$
\mathrm{J}\left(\mathrm{E}_{1}\right)+\mathrm{J}\left(\mathrm{E}_{2}\right)=a^{3}-b^{2}+1, \quad \mathrm{~J}\left(\mathrm{E}_{1}\right) \cdot \mathrm{J}\left(\mathrm{E}_{2}\right)=a^{3}
$$

### 4.7 Description of the Elliptic Fibration $\Upsilon_{2}$ on $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$

It is a standard fact that any given elliptic curve can be realized as a projective Legendre cubic

$$
\left\{y^{2} w=x(x-w)(x-\lambda w)\right\} \subset \mathbb{P}^{2}
$$

for some $\lambda \in \mathbb{C} \backslash\{0,1\}$. We shall assume therefore that $\alpha, \beta \in \mathbb{C}$ are chosen such that $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ are isomorphic with the above cubics for $\lambda=\alpha$ and $\lambda=\beta$, respectively. The J-invariants of the two curves can then be computed as:

$$
\mathrm{J}\left(\mathrm{E}_{1}\right)=\frac{4\left(\alpha^{2}-\alpha+1\right)^{3}}{27 \alpha^{2}(\alpha-1)^{2}}, \quad \mathrm{~J}\left(\mathrm{E}_{2}\right)=\frac{4\left(\beta^{2}-\beta+1\right)^{3}}{27 \beta^{2}(\beta-1)^{2}}
$$

In this context, an explicit model for the Kummer surface $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ can be constructed (see [7, 17]) by taking the minimal resolution of the quartic surface:

$$
\begin{equation*}
\left\{z^{2} x y=(x-w)(x-\alpha w)(y-w)(y-\beta w)\right\} \subset \mathbb{P}^{3} \tag{52}
\end{equation*}
$$

Note that, generically, the quartic surface (52) has seven rational double point singularities located at:

$$
\begin{gathered}
{[1,0,0,0], \quad[0,1,0,0], \quad[0,0,1,0]} \\
{[1,1,0,1],[\alpha, 1,0,1], \quad[1, \beta, 0,1], \quad[\alpha, \beta, 0,1] .}
\end{gathered}
$$

The first three are rational double points of type $A_{3}$. The last four are singularities of type $A_{1}$. One can therefore reconstruct the double Kummer pencil on the minimal resolution of (52) by taking:

$$
\begin{aligned}
\mathrm{H}_{0}+\mathrm{E}_{00}+\mathrm{G}_{0} & =\mathrm{A}_{3} \text { configuration associated to }[0,0,1,0] \\
\mathrm{E}_{21}+\mathrm{G}_{1}+\mathrm{E}_{31} & =\mathrm{A}_{3} \text { configuration associated to }[1,0,0,0] \\
\mathrm{E}_{12}+\mathrm{H}_{1}+\mathrm{E}_{13} & =\mathrm{A}_{3} \text { configuration associated to }[0,1,0,0] \\
\mathrm{E}_{22} & =\mathrm{A}_{1} \text { curve associated to }[1,1,0,1] \\
\mathrm{E}_{32} & =\mathrm{A}_{1} \text { curve associated to }[\alpha, 1,0,1] \\
\mathrm{E}_{23} & =\mathrm{A}_{1} \text { curve associated to }[1, \beta, 0,1] \\
\mathrm{E}_{33} & =\mathrm{A}_{1} \text { curve associated to }[\alpha, \beta, 0,1] \\
\mathrm{H}_{2} & =\text { proper transform of }\{x=w, z=0\} \\
\mathrm{H}_{3} & =\text { proper transform of }\{x=\alpha w, z=0\} \\
\mathrm{G}_{2} & =\text { proper transform of }\{y=w, z=0\}
\end{aligned}
$$

$$
\begin{gathered}
\mathrm{G}_{3}=\text { proper transform of }\{y=\beta w, z=0\} \\
\mathrm{E}_{01}=\text { proper transform of }\{x=w=0\} \\
\mathrm{E}_{10}=\text { proper transform of }\{y=w=0\} \\
\mathrm{E}_{11}=\text { proper transform of }\left\{w=0, z^{2}=x y\right\} \\
\mathrm{E}_{02}=\text { proper transform of }\{x=0, y=w\} \\
\mathrm{E}_{03}=\text { proper transform of }\{x=0, y=\beta w\} \\
\mathrm{E}_{20}=\text { proper transform of }\{y=0, x=w\} \\
\mathrm{E}_{30}=\text { proper transform of }\{y=0, x=\alpha w\}
\end{gathered}
$$

A simple analysis of the curves $\mathrm{H}_{i}, \mathrm{G}_{j}$ and locations of the intersections with $\mathrm{E}_{i j}$ allows one to conclude that the minimal resolution of (52) is canonically isomorphic to $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$.

Remark 4.12. The birational morphism $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right) \rightarrow \mathbb{P}^{3}$ whose image is the quartic surface (52) can also be defined directly from the double Kummer pencil by taking the projective morphism associated to the base-point free linear system $|\mathrm{V}|$ given by:


The advantage of the realization of $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ as the quartic surface (52), follows from the fact that, in this context, one can explicitly construct the jacobian fibration $\Upsilon_{2}$ of Lemma 4.8. This elliptic fibration is induced by the rational map:

$$
[x, y, z, w] \mapsto[\mathrm{R}(x, y, w), x y]
$$

where $\mathrm{R}(x, y, w)$ is the quadratic polynomial:

$$
\mathrm{R}(x, y, w)=\left(-\frac{1}{\alpha}\right) x^{2}+\left(-\frac{1}{\beta}\right) y^{2}+\left(\frac{\alpha+1}{\alpha}\right) x w+\left(\frac{\beta+1}{\beta}\right) y w-w^{2} .
$$

The $I_{6}^{*}$ fiber of $\Upsilon_{2}$ appears over the point $[1,0]$. Away from this location, the generic smooth elliptic fiber

$$
\Upsilon_{2}^{\lambda}:=\Upsilon_{2}^{-1}([\mu, 1])
$$

can be regarded as the double cover of the projective conic in $\mathbb{P}^{2}(x, y, w)$ :

$$
\begin{equation*}
\mathrm{R}(x, y, w)=\mu x y \tag{53}
\end{equation*}
$$

branched at the four points:

$$
\begin{align*}
& {[1,(1-\mu) \beta+1,1], \quad[\alpha,(1-\mu \alpha) \beta+1,1],}  \tag{54}\\
& {[(1-\mu) \alpha+1,1,1], \quad[(1-\mu \beta) \alpha+1, \beta, 1] .}
\end{align*}
$$

One encounters singular fibers if the conic (53) is singular, or if at least two of the above four branch points coincide. This argument allows one to conclude that the points on the base of the fibration $\Upsilon_{2}$ associated to singular fibers (away from the $\mathrm{I}_{6}^{*}$ fiber) are of type $[\mu, 1]$ with $\mu$ belonging to the set:

$$
\Sigma_{\Upsilon_{2}}:=\left\{1, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha \beta}, \frac{\alpha \beta+1}{\alpha \beta}, \frac{\alpha+\beta}{\alpha \beta}\right\} .
$$

Remark 4.13. For generic choices of $\alpha$ and $\beta$, the above set contains six distinct points, each of which determines an $\mathrm{I}_{2}$ fiber in $\Upsilon_{2}$. However, it may happen that two, or more, of the above six values coincide. This happens precisely when:

$$
\begin{equation*}
\beta \in\left\{\alpha, \frac{1}{\alpha}, 1-\alpha, \frac{1}{1-\alpha}, \frac{\alpha}{\alpha-1}, \frac{\alpha-1}{\alpha}\right\} \tag{55}
\end{equation*}
$$

a condition that is also equivalent to $\mathrm{J}\left(\mathrm{E}_{1}\right)=\mathrm{J}\left(\mathrm{E}_{2}\right)$.
Lemma 4.14. The functional invariant of the jacobian fibration $\Upsilon_{2}$ has the form:

$$
\begin{equation*}
\mathcal{J}_{\Upsilon_{2}}(\mu)=\frac{4\left(\alpha^{4} \beta^{4} \mathrm{D}(\mu)+(\alpha-1)^{2}(\beta-1)^{2}\right)^{3}}{27 \alpha^{8} \beta^{8}(\alpha-1)^{4}(\beta-1)^{4} \mathrm{D}^{2}(\mu)} \tag{56}
\end{equation*}
$$

where:

$$
\mathrm{D}(\mu):=(\mu-1)\left(\mu-\frac{1}{\alpha}\right)\left(\mu-\frac{1}{\beta}\right)\left(\mu-\frac{1}{\alpha \beta}\right)\left(\mu-\frac{\alpha \beta+1}{\alpha \beta}\right)\left(\mu-\frac{\alpha+\beta}{\alpha \beta}\right) .
$$

Proof. We accomplish the computation of $\mathcal{J}_{\Upsilon_{2}}(\mu)$ through the following sequence of steps.

1. Construct an explicit isomorphism $i_{\mu}$ between the conic (53) and $\mathbb{P}^{1}$.
2. Perform a projective automorphism of $\mathbb{P}^{1}$ such that the images through $i_{\mu}$ of the four branch points (54) are sent to $[0,1],[1,1],[r, 1]$ and $[1,0]$.
3. Evaluate $\mathcal{J}_{\Upsilon_{2}}(\mu)$ as:

$$
\begin{equation*}
\frac{4\left(r^{2}-r+1\right)^{3}}{27 r^{2}(r-1)^{2}} \tag{57}
\end{equation*}
$$

In order to complete the first step, let us note that:

$$
\begin{gathered}
\mathrm{R}(x, y, w)-\mu x y= \\
-\left[w-\left(\frac{\alpha+1}{2 \alpha}\right) x-\left(\frac{\beta+1}{2 \beta}\right) y\right]^{2}+\left(\frac{\alpha-1}{2 \alpha}\right)^{2} x^{2}+\left(\frac{\beta-1}{2 \beta}\right)^{2} y^{2}+\left(\frac{(\alpha+1)(\beta+1)}{2 \alpha \beta}-\mu\right) x y= \\
-\left[w-\left(\frac{\alpha+1}{2 \alpha}\right) x-\left(\frac{\beta+1}{2 \beta}\right) y\right]^{2}+\left[\left(\frac{\alpha-1}{2 \alpha}\right) x+\left(\frac{(\alpha+1)(\beta+1)}{2 \alpha \beta}-\mu\right)\left(\frac{\alpha}{\alpha-1}\right) y\right]^{2}+ \\
+\left[\left(\frac{\beta-1}{2 \beta}\right)^{2}-\left(\frac{(\alpha+1)(\beta+1)}{2 \alpha \beta}-\mu\right)^{2}\left(\frac{\alpha}{\alpha-1}\right)^{2}\right] y^{2}= \\
-\left[w-\left(\frac{\alpha+1}{2 \alpha}\right) x-\left(\frac{\beta+1}{2 \beta}\right) y\right]^{2}+\left[\left(\frac{\alpha-1}{2 \alpha}\right) x+\left(\frac{(\alpha+1)(\beta+1)}{2 \alpha \beta}-\mu\right)\left(\frac{\alpha}{\alpha-1}\right) y\right]^{2}+ \\
+\left[\left(\frac{(\alpha-1)(\beta-1)}{2 \alpha \beta}\right)^{2}-\left(\frac{(\alpha+1)(\beta+1)}{2 \alpha \beta}-\mu\right)^{2}\right]\left(\frac{\alpha}{\alpha-1}\right)^{2} y^{2}= \\
-\left[w-\left(\frac{\alpha+1}{2 \alpha}\right) x-\left(\frac{\beta+1}{2 \beta}\right) y\right]^{2}+\left[\left(\frac{\alpha-1}{2 \alpha}\right) x+\left(\frac{(\alpha+1)(\beta+1)}{2 \alpha \beta}-\mu\right)\left(\frac{\alpha}{\alpha-1}\right) y\right]^{2}+ \\
-\left[\left(\mu-\frac{\alpha+\beta}{\alpha \beta}\right)\left(\mu-\frac{\alpha \beta+1}{\alpha \beta}\right)\right]\left(\frac{\alpha}{\alpha-1}\right)^{2} y^{2}= \\
\quad=\left(w-\frac{1}{\alpha} x-\left(\mu-\frac{\beta+1}{\alpha \beta}\right)\left(\frac{\alpha}{\alpha-1}\right) y\right)\left(-w+x-\left(\mu-\frac{\beta+1}{\beta}\right)\left(\frac{\alpha}{\alpha-1}\right) y\right)-
\end{gathered}
$$

$$
-\left[\left(\mu-\frac{\alpha+\beta}{\alpha \beta}\right)\left(\mu-\frac{\alpha \beta+1}{\alpha \beta}\right)\right]\left(\frac{\alpha}{\alpha-1}\right)^{2} y^{2}
$$

The change in projective coordinates:

$$
\begin{gathered}
x_{1}=-w+x-\left(\mu-\frac{\beta+1}{\beta}\right)\left(\frac{\alpha}{\alpha-1}\right) y, \quad y_{1}=\left(\frac{\alpha}{\alpha-1}\right) y \\
w_{1}=w-\frac{1}{\alpha} x-\left(\mu-\frac{\beta+1}{\alpha \beta}\right)\left(\frac{\alpha}{\alpha-1}\right) y
\end{gathered}
$$

allows one to rewrite the conic (53) as:

$$
x_{1} w_{1}=\Delta y_{1}^{2}
$$

where:

$$
\Delta=\left(\mu-\frac{\alpha+\beta}{\alpha \beta}\right)\left(\mu-\frac{\alpha \beta+1}{\alpha \beta}\right)
$$

This yields the parametrization of (53) via the embedding:

$$
\mathbb{P}^{1} \hookrightarrow \mathbb{P}^{2}, \quad[u, v] \mapsto\left[u^{2}, u v, \Delta v^{2}\right]
$$

with the inverse map $i_{\mu}$ given by the analytic continuation of $\left[x_{1}, y_{1}, w_{1}\right] \mapsto\left[x_{1}, y_{1}\right]$. This procedure results in an identification between the conic (53) and $\mathbb{P}^{1}$ which sends the four branch points $(54)$ to

$$
\begin{aligned}
& {[\beta+1-\beta \mu, \beta], \quad[\alpha \beta \Delta, \beta+1-\mu \alpha \beta]} \\
& {[\alpha \beta+1-\mu \alpha \beta, \beta], \quad[\alpha+\beta-\mu \alpha \beta, \beta] .}
\end{aligned}
$$

In accordance with the plan presented earlier, we take:

$$
\begin{gathered}
r=\frac{((\alpha \beta+1-\mu \alpha \beta)-(\beta+1-\beta \mu))((\alpha \beta \Delta)-(\alpha+\beta-\mu \alpha \beta))}{((\alpha \beta+1-\mu \alpha \beta)-(\alpha+\beta-\mu \alpha \beta))((\alpha \beta \Delta)-(\beta+1-\beta \mu))}= \\
=\frac{(\mu-1)(\mu \alpha \beta-1)(\mu \alpha \beta-\alpha-\beta)}{(\alpha-1)(\beta-1)}
\end{gathered}
$$

Then the functional invariant is computed as:

$$
\mathcal{J}_{\Upsilon_{2}}(\mu)=\frac{4\left(r^{2}-r+1\right)^{3}}{27 r^{2}(r-1)^{2}}=\frac{4\left(\alpha^{4} \beta^{4} \mathrm{D}(\mu)+(\alpha-1)^{2}(\beta-1)^{2}\right)^{3}}{27 \alpha^{8} \beta^{8}(\alpha-1)^{4}(\beta-1)^{4} \mathrm{D}^{2}(\mu)}
$$

The above discussion also provides the homological invariant data of the fibration $\Upsilon_{2}$.
Corollary 4.15. In addition to the $\mathrm{I}_{6}^{*}$ singular fiber which appears over the point $[1,0]$, the elliptic fibrations $\Upsilon_{2}$ has singular fibers at the points $[\mu, 1]$ with $\mu$ belonging to the set:

$$
\Sigma_{\Upsilon_{2}}=\{\mu \mid \mathrm{D}(\mu)=0\}
$$

The following cases can occur:
(a) $\mathrm{J}\left(\mathrm{E}_{1}\right) \neq \mathrm{J}\left(\mathrm{E}_{2}\right)$. In this case $\Sigma_{\Upsilon_{2}}$ has six distinct points and each of them corresponds to an $\mathrm{I}_{2}$ singular fiber.
(b) $\mathrm{J}\left(\mathrm{E}_{1}\right)=\mathrm{J}\left(\mathrm{E}_{2}\right) \notin\{0,1\}$. In this case the polynomial $\mathrm{D}(\mu)$ has five distinct roots, one of which is of order two. The order-two root corresponds to a singular fiber of type $\mathrm{I}_{4}$. The remaining four roots correspond to $\mathrm{I}_{2}$ fibers.
(c) $\mathrm{J}\left(\mathrm{E}_{1}\right)=\mathrm{J}\left(\mathrm{E}_{2}\right)=1$. In this case the polynomial $\mathrm{D}(\mu)$ has four distinct roots, two of which have order two. The two roots of order two correspond to singular fibers of type $\mathrm{I}_{4}$. The remaining two roots correspond to fibers of type $\mathrm{I}_{2}$.
(d) $\mathrm{J}\left(\mathrm{E}_{1}\right)=\mathrm{J}\left(\mathrm{E}_{2}\right)=0$. In this case the polynomial $\mathrm{D}(\mu)$ has four distinct roots, one of which is of order three. The order-three root corresponds to singular fiber of type $\mathrm{I}_{3}$. The remaining three roots correspond to fibers of type $\mathrm{I}_{2}$.

### 4.8 Proof of Claim 4.11

Recall the analysis of Sections 4.5 and 4.7. Both fibrations $\Psi_{2}$ and $\Upsilon_{2}$ have the $I_{6}^{*}$ singular fiber located over the point $[1,0]$ and their respective functional invariants, as described in (49) and (56), are :

$$
\begin{gather*}
\mathcal{J}_{\Psi_{2}}(\lambda)=\frac{\left(\mathrm{P}^{2}(\lambda)+3\right)^{2}}{9\left(\mathrm{P}^{2}(\lambda)-1\right)^{2}}, \quad \mathrm{P}(\lambda)=4 \lambda^{3}-3 a \lambda-b .  \tag{58}\\
\mathcal{J}_{\Upsilon_{2}}(\mu)=\frac{4\left(\alpha^{4} \beta^{4} \mathrm{D}(\mu)+(\alpha-1)^{2}(\beta-1)^{2}\right)^{3}}{27 \alpha^{8} \beta^{8}(\alpha-1)^{4}(\beta-1)^{4} \mathrm{D}^{2}(\mu)} \tag{59}
\end{gather*}
$$

The main polynomial in the denominator of $\mathrm{J}_{\Psi_{2}}(\lambda)$ is $\mathrm{P}^{2}(\lambda)-1$. Its (generic) six roots are naturally divided into two sets of three roots, each three-set having the sum of its elements equal to zero. A similar feature can be observed in the denominator of $\mathrm{J}_{\Upsilon_{2}}(\mu)$. The main polynomial present there is $\mathrm{D}(\mu)$ whose (generic) six roots can be partitioned into two sets of three with identical sum.

$$
\begin{equation*}
\left\{1, \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\alpha \beta}, \frac{\alpha \beta+1}{\alpha \beta}, \frac{\alpha+\beta}{\alpha \beta}\right\}=\left\{1, \frac{1}{\alpha \beta}, \frac{\alpha+\beta}{\alpha \beta}\right\} \cup\left\{\frac{1}{\alpha}, \frac{1}{\beta}, \frac{\alpha \beta+1}{\alpha \beta}\right\} \tag{60}
\end{equation*}
$$

The two fibrations $\Psi_{2}$ and $\Upsilon_{2}$ have equivalent functional invariant and homological invariant data if and only if there exists an invertible affine transformation $\Xi(\lambda)=q \lambda+p$ with $p, q \in \mathbb{C}(q \neq 0)$ such that

$$
\begin{equation*}
\mathcal{J}_{\Psi_{2}}(\lambda)=\mathcal{J}_{\Upsilon_{2}}(\Xi(\lambda)) \tag{61}
\end{equation*}
$$

and $\Xi$ sends the roots of $\mathrm{P}(\lambda) \pm 1$ to the two subsets in (60) while preserving the homological type.
As a first observation, we note that it follows that $p=(\alpha+1)(\beta+1) / 3 \alpha \beta$. Let then:

$$
\begin{array}{r}
\mathrm{D}_{1}(\mu)=(\mu-1)\left(\mu-\frac{1}{\alpha \beta}\right)\left(\mu-\frac{\alpha+\beta}{\alpha \beta}\right) \\
\mathrm{D}_{2}(\mu)=\left(\mu-\frac{1}{\alpha}\right)\left(\mu-\frac{1}{\beta}\right)\left(\mu-\frac{\alpha \beta+1}{\alpha \beta}\right) .
\end{array}
$$

Two possibilities can occur:
(a) $q^{3}(\mathrm{P}(\lambda)-1)=4 \mathrm{D}_{1}(q \lambda+p)$ and $q^{3}(\mathrm{P}(\lambda)+1)=4 \mathrm{D}_{2}(q \lambda+p)$
(b) $q^{3}(\mathrm{P}(\lambda)-1)=4 \mathrm{D}_{2}(q \lambda+p)$ and $q^{3}(\mathrm{P}(\lambda)+1)=4 \mathrm{D}_{1}(q \lambda+p)$.

Making the constant terms coincide implies, in the two cases:

$$
b= \pm \frac{(\alpha-2)(\alpha+1)(2 \alpha-1)(\beta-2)(\beta+1)(2 \beta-1)}{27 \alpha(\alpha-1) \beta(\beta-1)}
$$

and

$$
\begin{equation*}
q^{3}=-\frac{2(\alpha-1)(\beta-1)}{\alpha^{2} \beta^{2}} \tag{62}
\end{equation*}
$$

The first equality, in turn, requires the a priori condition.

$$
\begin{equation*}
b^{2}=\frac{(\alpha-2)^{2}(\alpha+1)^{2}(2 \alpha-1)^{2}(\beta-2)^{2}(\beta+1)^{2}(2 \beta-1)^{2}}{729 \alpha^{2}(\alpha-1)^{2} \beta^{2}(\beta-1)^{2}}=\left(\mathrm{J}\left(\mathrm{E}_{1}\right)-1\right)\left(\mathrm{J}\left(\mathrm{E}_{2}\right)-1\right) \tag{63}
\end{equation*}
$$

Continuing the argument, we note that when imposing the equality of the linear terms in the above cases, one is led to:

$$
\begin{equation*}
a=\frac{4\left(\alpha^{2}-\alpha+1\right)\left(\beta^{2}-\beta+1\right)}{9 \alpha^{2} \beta^{2} q^{2}} . \tag{64}
\end{equation*}
$$

This constraint, in connection with (62), requires a second a priori condition:

$$
\begin{equation*}
a^{3}=\frac{16\left(\alpha^{2}-\alpha+1\right)^{3}\left(\beta^{2}-\beta+1\right)^{3}}{729 \alpha^{2}(\alpha-1)^{2} \beta^{2}(\beta-1)^{2}}=\mathrm{J}\left(\mathrm{E}_{1}\right) \cdot \mathrm{J}\left(\mathrm{E}_{2}\right) \tag{65}
\end{equation*}
$$

Then, depending on the case in question, one can eventually solve for $q$, yielding:

$$
q=\frac{-9 a(\alpha-1)(\beta-1)}{2\left(\alpha^{2}-\alpha+1\right)\left(\beta^{2}-\beta+1\right)} .
$$

The equivalence (61) of functional invariants is immediately verified. To finish the proof of Claim 4.11, we note that conditions (63) and (65) are equivalent to:

$$
\begin{equation*}
\mathrm{J}\left(\mathrm{E}_{1}\right)+\mathrm{J}\left(\mathrm{E}_{2}\right)=a^{3}-b^{2}+1, \quad \mathrm{~J}\left(\mathrm{E}_{1}\right) \cdot \mathrm{J}\left(\mathrm{E}_{2}\right)=a^{3} \tag{66}
\end{equation*}
$$

### 4.9 Connections with the Hodge Conjecture

Consider $\mathrm{X}(a, b)$ as above and let $\mathrm{A}(a, b)=\mathrm{E}_{1} \times \mathrm{E}_{2}$ where $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ are the two corresponding elliptic curves. The general form of the Hodge conjecture predicts in this case (see for instance section 7 of [24]) the existence of a special correspondence:

$$
\begin{equation*}
\mathcal{Z} \subset \mathrm{A}(a, b) \times \mathrm{X}(a, b) \tag{67}
\end{equation*}
$$

inducing the canonical Hodge isomorphism between the M-polarized Hodge structure of $\mathrm{X}(a, b)$ and the natural H-polarized Hodge structure of $\mathrm{A}(a, b)$. The correspondence $\mathcal{Z}$ gives, in turn, a class:

$$
\begin{equation*}
[\mathcal{Z}] \in \mathrm{H}^{4}(\mathrm{~A}(a, b) \times \mathrm{X}(a, b)) \cap \mathrm{H}^{4}(\mathrm{~A}(a, b) \times \mathrm{X}(a, b), \mathbb{Q}) . \tag{68}
\end{equation*}
$$

The computation undertaken in this section leads to an explicit description of $\mathcal{Z}$.
Recall that $\mathrm{X}(a, b)$ is the minimal resolution of the quartic surface

$$
\mathrm{Q}(a, b) \subset \mathbb{P}^{3}(x, y, z, w)
$$

defined by (34). On the other hand, $\mathrm{E}_{1}$ and $\mathrm{E}_{2}$ are given by the Legendre presentations:

$$
\begin{aligned}
& \left\{y_{1}^{2} w_{1}=x_{1}\left(x_{1}-w_{1}\right)\left(x_{1}-\alpha w_{1}\right)\right\} \subset \mathbb{P}^{2}\left(x_{1}, y_{1}, w_{1}\right) \\
& \left\{y_{2}^{2} w_{2}=x_{2}\left(x_{2}-w_{2}\right)\left(x_{2}-\beta w_{2}\right)\right\} \subset \mathbb{P}^{2}\left(x_{2}, y_{2}, w_{2}\right)
\end{aligned}
$$

with the parameters $\alpha, \beta$ being related to $a, b$ by the conditions (66). The correspondence $\mathcal{Z}$ is then the pull-back on $\mathrm{A}(a, b) \times \mathrm{X}(a, b)$ of the intersection between $\mathrm{E}_{1} \times \mathrm{E}_{2} \times \mathrm{Q}(a, b)$ and two special hypersurfaces in $\mathbb{P}^{2}\left(x_{1}, y_{1}, w_{1}\right) \times \mathbb{P}^{2}\left(x_{2}, y_{2}, w_{2}\right) \times \mathbb{P}^{3}(x, y, z, w)$.

In order to describe these two hypersurfaces, note that the rational double cover map

$$
\mathrm{E}_{1} \times \mathrm{E}_{2} \rightarrow \mathrm{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)
$$

is induced by the rational map:

$$
\begin{gather*}
\kappa: \mathbb{P}^{2} \times \mathbb{P}^{2} \rightarrow \mathbb{P}^{3}  \tag{69}\\
\kappa\left(\left[x_{1}, y_{1}, w_{1}\right],\left[x_{2}, y_{2} . w_{2}\right]\right)=\left[x_{1}^{2} x_{2} w_{2}, x_{1} x_{2}^{2} w_{1}, y_{1} y_{2} w_{1} w_{2}, x_{1} x_{2} w_{1} w_{2}\right]
\end{gather*}
$$

the image $\kappa\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ being given by the quartic surface:

$$
\begin{equation*}
\left\{z_{3}^{2} x_{3} y_{3}=\left(x_{3}-w_{3}\right)\left(x_{3}-\alpha w_{3}\right)\left(y_{3}-w_{3}\right)\left(y_{3}-\beta w_{3}\right)\right\} \subset \mathbb{P}^{3}\left(x_{3}, y_{3}, z_{3}, w_{3}\right) \tag{70}
\end{equation*}
$$

Each of the two surfaces $\mathrm{X}(a, b)$ and $\operatorname{Km}\left(\mathrm{E}_{1} \times \mathrm{E}_{2}\right)$ carries a special elliptic fibration, $\Theta_{2}$ and $\Upsilon_{2}$, respectively. By sections 4.4 and 4.6 , these fibrations are induced by the rational maps:

$$
\begin{gathered}
\widetilde{\Theta}_{2}: \mathbb{P}^{3} \rightarrow \mathbb{P}^{1}, \widetilde{\Theta}_{2}([x, y, z, w])=[x, w] \\
\widetilde{\Upsilon}_{2}: \mathbb{P}^{3} \rightarrow \mathbb{P}^{1}, \widetilde{\Upsilon}_{2}\left(\left[x_{3}, y_{3}, z_{3}, w_{3}\right]\right)=\left[\mathrm{R}\left(x_{3}, y_{3}, w_{3}\right), x_{3} y_{3}\right]
\end{gathered}
$$

One has therefore the following diagram.


The first hypersurface in $\mathbb{P}^{2}\left(x_{1}, y_{1}, w_{1}\right) \times \mathbb{P}^{2}\left(x_{2}, y_{2}, w_{2}\right) \times \mathbb{P}^{3}(x, y, z, w)$ is obtained by identifying the two base spaces of the birational projections $\widetilde{\Upsilon}_{2} \circ \kappa$ and $\widetilde{\Theta}_{2}$. As explained in section 4.8, this is achieved through the isomorphism:

$$
\begin{gathered}
\Xi: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}, \quad \Xi\left(\left[\lambda_{1}, \lambda_{2}\right]\right)=\left[q \lambda_{1}+p \lambda_{2}, \lambda_{2}\right] \\
p=\frac{(\alpha+1)(\beta+1)}{3 \alpha \beta}, \quad q=\frac{-9 a(\alpha-1)(\beta-1)}{2\left(\alpha^{2}-\alpha+1\right)\left(\beta^{2}-\beta+1\right)} .
\end{gathered}
$$

One has therefore the correspondence:

$$
\left\{(\mu, \lambda) \in \mathbb{P}^{1} \times \mathbb{P}^{1} \mid \Xi(\lambda)=\mu\right\}
$$

and the pull-back of this through $\left(\widetilde{\Upsilon}_{2} \times \kappa\right) \times \widetilde{\Theta}_{2}$ yields the degree eight hypersurface:

$$
\begin{gather*}
\left\{\mathcal{H}_{1}=0\right\} \subset \mathbb{P}^{2}\left(x_{1}, y_{1}, w_{1}\right) \times \mathbb{P}^{2}\left(x_{2}, y_{2}, w_{2}\right) \times \mathbb{P}^{3}(x, y, z, w)  \tag{71}\\
\mathcal{H}_{1}\left(x_{1}, y_{1}, w_{1}, x_{2}, y_{2}, w_{2}, x, y, z, w\right)=x_{1}^{2} x_{2}^{2}\left[q \alpha \beta x x_{1} x_{2} w_{1} w_{2}+\right. \\
\left.+w\left(\alpha x_{2} w_{1}^{2}\left(x_{2}-w_{2}\right)+\beta x_{1} w_{2}^{2}\left(x_{1}-w_{1}\right)+\alpha \beta w_{1} w_{2}\left(w_{1} w_{2}-x_{1} w_{2}-x_{2} w_{1}+p x_{1} x_{2}\right)\right)\right] .
\end{gather*}
$$

The second hypersurface cutting out the Hodge cycle can be obtained by, essentially, making a fiberwise identification between the elliptic fibrations $\Upsilon_{2}$ and $\Psi_{2}$. By the discussion in Section 4.5, the double cover rational map $\pi: \mathrm{X}(a, b) \rightarrow \mathrm{Y}(a, b)$ closes the commutative diagram:

which is induced by the diagram of rational maps:

with:

$$
\begin{gathered}
\widetilde{\pi}([x, y, z, w])=\left[z^{4} w\left(y^{2} w-\mathrm{P}(x, w)\right), \frac{1}{2} y z^{5} w^{4}\left(z^{2}-w^{2}\right), z w, z x\right], \quad \mathrm{P}(x, w)=4 x^{3}-3 a x w^{2}-b w^{3} \\
\widetilde{\Theta}_{2}([x, y, z, w])=[x, w], \quad \widetilde{\Psi}_{2}([u, v, t, r])=[r, t]
\end{gathered}
$$

The K3 surface $\mathrm{Y}(a, b)$ appears as the resolution of the degree-twelve weighted Legendre surface:

$$
v^{2}=(u+t \mathrm{P}(r, t))\left(u+t^{4}\right)\left(u-t^{4}\right)
$$

Fixing then $[\lambda, 1] \in \mathbb{P}^{1}$, the elliptic fiber of $\Psi_{2}$ over $[\lambda, 1]$ can be seen naturally as the double cover of $\mathbb{P}^{1}$ branched at

$$
\begin{equation*}
\left[-4 \lambda^{3}+3 a \lambda+b, 1\right],[1,1],[1,-1] \text { and }[0,1] \tag{72}
\end{equation*}
$$

By Section 4.8, the corresponding elliptic fiber on $\Upsilon_{2}$ is the one over $\Xi([\lambda, 1])=[q \lambda+p, 1]$ and, according to Section 4.7, this fiber appears in $\mathbb{P}^{3}\left(x_{3}, y_{3}, z_{3}, w_{3}\right)$ as the intersection of the two surfaces:

$$
z_{3}^{2} x_{3} y_{3}=\left(x_{3}-w_{3}\right)\left(x_{3}-\alpha w_{3}\right)\left(y_{3}-w_{3}\right)\left(y_{3}-\beta w_{3}\right), \quad \mathrm{R}\left(x_{3}, y_{3}, w_{3}\right)=\mu x_{3} y_{3}
$$

where $\mu=q \lambda+p$ and

$$
\mathrm{R}\left(x_{3}, y_{3}, w_{3}\right)=\left(-\frac{1}{\alpha}\right) x_{3}^{2}+\left(-\frac{1}{\beta}\right) y_{3}^{2}+\left(\frac{\alpha+1}{\alpha}\right) x_{3} w_{3}+\left(\frac{\beta+1}{\beta}\right) y_{3} w_{3}-w_{3}^{2} .
$$

This elliptic curve is then naturally the double cover of the conic

$$
\begin{equation*}
\left\{\mathrm{R}\left(x_{3}, y_{3}, w_{3}\right)=\mu x_{3} y_{3}\right\} \subset \mathbb{P}^{2}\left(x_{3}, y_{3}, w_{3}\right) \tag{73}
\end{equation*}
$$

Moreover, as computed in Section 4.7, under the change of projective coordinates:

$$
\begin{gather*}
x_{4}=-w_{3}+x_{3}-\left(\mu-\frac{\beta+1}{\beta}\right)\left(\frac{\alpha}{\alpha-1}\right) y_{3}  \tag{74}\\
y_{4}=\left(\frac{\alpha}{\alpha-1}\right) y_{3} \\
w_{4}=w_{3}-\frac{1}{\alpha} x_{3}-\left(\mu-\frac{\beta+1}{\alpha \beta}\right)\left(\frac{\alpha}{\alpha-1}\right) y_{3}
\end{gather*}
$$

the conic (73) becomes:

$$
\left\{x_{4} w_{4}=\Delta y_{4}^{2}\right\} \subset \mathbb{P}^{2}\left(x_{4}, y_{4}, w_{4}\right), \quad \Delta=\left(\mu-\frac{\alpha+\beta}{\alpha \beta}\right)\left(\mu-\frac{\alpha \beta+1}{\alpha \beta}\right)
$$

and one can naturally identify it with $\mathbb{P}^{1}$ through the the restriction of the rational map $\left[x_{4}, y_{4}, w_{4}\right] \mapsto\left[x_{4}, y_{4}\right]$. This (non-canonical) procedure realizes the elliptic curve in question as the double cover of $\mathbb{P}^{1}$ branched at the four points:

$$
\begin{align*}
& {[\beta+1-\beta \mu, \beta], \quad[\alpha \beta \Delta, \beta+1-\mu \alpha \beta]}  \tag{75}\\
& {[\alpha \beta+1-\mu \alpha \beta, \beta], \quad[\alpha+\beta-\mu \alpha \beta, \beta]}
\end{align*}
$$

In order to identify the two elliptic curves (of the fibrations $\Psi_{2}$ and $\Upsilon_{2}$, respectively), one needs the to pick $\mu=q \lambda+p$ and construct an isomorphism:

$$
\chi: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}
$$

that maps the four branch points in (72) to the four points of (72).

Note that the condition $\mu=q \lambda+p$ is equivalent to the equality of the cross-ratio of the two quadruples (72) and (75) and, therefore, any isomorphism mapping three points to three points will automatically map the remaining fourth point on one side to the fourth point on the other side.

An isomorphism $\chi: \mathbb{P}^{1} \rightarrow \mathbb{P}^{1}$ such that

$$
\begin{gathered}
\chi([0,1])=[\beta+1-\beta \mu, \beta], \quad \chi([1,1])=[\alpha \beta+1-\mu \alpha \beta, \beta], \\
\chi([-1,1])=[\alpha+\beta-\mu \alpha \beta, \beta]
\end{gathered}
$$

is given by $\chi\left(\left[\lambda_{1}, \lambda_{2}\right]\right)=\left[A \lambda_{1}+B \lambda_{2}, C \lambda_{1}+D \lambda_{2}\right]$ with:

$$
\begin{gather*}
A=\frac{2 \alpha \beta(\mu-1)(\mu \beta-1)-\beta \mu-\beta^{2}(\mu-1)+1}{\beta-1}  \tag{76}\\
B=\beta+1-\mu \beta, \quad C=\frac{\beta(\beta+1-2 \beta \mu)}{\beta-1}, \quad D=\beta
\end{gather*}
$$

The identification of the two elliptic fibers therefore requires:

$$
\begin{gathered}
\mu=q \lambda+p \\
{\left[A\left(y^{2} w-\mathrm{P}(x, w)\right)+B w^{3}, C\left(y^{2} w-\mathrm{P}(x, w)\right)+D w^{3}\right]=\left[x_{4}, y_{4}\right]}
\end{gathered}
$$

with $x_{4}, y_{4}$ defined in (74). The first condition leads to the hypersurface $\mathcal{H}_{1}$ described in (71). The second condition leads to a second hypersurface:

$$
\begin{gather*}
\left\{\mathcal{H}_{2}=0\right\} \subset \mathbb{P}^{2}\left(x_{1}, y_{1}, w_{1}\right) \times \mathbb{P}^{2}\left(x_{2}, y_{2}, w_{2}\right) \times \mathbb{P}^{3}(x, y, z, w)  \tag{77}\\
\mathcal{H}_{2}\left(x_{1}, y_{1}, w_{1}, x_{2}, y_{2}, w_{2}, x, y, z, w\right)=x_{4}\left(C\left(y^{2} w-\mathrm{P}(x, w)\right)+D w^{3}\right)-y_{4}\left(A\left(y^{2} w-\mathrm{P}(x, w)\right)+B w^{3}\right) .
\end{gather*}
$$

## 5 A String Duality Point of View

Following the works of Vafa [35] and Sen [32] in 1996, it was noted that the geometry underlying elliptic $K 3$ surfaces with section is related to the geometry of elliptic curves endowed with certain flat principal $G$ bundles and an additional parameter called the B-field. This non-trivial connection appears in string theory as the eight-dimensional manifestation of the phenomenon called F-theory/heterotic string duality. Over the past ten years the correspondence has been analyzed extensively ( $[6,10]$ ) from a purely mathematical point of view. As it turns out, it leads to a beautiful geometric picture which links together moduli spaces for these two seemingly distinct types of geometrical objects: elliptic $K 3$ surfaces with section and flat bundles over elliptic curves.

In brief, what happens is the following. On the F-theory side, one has the moduli space $\mathcal{M}_{\mathrm{K} 3}$ of elliptic K 3 surfaces with section. This is a quasi-projective analytic variety of complex dimension eighteen. However, $\mathcal{M}_{\mathrm{K} 3}$ is not compact. Nevertheless, there exists a nice smooth partial compactification $\mathcal{M}_{\mathrm{K} 3} \subset \overline{\mathcal{M}}_{\mathrm{K} 3}$, consisting of an enlargement of the original space by adding two Type II Mumford boundary divisors $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$. Geometrically, the points of the two compactifying divisors correspond to Type II stable elliptic K3 surfaces. These are special degenerations of $K 3$ surfaces realized as a union $V_{1} \cup V_{2}$ of two rational surfaces meeting over a common elliptic curve $E$ which is anti-canonical on both $V_{1}$ and $V_{2}$.

On the heterotic side, one has to consider two moduli spaces $\mathcal{M}_{\text {het }}^{\mathrm{G}}$ of triples ( $\mathrm{E}, \mathrm{P}, \mathrm{B}$ ) consisting of elliptic curves, flat G-bundles, and B-fields. There are two choices of Lie groups G:

$$
\begin{equation*}
\left(E_{8} \times E_{8}\right) \rtimes \mathbb{Z}_{2} \quad \operatorname{Spin}(32) / \mathbb{Z}_{2} \tag{78}
\end{equation*}
$$

The moduli space $\mathcal{M}_{\mathrm{E}, \mathrm{G}}$ associated to the first two components ( $\mathrm{E}, \mathrm{P}$ ) of the above triples is (as described in [14]) a quasi-projective analytic space of complex dimension seventeen. The actual heterotic moduli space $\mathcal{M}_{\text {het }}^{\mathrm{G}}$ (described in [8]) fibers naturally as a holomorphic $\mathbb{C}^{*}$ fibration over $\mathcal{M}_{\mathrm{E}, \mathrm{G}}$.

In this context, the mathematical facts underlying the string duality can be summarized as follows. Each of the two moduli spaces $\mathcal{M}_{\mathrm{E}, \mathrm{G}}$ associated to the two choices of possible Lie groups on the heterotic side is naturally isomorphic to one of the corresponding Type II Mumford boundary divisors $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ from the F-theory side. Moreover, there exists a holomorphic identification between an open subset of $\mathcal{M}_{\text {het }}^{\mathrm{G}}$ neighboring the cusps of the $\mathbb{C}^{*}$-fibration over $\mathcal{M}_{\mathrm{E}, \mathrm{G}}$ and a special subset (of large complex structures) of $\mathcal{M}_{\mathrm{K} 3}$ which makes an open neighborhood of the corresponding boundary divisor. We refer the reader to [ 9,10 ] for further details and proofs.

The above holomorphic identification between the appropriate regions of $\mathcal{M}_{\mathrm{K} 3}$ and $\mathrm{M}_{\mathrm{het}}^{\mathrm{G}}$ is however defined Hodge-theoretically and therefore is not fully satisfactory from a geometer's point of view. As with any string duality, one would like to have a purely geometrical pattern that connects the spaces and structures appearing on the two sides of the duality correspondence.

Such a geometric connection, in the context of the above duality, has been known for some time, but only in the stable limit, i.e. on the boundary of the moduli spaces [1, 13]. As mentioned earlier, on the F-theory side this limit corresponds to stable K3 surfaces, whereas on the heterotic side it corresponds to $B=0$. Given a Type II stable K3 surface $V_{1} \cup V_{2}$, one can obtain the heterotic elliptic curve E by just taking the common curve $\mathrm{V}_{1} \cap \mathrm{~V}_{2}$ and the heterotic G-bundle can also be derived explicitly from the geometry of the rational surfaces $V_{1}$ and $V_{2}$.

It is natural to ask whether there exists a geometrical transformation underlying the Hodge theoretic duality away from the stable boundary, i.e. in the bulk of the two moduli spaces involved $\mathcal{M}_{\mathrm{K} 3}$ and $\mathcal{M}_{\mathrm{het}}^{\mathrm{G}}$, or at least in the large complex structure region [9]. The simplest case to consider is the restriction on the heterotic side to the $\mathrm{P}=0$ locus. From the Hodge-theoretic correspondence, one knows that this restriction corresponds on the F-theory side to K 3 surfaces with a special lattice polarization of type $\mathrm{M}=\mathrm{H} \oplus \mathrm{E}_{8} \oplus \mathrm{E}_{8}$. These are precisely the M-polarized K3 surfaces that form the main focus of this paper. Moreover, on the heterotic side, under the vanishing of the flat bundle, the B-field has the same properties as a second elliptic curve. Therefore this special case of the duality can be regarded as relating Hodge-theoretically M-polarized K3 surfaces to pairs of elliptic curves:

$$
\mathrm{X} \longleftrightarrow(\mathrm{E}, \mathrm{~B}) .
$$

This is the precisely the Hodge-theoretic identification from equation (10).
From this point of view, the transformation we described in Section 3, and for which we have computed explicit formulas in Section 4, provides the proper geometrical description of the F-theory/heterotic string duality for M-polarized K3 surfaces.

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[^0]:    *Department of Mathematics, Stanford University, Stanford, CA 94305. e-mail: clingher@math.stanford.edu
    ${ }^{\dagger}$ Department of Mathematics, University of Washington, Seattle, WA 98195. e-mail: doran@math.washington.edu

[^1]:    ${ }^{1}$ In fact, the use of this terminology for (2) is quite natural. The identification of Hodge structures given by (2) is a particular case of a more general Hodge-theoretic phenomenon which, surprisingly, was predicted by physics. In string theory this relationship is known as the F-Theory/Heterotic String Duality in eight dimensions. We refer the reader to Section 5 for a brief discussion of this aspect.
    ${ }^{2}$ An equivalent two-parameter family is known in the physics literature as the Morrison-Vafa family [25].

[^2]:    ${ }^{3}$ Such an approach via period computations have been taken in the physics literature [3, 6].

[^3]:    ${ }^{4}$ The function J is normalized such that $\mathrm{J}(i)=1$ and $\mathrm{J}\left(e^{\frac{2 \pi i}{3}}\right)=0$.

[^4]:    ${ }^{5}$ See, for instance, Chapter 1 of [12] for a proof of this result.

[^5]:    ${ }^{6} \mathrm{~A}$ root of $\mathrm{NS}(\mathrm{X})$ is an algebraic class of self-intersection -2 .

[^6]:    ${ }^{7}$ There is also an interesting toric reinterpretation of $\Theta_{1}$ and $\Theta_{2}$. They are induced from toric fibrations on a particular toric Fano three-fold by restriction to the anti-canonical hypersurface. These two toric fibrations are beautifully illustrated in Figure 1 of [5].
    ${ }^{8}$ This is also the reason why we decided to use the terms standard for $\Theta_{1}$ and alternate for $\Theta_{2}$.

[^7]:    ${ }^{9}$ We wish to thank Afsaneh Mehran for pointing out an innacuracy in a previous formulation of this definition.

[^8]:    ${ }^{10}$ Here by the term isomorphism we mean an isomorphism of M-polarized K3 surfaces.
    ${ }^{11}$ Recall that $J$ is normalized such that the two orbifold points of $\mathbb{H} / \operatorname{PSL}(2, \mathbb{Z})$ are mapped to 0 and 1 .

[^9]:    ${ }^{12}$ For explicit details, we refer the reader to sections 1.3.3 and 1.3.4 of [12].

