Note on a Geometric Isogeny of K3 Surfaces

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Abstract

The paper establishes a correspondence relating two specific classes of complex algebraic K3 surfaces. The first class consists of K3 surfaces polarized by the rank-sixteen lattice $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$. The second class consists of K3 surfaces obtained as minimal resolutions of double covers of the projective plane branched over a configuration of six lines. The correspondence underlies a geometric two-isogeny of K3 surfaces.

1 Geometric Two-Isogenies on K3 Surfaces

Let X be an algebraic K3 surface defined over the field of complex numbers. A *Nikulin (or symplectic) involution* on X is an analytic automorphism of order two $\Phi\colon \mathrm{X} \to \mathrm{X}$ such that $\Phi^*(\omega) = \omega$ for any holomorphic two-form ω on X. This type of involution has many interesting properties (see [17, 18]), amongst which the most important are: (a) the fixed locus of Φ consists of precisely eight distinct points, and (b) the surface Y obtained as the minimal resolution of the quotient X/Φ is a K3 surface. Equivalently, one can construct Y as follows. Blow up the eight fixed points on X obtaining a new surface $\widetilde{\mathrm{X}}$. The Nikulin involution Φ extends to an involution $\widetilde{\Phi}$ on $\widetilde{\mathrm{X}}$ which has as fixed locus the disjoint union of the eight resulting exceptional curves. The quotient $\widetilde{\mathrm{X}}/\widetilde{\Phi}$ is smooth and recovers the surface Y from above.

In the context of the above construction, one has a degree-two rational map $\mathrm{p}_\Phi\colon \mathrm{X} \dashrightarrow \mathrm{Y}$ with a branch locus given by eight disjoint rational curves (the even eight configuration in the sense of Mehran [16]). In addition, there is a push-forward morphism (see [12, 17])

$$(\mathrm{p}_\Phi)_*\colon \mathrm{H}^2(\mathrm{X}, \mathbb{Z}) \to \mathrm{H}_\mathrm{Y} \tag{1}$$

mapping into the orthogonal complement in $\mathrm{H}^2(\mathrm{Y}, \mathbb{Z})$ of the even eight curves. The metamorphosis of the surface X into Y is referred to in the literature as the *Nikulin construction*.

The most well-known class of Nikulin involutions is given by the *Shioda-Inose structures* [12, 17, 18]. These consist of Nikulin involutions that satisfy two additional requirements. The first condition asks for the surface Y to be Kummer. The second requirement asserts that the morphism (1) induces a Hodge isometry between the lattices of transcendental cocycles $\mathrm{T}_\mathrm{X}(2)$ and T_Y. An effective criterion for a particular K3 surface X to admit a Shioda-Inose structure was given by Morrison [17].

In this paper, we shall work with another class of Nikulin involutions: fiber-wise translations by a section of order two in a jacobian elliptic fibration. This class of involutions was discussed by Van Geemen and Sarti [9]. Let us be precise:

Definition 1.1. *A* Van Geemen-Sarti involution *is an automorphism* $\Phi_\mathrm{X}\colon \mathrm{X} \to \mathrm{X}$ *for which there exists a triple* $(\varphi_\mathrm{X}, S_1, S_2)$ *such that:*

(a) $\varphi_\mathrm{X}\colon \mathrm{X} \to \mathbb{P}^1$ *is an elliptic fibration on* X*,*

(b) S_1 *and* S_2 *are disjoint sections of* φ_X*,*

(c) S_2 *is an element of order two in the Mordell-Weil group* $\mathrm{MW}(\varphi_\mathrm{X}, \mathrm{S}_1)$*,*

(d) Φ_X *is the involution obtained by extending the fiber-wise translations by* S_2 *in the smooth fibers of* φ_X *using the group structure with neutral element given by* S_1*.*

Under the above conditions, one says that the triple $(\varphi_\mathrm{X}, S_1, S_2)$ *is* compatible *with the involution* Φ_X*.*

Any given Van Geemen-Sarti involution is, in particular, a Nikulin involution. One can naturally regard a Van Geemen-Sarti involution Φ_X as a fiber-wise two-isogeny between the original K3 surface X and the newly constructed K3 surface Y. Since Φ_X acts as a translation by an element of order two in each of the smooth fibers

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of $\varphi_{\rm X}$, there is a canonically induced elliptic fibration $\varphi_{\rm Y}\colon {\rm Y} \to \mathbb{P}^1$. The new fibration $\varphi_{\rm Y}$ carries two special sections, ${\rm S}_1'$ and ${\rm S}_2'$, as follows. The section ${\rm S}_1'$ is the image under the map ${\rm p}_{\Phi_{\rm X}}$ of the two sections ${\rm S}_1$ and ${\rm S}_2$ of φ. The section ${\rm S}_2'$ is the image under ${\rm p}_{\Phi_{\rm X}}$ of the divisor on X obtained by compactifying the curve obtained by taking the union of remaining two order-two points in the smooth fibers of $\varphi_{\rm X}$. The two sections ${\rm S}_1'$ and ${\rm S}_2'$ are disjoint and ${\rm S}_2'$ represents an element of order two in the Mordell-Weil group ${\rm MW}(\varphi_{\rm Y}, {\rm S}_1')$. Then, by standard results [21], the fiber-wise translations by the order-two section ${\rm S}_2'$ extend to determine an involution $\Phi_{\rm Y}\colon {\rm Y} \to {\rm Y}$ which is a Van Geemen-Sarti involution on Y.

The same procedure applied initially to the involution $\Phi_{\rm Y}$ recovers the K3 surface X together with the triple $(\varphi_{\rm X}, {\rm S}_1, {\rm S}_2)$ and the involution $\Phi_{\rm X}$. One has therefore the following commutative diagram.

$$\begin{tikzcd}
{\rm Y} \arrow[loop left, "\Phi_{\rm Y}"] \arrow[rr, dashed, bend right=10, "{\rm p}_{\Phi_{\rm Y}}"'] \arrow[dr, "\varphi_{\rm Y}"'] & & {\rm X} \arrow[loop right, "\Phi_{\rm X}"] \arrow[ll, dashed, bend right=10, "{\rm p}_{\Phi_{\rm X}}"'] \arrow[dl, "\varphi_{\rm X}"] \\
& \mathbb{P}^1 &
\end{tikzcd} \tag{2}$$

The rational maps ${\rm p}_{\Phi_{\rm X}}$ and ${\rm p}_{\Phi_{\rm Y}}$ are of degree two. Hence, $({\rm p}_{\Phi_{\rm X}}, {\rm p}_{\Phi_{\rm Y}})$ can be be seen as forming *a pair of dual two-isogenies*[1] between the surfaces X and Y.

Note that, by standard results [2, 13, 19, 21] on elliptic fibrations on K3 surfaces, once a K3 surface X is endowed with an elliptic fibration $\varphi_{\rm X}\colon {\rm X} \to \mathbb{P}^1$ with two disjoint sections S_1 and S_2, the condition for the triple $(\varphi_{\rm X}, S_1, S_2)$ to define a Van Geemen-Sarti involution can be formulated entirely in terms of cohomology. One first considers the cohomology class F of the fiber of $\varphi_{\rm X}$ as well as the class of S_1. These classes span a primitive lattice embedding ${\rm H} \hookrightarrow {\rm NS}({\rm X})$. In fact, the Neron-Severi lattice factors into an orthogonal direct product

$$ {\rm NS}({\rm X}) \ = \ {\rm H} \oplus \mathcal{W} \tag{3}$$

where $\mathcal{W}$ is a negative definite lattice of rank ${\rm p}_{\rm X} - 2$. Denote by $\mathcal{W}_{\rm root}$ the sub-lattice spanned by the roots of $\mathcal{W}$. This sub-lattice is actually spanned by the irreducible components of the singular fibers of $\varphi_{\rm X}$ not meeting S_1. As proved by Shioda [21], one has then an isomorphism of abelian groups:

$$ {\rm MW}(\varphi, S_1) \ \simeq \ \mathcal{W}/\mathcal{W}_{\rm root}. \tag{4}$$

Let $S_2^w \in \mathcal{W}$ be the image of the class S_2 under the projection ${\rm NS}({\rm X}) \to \mathcal{W}$ associated with the factorization (3). Note that $S_2^w = S_2 - S_1 - 2F$ and S_2^w has self-intersection -4. One obtains the following criterion:

Proposition 1.2. *The triple $(\varphi_{\rm X}, S_1, S_2)$ defines a Van Geemen-Sarti involution $\Phi\colon {\rm X} \to {\rm X}$ if and only if $2S_2^w \in \mathcal{W}_{\rm root}$.*

We also note that a Van Geemen-Sarti involution on a K3 surface X is equivalent to a pseudo-ample polarization by the rank-ten lattice H$\oplus$N where N is the rank-eight Nikulin lattice as defined by [17]. The Nikulin construction defines a natural involution on the ten-dimensional moduli space of ${\rm H} \oplus {\rm N}$-polarized K3 surfaces.

2 Outline of the Paper

In this work we construct Van Geemen-Sarti involutions on two specific classes of algebraic K3 surfaces. The first class consists of algebraic K3 surfaces X endowed with a pseudo-ample lattice polarization:

$$ i\colon {\rm H} \oplus {\rm E}_7 \oplus {\rm E}_7 \ \hookrightarrow \ {\rm NS}({\rm X}). $$

This polarization structure is equivalent geometrically to a jacobian elliptic fibration on X that has two singular fibers of Kodaira type III* or higher. For details regarding the concept of lattice polarization, we refer the reader to Dolgachev's paper [7] or the previous work [2] of the authors. For the purposes of this paper, an additional genericity condition is introduced (Definition 4.3 of Section 4).

The second class of K3 surfaces consists of a special collection of double sextic surfaces - we consider surfaces Z obtained as minimal resolutions of double covers of the projective plane $\mathbb{P}^2$ branched over a configuration $\mathcal{L}$ of six distinct lines. The lines are assumed to be so located that no three of them pass through the same common point. We also introduce an explicit condition for genericity of $\mathcal{L}$, as given by Definition 3.4 of Section 3.

The main results of this paper are as follows:

[1]The rational maps ${\rm p}_{\Phi_{\rm X}}$ and ${\rm p}_{\Phi_{\rm Y}}$ are not isogenies in the traditional sense (finite and etale morphism). The first author would like to thank Mohan Kumar for pointing out this fact.

Theorem 2.1. *The K3 surfaces* Z *and* X *introduced above carry canonically-defined Van Geemen-Sarti involutions, denoted* Φ_{Z} *or* Φ_{X}*, respectively.*

Theorem 2.2. *If genericity is assumed on both sides, then one has a bijective correspondence:*

$$(\mathrm{Z}, \mathcal{L}) \longleftrightarrow (\mathrm{X}, i) \tag{5}$$

between the two classes of surfaces, with the two K3 surfaces involved being related by a pair of dual geometric two-isogenies

$$\begin{tikzcd} \mathrm{Z} \arrow[loop left, "\Phi_{\mathrm{Z}}"] \arrow[r, dashed, bend right, "p_{\Phi_{\mathrm{Z}}}"'] & \mathrm{X} \arrow[l, dashed, bend right, "p_{\Phi_{\mathrm{X}}}"'] \arrow[loop right, "\Phi_{\mathrm{X}}"] \end{tikzcd} \tag{6}$$

as described in Section 1.

Theorem 2.2 remains true if the genericity conditions are removed. However, in that case, in order to account for all possible $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$-polarized K3 surfaces (X, i), one has to allow for surfaces Z to degenerate to situations when at least three of the six lines in the configuration $\mathcal{L}$ are meeting at a point. The proofs associated with these degenerate cases will be included in a subsequent paper.

The present work builds on ideas from paper [3], where the authors have shown that dual pairs of geometric two-isogenies as in Section 1 relate K3 surfaces X polarized by the rank-eighteen lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$ to Kummer surfaces Z associated to a cartesian product of two elliptic curves. In this situation, the Van Geemen-Sarti involution Φ_{X} is a Shioda-Inose structure. This case was also considered by T. Shioda in [22]. In an earlier work motivated by arithmetic considerations, B. Van Geemen and J. Top [10] have presented a particular variant of the $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$ case - an isogeny between a one-dimensional family of K3 surfaces polarized by $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \mathrm{A}_1(2)$ and Kummer surfaces associated to a cartesian product a pair of of two-isogeneous elliptic curves.

In the appendix section to paper [8] by F. Galluzzi and G. Lombardo, I. Dolgachev argued that any K3 surface Z with the Neron-Severi lattice $\mathrm{NS}(\mathrm{X})$ isomorphic to $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$ carries a canonical Shioda-Inose structure and the associated Nikulin construction leads to a Kummer surface associated with the Jacobian $\mathrm{Jac}(\mathrm{C})$ of a genus-two curve. This situation appears here as a particular case of Theorem 2.2. Polarized K3 surfaces (X, i) for which the lattice polarization extends to $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$ correspond, under (5), to configurations $\mathcal{L}$ in which the six lines are tangent to a common conic. An explicit formula for determining the $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$-polarized K3 surface X has been given by A. Kumar [15].

We shall also note that the geometric setting of Theorems 2.1 and 2.2 is ideal for performing explicit Kuga-Satake type constructions [14] without relying on period computations. In the companion paper [4], the authors use the results of this work in order to give a full classification of the K3 surfaces polarized by the lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$ in terms of Siegel modular forms.

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3 Double Covers of the Projective Plane

Let $\mathcal{L} = \{\mathrm{L}_1, \mathrm{L}_2, \cdots \mathrm{L}_6\}$ be a configuration[2] of six distinct lines in $\mathbb{P}^2$. We shall assume that no three of the six lines are concurrent. Denote by q_{ij}, with $1 \leq i < j \leq 6$, the fifteen resulting intersection points. Let $\rho \colon \mathrm{R} \to \mathbb{P}^2$ be the blow-up of the projective plane at the points q_{ij} and denote by $\mathrm{L}'_1, \mathrm{L}'_2, \cdots \mathrm{L}'_6$ the rational curves in R obtained as the proper transforms of the six lines $\mathrm{L}_1, \mathrm{L}_1, \cdots \mathrm{L}_6$. Since

$$\frac{1}{2} \sum_{i=1}^{6} \mathrm{L}'_i \in \mathrm{NS}(\mathrm{R}),$$

one has that there exists a double cover $\pi \colon \mathrm{Z} \to \mathrm{R}$ branched over $\mathrm{L}'_1, \mathrm{L}'_2, \cdots \mathrm{L}'_6$. The surface Z is a smooth algebraic K3 surface of Picard rank sixteen or higher. In this section, we prove that the K3 surface Z so defined carries a canonical Van Geemen-Sarti involution denoted Φ_{Z}. Moreover, the K3 surface W resulting from the Nikulin construction associated to the involution Φ_{Z} is endowed with a canonical $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$ polarization.

In order to define the involution Φ_{Z}, we follow the guidelines of Section 1. We introduce first an underlying elliptic fibration $\varphi_{\mathrm{Z}} \colon \mathrm{Z} \to \mathbb{P}^1$ with two sections. We then show that fiber-wise translation by the second section determines a Van Geemen-Sarti involution.

[2] The arguments of this section require a choice of labeling for the configuration of lines $\mathcal{L}$. However, as explained in Remark 3.9, the final outcome is independent of the labeling.

3.1 A Special Elliptic Fibration on Z

By construction, the surface Z comes endowed with a non-symplectic involution $\sigma\colon \mathrm{Z} \to \mathrm{Z}$. The fixed locus of σ is given by six rational curves $\Delta_1, \Delta_2, \cdots \Delta_6$, representing the ramification locus of the double cover map $\pi\colon \mathrm{Z} \to \mathrm{R}$. We denote by E_{ij} the fifteen exceptional curves on the surface R and by G_{ij} their respective strict transforms on Z. Set also $\mathrm{T} = (\pi \circ \rho)^* \mathrm{H}$ where H is a hyperplane divisor on $\mathbb{P}^2$.

The following divisor on R will prove to be instrumental:

$$\mathrm{D} \ = \ 5\rho^*(\mathrm{H}) - 3\mathrm{E}_{13} - 2\left(\mathrm{E}_{14} + \mathrm{E}_{25} + \mathrm{E}_{26}\right) - \left(\mathrm{E}_{24} + \mathrm{E}_{35} + \mathrm{E}_{36} + \mathrm{E}_{56}\right). \tag{7}$$

The linear system $|\mathrm{D}|$ corresponds to curves of degree five in $\mathbb{P}^2$ passing through the four points $q_{24}, q_{35}, q_{36}, q_{56}$, having double points at q_{14}, q_{25}, q_{26} and a triple point at q_{13}.

Proposition 3.1. *One has $h^1(\mathrm{R}, \mathrm{D}) = 2$. The pencil $|\mathrm{D}|$ is base-point free and its generic member is a smooth rational curve. The induced morphism*

$$\varphi_{|\mathrm{D}|}\colon \mathrm{R} \to \mathbb{P}^1 \tag{8}$$

is a ruling.

Proof. Note that it suffices to prove the above statement assuming that R is the blow-up of $\mathbb{P}^2$ at the eight points q_{13}, q_{14}, q_{24}, q_{25}, q_{26}, q_{35}, q_{36}, q_{56}. The eight points in question are in *almost general position* (as defined in [6]). Then, as proved in [6, 5], the rational surface R is a generalized Del Pezzo surface with the anticanonical line bundle $-\mathrm{K}_\mathrm{R}$ having the big and nef properties.

Since $\mathrm{D}^2 = 0$ and $\mathrm{D} \cdot \mathrm{K}_\mathrm{R} = -2$, one obtains, via the Riemann-Roch formula:

$$h^0(\mathrm{R}, \mathrm{D}) - h^1(\mathrm{R}, \mathrm{D}) + h^2(\mathrm{R}, \mathrm{D}) \ = \ 2.$$

But $h^2(\mathrm{R}, \mathrm{D}) = h^0(\mathrm{R}, \mathrm{K}_\mathrm{R} - \mathrm{D}) = 0$. In particular $h^0(\mathrm{R}, \mathrm{D}) \geq 2$.

Let C be the unique conic in $\mathbb{P}^2$ passing through the five points $q_{13}, q_{14}, q_{25}, q_{26}, q_{56}$. The conic C is smooth. Denote by C' the rational curve on R obtained as the proper transform of C. Then:

$$\mathrm{L}_1' + \mathrm{L}_2' + \mathrm{L}_3' + \mathrm{C}' \tag{9}$$

is a special member of $|\mathrm{D}|$. As $\mathrm{D} \cdot \mathrm{L}_1' = \mathrm{D} \cdot \mathrm{L}_2' = \mathrm{D} \cdot \mathrm{L}_3' = \mathrm{D} \cdot \mathrm{C}' = 0$, if $|\mathrm{D}|$ were to have base points, then the entire divisor (9) would be part of the base locus. This would imply $h^0(\mathrm{R}, \mathrm{D}) = 1$, contradicting the above estimation. The pencil $|\mathrm{D}|$ has therefore no base points. By Bertini's Theorem, the generic member of $|\mathrm{D}|$ is smooth and irreducible, and by the degree-genus formula we obtain that the generic member is a smooth rational curve.

It remains to be shown that $h^1(\mathrm{R}, \mathrm{D}) = 0$. One has $h^1(\mathrm{R}, \mathrm{D}) = h^1(\mathrm{R}, \mathrm{K}_\mathrm{R} - \mathrm{D})$. But $(\mathrm{D} - \mathrm{K}_\mathrm{R})^2 = 5$ and since both D and $-\mathrm{K}_\mathrm{R}$ are nef, one has that $\mathrm{D} - \mathrm{K}_\mathrm{R}$ is nef. By Ramanujam's Vanishing Theorem [20], one obtains $h^1(\mathrm{R}, \mathrm{K}_\mathrm{R} - \mathrm{D}) = 0$. □

Note that the lines L_5' and L_6' are disjoint sections of the ruling (8), while L_4' is a bi-section. The entire construction lifts then to the level of the K3 surface Z where one obtains:

Lemma 3.2. *The pull-back under the double cover $\rho\colon \mathrm{Z} \to \mathrm{R}$ of the linear system associated to (7), i.e.*

$$|\ 5\mathrm{T} - 3\mathrm{G}_{13} - 2\left(\mathrm{G}_{14} + \mathrm{G}_{25} + \mathrm{G}_{26}\right) - \left(\mathrm{G}_{24} + \mathrm{G}_{35} + \mathrm{G}_{36} + \mathrm{G}_{56}\right)\ |,$$

determines an elliptic fibration $\varphi_\mathrm{Z}\colon \mathrm{Z} \to \mathbb{P}^1$ with the smooth rational curves Δ_5 and Δ_6 as distinct sections.

The smooth fibers of φ_Z appear as double covers of the smooth rational curves of the ruling (8), with the branch locus given by the four points of intersection with L_4', L_5' and L_6'.

Let us discuss the basic properties of the elliptic fibration φ_Z. We shall differentiate between the following two possibilities:

(a) the six lines of the configuration $\mathcal{L}$ are tangent to a common smooth conic in $\mathbb{P}^2$,

(b) there is no smooth conic tangent to all the six lines of the configuration $\mathcal{L}$.

In situation (a) the surface Z is a Kummer surface associated to the Jacobian of a genus-two curve. We shall refer to such a six-line configuration as **special** or **Kummer**. If the six-line configuration $\mathcal{L}$ is in situation (b), we shall refer to it as **non-special** or **non-Kummer**.

Proposition 3.3. *If the six-line configuration $\mathcal{L}$ is non-Kummer then the elliptic fibration $\varphi_\mathrm{Z}\colon \mathrm{Z} \to \mathbb{P}^1$ has a singular fiber of type I_4^*. This special fiber becomes of type I_5^* in the Kummer case.*

Proof. Let C and C′ be the curves defined within the proof of Proposition 3.1. Note that, as a consequence of the classical theorems of Pascal and Brianchon, the six-line configuration $\mathcal{L}$ is Kummer if and only if the conic curve C passes through q_{34}.

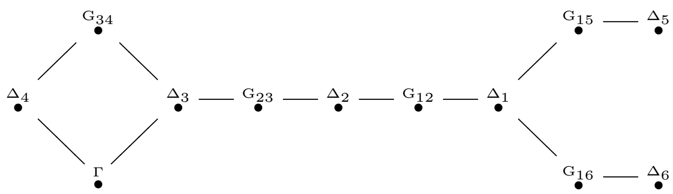
If the configuration $\mathcal{L}$ is non-Kummer, then under the double cover map $\pi\colon \mathrm{Z} \to \mathrm{R}$, one has $\pi^*\mathrm{C}' = \Gamma$ where Γ is smooth rational curve. The involution σ maps the curve Γ to itself, with two fixed points located at the points of intersection with Δ_3 and Δ_4, respectively.

However, if the six-line configuration $\mathcal{L}$ is Kummer, then one has:

$$\pi^*\mathrm{C}' = \Gamma_1 + \Gamma_2$$

where Γ_1, Γ_2 are two disjoint smooth rational curves. The two curves Γ_1, Γ_2 are mapped one onto the other by the involution σ.

One obtains in this way a special configuration of rational curves on the K3 surface Z. If $\mathcal{L}$ is not Kummer, we have the following dual diagram:



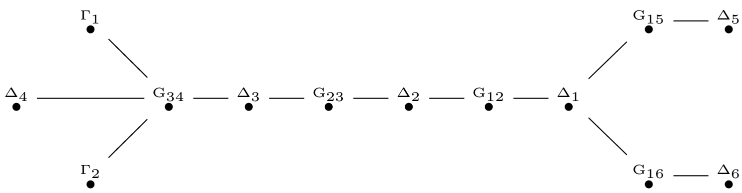
(10)

The special divisor:

$$\mathrm{G}_{34} + \Gamma + 2\,(\Delta_3 + \mathrm{G}_{23} + \Delta_2 + \mathrm{G}_{12} + \Delta_1) + \mathrm{G}_{15} + \mathrm{G}_{16}$$

is the pull-back on Z of the special quintic curve (9) and is a singular fiber of Kodaira type I_4^* for the elliptic fibration φ_{Z}. The diagram above also includes the two sections Δ_5 and Δ_6, as well as the bi-section Δ_4.

If the configuration $\mathcal{L}$ is Kummer, the dual diagram of rational curves gets modified as follows:



(11)

The pull-back to Z of the quintic curve (9) is now the divisor:

$$\Gamma_1 + \Gamma_2 + 2\,(\mathrm{G}_{34} + \Delta_3 + \mathrm{G}_{23} + \Delta_2 + \mathrm{G}_{12} + \Delta_1) + \mathrm{G}_{15} + \mathrm{G}_{16}$$

which forms a singular fiber of type I_5^* for the elliptic fibration φ_{Z}. ☐

In addition to the special singular fiber of Proposition 3.3, the elliptic fibration φ_{Z} carries additional singular fibers. In the generic situation, one has six additional I_2 fibers plus two singular fibers of type I_1 in the non-Kummer case, or a single fiber of type I_1 in the Kummer case, respectively. This genericity condition can be made precise. Consider the following divisors on the surface R:

$$\begin{aligned}
\Lambda_1 &= 5\rho^*(\mathrm{H}) - 3\mathrm{E}_{13} - 2\,(\mathrm{E}_{14} + \mathrm{E}_{25} + \mathrm{E}_{26}) - (\mathrm{E}_{24} + \mathrm{E}_{35} + \mathrm{E}_{36} + \mathrm{E}_{45} + \mathrm{E}_{56})\\
\Lambda_2 &= 4\rho^*(\mathrm{H}) - 2\,(\mathrm{E}_{13} + \mathrm{E}_{14} + \mathrm{E}_{25}) - (\mathrm{E}_{24} + \mathrm{E}_{26} + \mathrm{E}_{35} + \mathrm{E}_{36} + \mathrm{E}_{56})\\
\Lambda_3 &= 3\rho^*(\mathrm{H}) - 2\mathrm{E}_{13} - (\mathrm{E}_{14} + \mathrm{E}_{24} + \mathrm{E}_{25} + \mathrm{E}_{26} + \mathrm{E}_{35} + \mathrm{E}_{56})\\
\Lambda_4 &= 2\rho^*(\mathrm{H}) - (\mathrm{E}_{13} + \mathrm{E}_{14} + \mathrm{E}_{25} + \mathrm{E}_{26} + \mathrm{E}_{35})\\
\Lambda_5 &= \rho^*(\mathrm{H}) - (\mathrm{E}_{13} + \mathrm{E}_{25})\\
\Lambda_6 &= \mathrm{E}_{46}\ .
\end{aligned}$$

These classes have intersection numbers: $\Lambda_i^2 = \Lambda_i \cdot \mathrm{K}_{\mathrm{R}} = (\mathrm{D} - \Lambda_i)^2 = (\mathrm{D} - \Lambda_i) \cdot \mathrm{K}_{\mathrm{R}} = -1$. In addition, one has that:

$$h^0(\mathrm{R}, \Lambda_i) = h^0(\mathrm{R}, \mathrm{D} - \Lambda_i) = 1$$

for all indices i with $1 \leq i \leq 6$.

Definition 3.4. *The six-line configuration $\mathcal{L}$ is called generic if, for all $1 \leq i \leq 6$, the linear systems $|\Lambda_i|$ and $|\mathrm{D} - \Lambda_i|$ each consist of a single smooth rational curve.*

Assuming then a generic six-line configuration, one obtains that, for each $1 \leq i \leq 6$, the pull-back under the double-cover map $\pi\colon \mathrm{Z} \to \mathrm{R}$ of the two rational curves associated with Λ_i and $\mathrm{D} - \Lambda_i$ provides a pair of rational curves on the K3 surface Z that form an I_2 singular fiber for the elliptic fibration φ_{Z}.

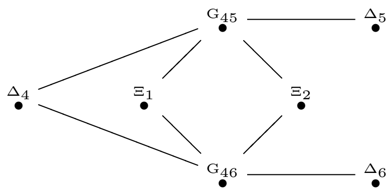
Remark 3.5. *Let us provide one example of non-generic situation. Consider the case when there exists an irreducible quintic curve in $\mathbb{P}^2$ with a triple point at q_{13}, three double points at q_{14}, q_{25}, q_{26} and passing through q_{24}, q_{35}, q_{36}, q_{45}, q_{46}, q_{56}. Then, the linear system*

$$\mid 5\rho^*(\mathrm{H}) - 3\mathrm{E}_{13} - 2\left(\mathrm{E}_{14} + \mathrm{E}_{25} + \mathrm{E}_{26}\right) - \left(\mathrm{E}_{24} + \mathrm{E}_{35} + \mathrm{E}_{36} + \mathrm{E}_{45} + \mathrm{E}_{46} + \mathrm{E}_{56}\right) \mid$$

contains a single rational curve M. *The curve* M *does not meet* L'_i *for any* $1 \leq i \leq 6$ *and* $\pi^*\mathrm{M} = \Xi_1 + \Xi_2$ *where* Ξ_1, Ξ_2 *are disjoint rational curves on the K3 surface* Z. *The divisor:*

$$\Xi_1 + \mathrm{G}_{45} + \Xi_2 + \mathrm{G}_{46}$$

is then a fiber of type I_4 *in the elliptic fibration* φ_{Z}.



Theorem 3.6. *Let* Z *be a K3 surface associated with a generic six-line configuration* $\mathcal{L}$. *The section* Δ_6, *interpreted as an element of the Mordell-Weil group* $\mathrm{MW}(\varphi_{\mathrm{Z}}, \Delta_5)$, *has order two. Fiber-wise translations by* Δ_6 *in the smooth fibers of* φ_{Z} *extend to form a Van Geemen-Sarti involution* $\Phi_{\mathrm{Z}}\colon \mathrm{Z} \to \mathrm{Z}$.

Proof. In order to prove the above statement, one needs to verify the condition of Proposition 1.2. We shall perform this verification here for the case of a non-Kummer line configuration $\mathcal{L}$. One can check that similar computation holds in the case of a Kummer configuration.

Assume therefore that the six-line configuration $\mathcal{L}$ is non-Kummer. Let F be the cohomology class of the fiber in φ_{Z}, i.e.

$$\mathrm{F} = 5\mathrm{T} - 3\mathrm{G}_{13} - 2\left(\mathrm{G}_{14} + \mathrm{G}_{25} + \mathrm{G}_{26}\right) - \left(\mathrm{G}_{24} + \mathrm{G}_{35} + \mathrm{G}_{36} + \mathrm{G}_{56}\right).$$

One has then the orthogonal direct product

$$\mathrm{NS}(\mathrm{Z}) = \langle \mathrm{F}, \Delta_5 \rangle \oplus \mathcal{W}.$$

The root sub-lattice $\mathcal{W}_{\mathrm{root}} \subset \mathcal{W}$ is spanned by the cohomology classes associated with the irreducible components of the singular fibers of φ_{Z} not meeting Δ_5. The factorization of $\mathcal{W}_{\mathrm{root}}$ includes then the following:

$$\langle \Upsilon_1 \rangle \oplus \langle \Upsilon_2 \rangle \oplus \langle \Upsilon_3 \rangle \oplus \ \cdots \ \oplus \langle \Upsilon_6 \rangle \oplus \langle \Upsilon_7,\ \Upsilon_8,\ \Delta_3,\ \mathrm{G}_{23},\ \Delta_2,\ \mathrm{G}_{12},\ \Delta_1,\ \mathrm{G}_{16} \rangle$$

where $\Upsilon_i = \pi^*\Lambda_i$, for $1 \leq i \leq 6$, and

$$\Upsilon_7 = \mathrm{G}_{34}\ , \quad \Upsilon_8 = \Gamma = 2\mathrm{T} - \left(\mathrm{G}_{13} + \mathrm{G}_{14} + \mathrm{G}_{25} + \mathrm{G}_{26} + \mathrm{G}_{56}\right).$$

The six classes $\Upsilon_1,\ \Upsilon_2\ \cdots\ \Upsilon_6$ represent the rational curves in the I_2 singular fibers which do not meet Δ_5. In this context, one has:

$$\Delta_6^w \ = \ \Delta_6 - \Delta_5 - 2\mathrm{F} \ = \ -\left(\Delta_3 + \mathrm{G}_{23} + \Delta_2 + \mathrm{G}_{12} + \Delta_1 + \mathrm{G}_{16}\right) - \frac{1}{2}\left(\Upsilon_1 + \Upsilon_2 + \cdots + \Upsilon_7 + \Upsilon_8\right) \ .$$

Hence $2\Delta_6^w \in \mathcal{W}_{\mathrm{root}}$. □

The above theorem remains true if one removes the genericity condition. Proofs for the non-generic cases will however not be included here.

Remark 3.7. *Note that, on each of the smooth fibers of the elliptic fibration* φ_{Z}, *one has four distinct points given by the intersections with* Δ_5, Δ_6 *and* Δ_4. *Consider the elliptic curve group law with center at* Δ_5. *The intersection with* Δ_6 *provides a special point of order two. The remaining two points of order two are located at the intersections with* Δ_4.

3.2 Properties of the Involution Φ_Z

Let us discuss the Nikulin construction associated with the Van Geemen-Sarti involution Φ_Z. Note that, by construction, the involution Φ_Z commutes with the non-symplectic involution σ. A second important feature is given by the fixed locus $\{p_1, p_2, \cdots p_8\}$ of Φ_Z. For simplicity of exposition, we shall assume that the six-line configuration is generic.

Consider the case of a non-Kummer configuration $\mathcal{L}$. The rational curves Δ_4, Δ_3, G_{23}, Δ_2, G_{21}, Δ_1 get mapped to themselves under Φ_Z and each of these six curves contains two of the fixed points. We denote by p_2, p_3, p_4, p_5 the following four intersection points:

$$\Delta_1 \cap G_{21}, \ \Delta_2 \cap G_{21}, \ \Delta_2 \cap G_{23}, \ \Delta_3 \cap G_{23}.$$

There are two additional fixed points p_1, p_6 on Δ_1, Δ_3, respectively. The last two points p_7, p_8 are given by the singularities of the I_1 fibers. Note that p_7, p_8 lie on Δ_4.

If the six-line configuration $\mathcal{L}$ is Kummer, then the above set-up of the fixed locus $\{p_1, p_2, \cdots p_8\}$ gets modified slightly. One obtains p_6 as the intersection $\Delta_3 \cap G_{34}$, the point p_7 lies on G_{34}, and p_8 is the singularity of the single I_1 fiber. It is still the case that p_7, p_8 lie on Δ_4.

Denote by W the K3 surface obtained from the Nikulin construction associated to the involution Φ_Z. By the general framework presented in Section 1, the surface W inherits a Jacobian elliptic fibration φ_W. The singular fiber types of the fibration φ_W are: $I_8^* + 2 \times I_2 + 6 \times I_1$ in the non-Kummer case and $I_{10}^* + I_2 + 6 \times I_1$ in the Kummer case, respectively. In order to be precise, consider the case of a non-Kummer configuration. In such a situation, the six rational curves

$$\Delta_1, \ \Delta_2, \ \Delta_3, \ \Delta_4, \ G_{12}, \ G_{23} \tag{12}$$

are mapped to themselves by the involution φ_Z. The three pairs of disjoint curves

$$(\Delta_5, \Delta_6), \ (G_{15}, G_{16}), \ (G_{34}, \Gamma) \tag{13}$$

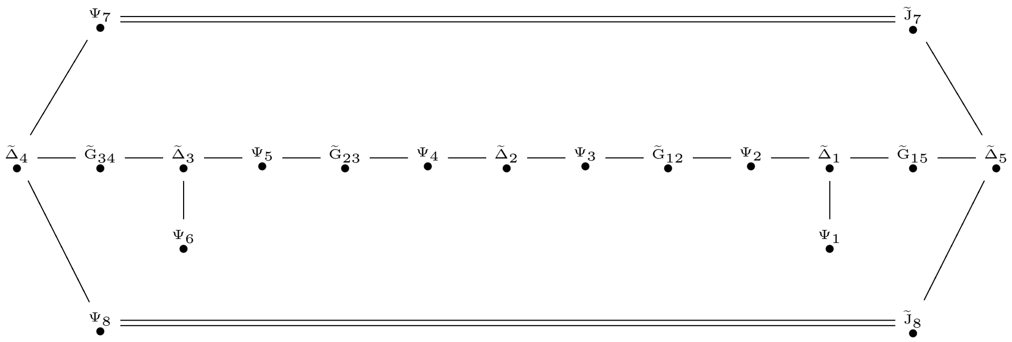
are exchanged by φ_Z. We denote by

$$\widetilde{\Delta}_1, \ \widetilde{\Delta}_2, \ \widetilde{\Delta}_3, \ \widetilde{\Delta}_4, \ \widetilde{G}_{12}, \ \widetilde{G}_{23}, \ \widetilde{\Delta}_5, \ \widetilde{G}_{15}, \ \widetilde{\Gamma}$$

the nine rational curves on the surface W that arise as push-forward of the curves in (12) and (13). Let also

$$\Psi_1, \ \Psi_2, \ \Psi_3, \Psi_4, \ \Psi_5, \ \Psi_6, \ \Psi_7, \ \Psi_8 \tag{14}$$

be the eight exceptional curves associated with the fixed locus. Two additional rational curves $\widetilde{J}_7$, $\widetilde{J}_8$ appear from resolving the quotients of singular curves of the I_1 fibers of φ_Z with singularities at p_7 and p_8. One obtains therefore nineteen rational curves on W that intersect according to the following dual diagram.



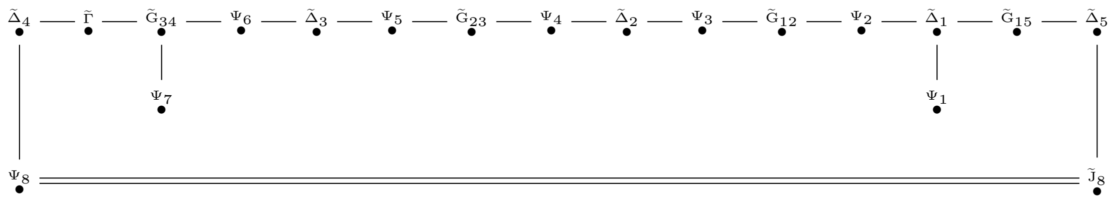
(15)

The I_8^* singular fiber of φ_W is given by:

$$\widetilde{G}_{34} + \Psi_6 + 2\left(\widetilde{\Delta}_3 + \Psi_5 + \widetilde{G}_{23} + \Psi_4 + \widetilde{\Delta}_2 + \Psi_3 + \widetilde{G}_{12} + \Psi_2 + \widetilde{\Delta}_1\right) + \Psi_1 + \widetilde{G}_{15},$$

whereas $\Psi_j + \widetilde{J}_j$ with $j = 7, 8$ are fibers of type I_2. The rational curves $\widetilde{\Delta}_4$, $\widetilde{\Delta}_5$ are sections in φ_W.

If the six-line configuration is Kummer, then, using a notation along the same lines as before, the nineteen rational curves intersect in a slightly different manner.



(16)

In diagram (16), $\widetilde{\Gamma}$ represents the push-forward of the rational curves Γ_1 and Γ_2. The I_{10}^* singular fiber of φ_{W} is given by:

$$\widetilde{\Gamma} + \Psi_7 + 2\left(\widetilde{\mathrm{G}}_{34} + \Psi_6 + \widetilde{\Delta}_3 + \Psi_5 + \widetilde{\mathrm{G}}_{23} + \Psi_4 + \widetilde{\Delta}_2 + \Psi_3 + \widetilde{\mathrm{G}}_{12} + \Psi_2 + \widetilde{\Delta}_1\right) + \Psi_1 + \widetilde{\mathrm{G}}_{15},$$

with $\Psi_8 + \widetilde{\mathrm{J}}_8$ being a fiber of type I_2. The rational curves $\widetilde{\Delta}_4$, $\widetilde{\Delta}_5$ are still sections in φ_{W}.

Theorem 3.8. *The K3 surface* W *associated to the involution* Φ_{Z} *by the Nikulin construction carries a canonical pseudo-ample lattice polarization*

$$i\colon \mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7 \hookrightarrow \mathrm{NS}(\mathrm{W}). \tag{17}$$

If the six-line configuration $\mathcal{L}$ *is Kummer, the lattice polarization* (17) *extends canonically to a polarization by the rank-seventeen lattice* $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$.

Proof. We use the notation from diagrams (15) and (16). In the case of a Kummer six-line configuration, the primitive embedding of the orthogonal direct product $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$ in $\mathrm{NS}(\mathrm{W})$ is given by:

$$\begin{aligned}
\mathrm{H} &= \langle\, \widetilde{\Delta}_2,\ \Psi_7 + 2\widetilde{\Delta}_4 + 3\widetilde{\mathrm{G}}_{34} + 4\widetilde{\Delta}_3 + 2\Psi_6 + 3\Psi_5 + 2\widetilde{\mathrm{G}}_{23} + \Psi_4 \,\rangle \\
\mathrm{E}_7 &= \langle\, \Psi_7, \widetilde{\Delta}_4, \widetilde{\mathrm{G}}_{34}, \widetilde{\Delta}_3, \Psi_6, \Psi_5, \widetilde{\mathrm{G}}_{23} \,\rangle, \\
\mathrm{E}_7 &= \langle\, \widetilde{\mathrm{J}}_8, \widetilde{\Delta}_5, \widetilde{\mathrm{G}}_{15}, \widetilde{\Delta}_1, \Psi_1, \Psi_2, \widetilde{\mathrm{G}}_{12} \,\rangle.
\end{aligned}$$

In the special case, one has a copy $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$ naturally embedded in $\mathrm{NS}(\mathrm{W})$ as:

$$\begin{aligned}
\mathrm{H} &= \langle\, \widetilde{\Delta}_2,\ 2\widetilde{\Delta}_4 + 4\widetilde{\Gamma} + 6\widetilde{\mathrm{G}}_{34} + 3\Psi_7 + 5\Psi_6 + 4\widetilde{\Delta}_3 + 3\Psi_5 + 2\widetilde{\mathrm{G}}_{23} + \Psi_4 \,\rangle \\
\mathrm{E}_8 &= \langle\, \widetilde{\Delta}_4, \widetilde{\Gamma}, \widetilde{\mathrm{G}}_{34}, \Psi_7, \Psi_6, \widetilde{\Delta}_3, \Psi_5, \widetilde{\mathrm{G}}_{23} \,\rangle, \\
\mathrm{E}_7 &= \langle\, \widetilde{\mathrm{J}}_8, \widetilde{\Delta}_5, \widetilde{\mathrm{G}}_{15}, \widetilde{\Delta}_1, \Psi_1, \Psi_2, \widetilde{\mathrm{G}}_{12} \,\rangle.
\end{aligned}$$

□

Remark 3.9. *Let us note that, by its construction, the Van Geemen-Sarti involution* Φ_{Z} *acts as a transposition on the set of six rational curves*

$$\{\ \Delta_1,\ \Delta_2,\ \Delta_3,\ \Delta_4,\ \Delta_5,\ \Delta_6\ \}$$

of the K3 surface Z. *The first four curves* Δ_1, Δ_2, Δ_3, Δ_4 *get mapped to themselves under* Φ_{Z}, *whereas* Δ_5, Δ_6 *are interchanged. One concludes that for any re-labeling of the defining six-line configuration* $\mathcal{L} = \{\mathrm{L}_1, \mathrm{L}_2, \cdots \mathrm{L}_6\}$, *there exists an involution of the surface* Z *realizing such a change. In particular, the isomorphism type of the elliptic fibration* φ_{Z} *does not depend on the labeling of the configuration* $\mathcal{L}$. *Hence, the isomorphism class of the* $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$*-polarized K3 surface* W *constructed above is independent of the labeling of the six-line configuration* $\mathcal{L}$.

The results of this section show that every K3 surface Z, obtained as the minimal resolution of a double cover of the projective plane $\mathbb{P}^2$ branched over a six-line configuration $\mathcal{L}$, is part of a geometric two-isogeny, in the sense of Section 1. The geometric counterpart of Z under this isogeny is a K3 surface W carrying a canonical polarization by the rank-sixteen lattice $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$. However, the results do not imply that all K3 surfaces endowed with $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$-polarizations can be realized in this manner. This is clarified by the following section.

4 K3 Surfaces Polarized by the Lattice $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$

In this section X is an algebraic K3 surface endowed with a pseudo-ample lattice polarization

$$i\colon \mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7 \hookrightarrow \mathrm{NS}(X). \tag{18}$$

We shall also assume that the lattice polarization (18) cannot be extended to a polarization by the rank-eighteen lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$. It is known that a geometric two-isogeny as in Section 1 links any given K3 surface polarized by $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$ with the Kummer surface of a product of two elliptic curves and that the correspondence is bijective. This case was treated with full details in earlier works by the authors [3] as well as others [11, 22].

In a manner similar to the presentation in the previous section, we shall distinguish between the following two possibilities:

(a) the lattice polarization i can be extended to a polarization by the rank-seventeen lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$

(b) the polarization i cannot be extended to a polarization by the lattice $\mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_7$.

We shall refer to a polarized K3 surface (X, i) in situation (a) as **special**. A polarized K3 surface (X, i) satisfying condition (b) will be referred to as **non-special**.

4.1 Elliptic Fibrations on X

By standard results on elliptic fibrations on K3 surfaces (see discussion in [2] or related works [13, 19]), jacobian elliptic fibrations on X are in one-to-one correspondence with isomorphism classes of primitive lattice embeddings of the rank-two hyperbolic lattice H into the Neron-Severi lattice $\mathrm{NS}(\mathrm{X})$. There are at least four non-isomorphic primitive embeddings $\mathrm{H} \hookrightarrow \mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$, each of these embeddings leading via the polarization i to a specific jacobian elliptic fibration on X. Two of these embeddings/fibrations are particularly important for the discussion here.

Theorem 4.1. *Let (X, i) be a K3 surface endowed with a pseudo-ample lattice polarization of type $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$. Then X carries two canonically defined jacobian elliptic fibrations*

$$\varphi^{\mathrm{s}}_{\mathrm{X}}, \varphi^{\mathrm{a}}_{\mathrm{X}} \colon \mathrm{X} \to \mathbb{P}^1,$$

which we shall refer as **standard** *and* **alternate**. *The standard fibration carries a section S^{s}. The alternate fibration carries two disjoint sections $\mathrm{S}^{\mathrm{a}}_1$ and $\mathrm{S}^{\mathrm{a}}_2$.*

If the polarized pair (X, i) is non-special, then the standard fibration has two singular fibers of type III^*. In such a case the alternate fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ has a singular fiber of type I^*_8

If (X, i) is special, then the standard fibration has a singular fiber of type II^* and another fiber of type III^*. The alternate fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ carries a fiber of type I^*_{10} in this case.

Proof. The first primitive lattice embedding of H is obvious - the first factor in the orthogonal decomposition of $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$. This embedding induces then the canonical **standard** elliptic fibration $\varphi^{\mathrm{s}}_{\mathrm{X}} \colon \mathrm{X} \to \mathbb{P}^1$ with a section S^{s} and two special fibers of Kodaira type III^* or higher. The pair $(\varphi^{\mathrm{s}}_{\mathrm{X}}, \mathrm{S}^{\mathrm{s}})$ is uniquely defined, up to an automorphism of X. If (X, i) is non-special then $\varphi^{\mathrm{s}}_{\mathrm{X}}$ has two singular fibers of type III^* and one obtains a configuration of seventeen smooth rational curves as in the following dual diagram.

$$\begin{array}{ccccccccccccccccccccccccccccc} a_1 & - & a_2 & - & a_3 & - & a_4 & - & a_6 & - & a_7 & - & a_8 & - & S^{\mathrm{s}} & - & b_8 & - & b_7 & - & b_6 & - & b_4 & - & b_3 & - & b_2 & - & b_1 \\ &&&&&& | &&&&&&&&&&&&&&&& | &&&&&& \\ &&&&&& a_5 &&&&&&&&&&&&&&&& b_5 &&&&&& \end{array} \tag{19}$$

The standard embedding of H is spanned by $\{\mathrm{F}^{\mathrm{s}}, \mathrm{S}^{\mathrm{s}}\}$ where

$$\mathrm{F}^{\mathrm{s}} = a_1 + 2a_2 + 3a_3 + 4a_4 + 2a_5 + 3a_6 + 2a_7 + a_8 = b_1 + 2b_2 + 3b_3 + 4b_4 + 2b_5 + 3b_6 + 2b_7 + b_8, \tag{20}$$

and the two E_7 sub-lattices are spanned by $\{a_1, a_2, \cdots a_7\}$ and $\{b_1, b_2, \cdots b_7\}$, respectively.

In the case where (X, i) is special, the standard fibration $\varphi^{\mathrm{s}}_{\mathrm{X}}$ has two singular fibers of types II^* and III^*, respectively. An extra rational curve appears on the dual diagram.

$$\begin{array}{ccccccccccccccccccccccccccccccc} a_1 & - & a_2 & - & a_3 & - & a_5 & - & a_6 & - & a_7 & - & a_8 & - & a_9 & - & S^{\mathrm{s}} & - & b_8 & - & b_7 & - & b_6 & - & b_4 & - & b_3 & - & b_2 & - & b_1 \\ &&&& | &&&&&&&&&&&&&&&&&&&& | &&&&&& \\ &&&& a_4 &&&&&&&&&&&&&&&&&&&& b_5 &&&&&& \end{array} \tag{21}$$

In both diagrams (19) and (21) one sees a singular fiber of D-type. This fact leads one to a second primitive lattice embedding of H into $\mathrm{NS}(\mathrm{X})$. The image of this embedding is spanned by $\{\mathrm{F}^{\mathrm{a}}, \mathrm{S}^{\mathrm{a}}_1\}$ with these classes given, if (X, i) is non-special situation, by:

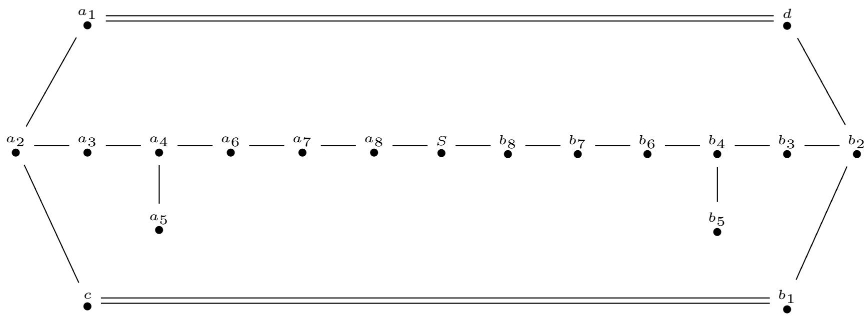
$$\mathrm{S}^{\mathrm{a}}_1 = a_2, \quad \mathrm{F}^{\mathrm{a}} = a_3 + a_5 + 2(a_4 + a_6 + a_7 + a_8 + S^{\mathrm{s}} + b_8 + b_7 + b_6 + b_4) + b_3 + b_5 \ . \tag{22}$$

In the special case one rather has:

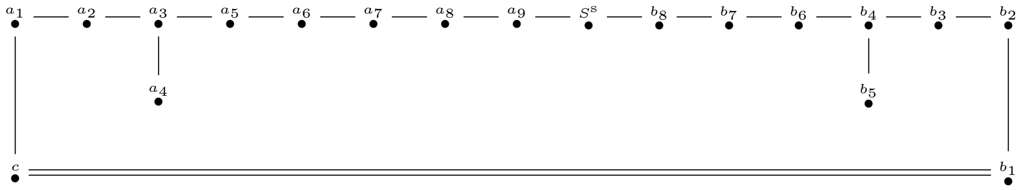
$$\mathrm{S}^{\mathrm{a}}_1 = a_1, \quad \mathrm{F}^{\mathrm{a}} = a_2 + a_4 + 2(a_3 + a_5 + a_6 + a_7 + a_8 + a_9 + S^{\mathrm{s}} + b_8 + b_7 + b_6 + b_4) + b_3 + b_5 \ .$$

This new embedding determines an alternate jacobian elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}} \colon \mathrm{X} \to \mathbb{P}^1$. This fibration has two disjoint sections $\mathrm{S}^{\mathrm{a}}_1$ and $\mathrm{S}^{\mathrm{a}}_2$ obtained as $\mathrm{S}^{\mathrm{a}}_1 = a_2$, $\mathrm{S}^{\mathrm{a}}_2 = b_2$, in the non-special case, and as $\mathrm{S}^{\mathrm{a}}_1 = a_1$, $\mathrm{S}^{\mathrm{a}}_2 = b_2$ in the special case. □

The alternate fibration $\varphi^{\mathrm{a}}_{\mathrm{X}} \colon \mathrm{X} \to \mathbb{P}^1$ plays a central role in the next results. In addition to the singular fiber of type I^*_8 (or I^*_{10} if the polarization is special), the fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ carries additional singular fibers. Generally, the singular fiber types of $\varphi^{\mathrm{a}}_{\mathrm{X}}$ are $\mathrm{I}^*_8 + 2 \times \mathrm{I}_2 + 6 \times \mathrm{I}_1$ in the case of a non-special polarized pair (X, i), and $\mathrm{I}^*_{10} + \mathrm{I}_2 + 6 \times \mathrm{I}_1$ for a special (X, i), respectively. In such a general situation, the dual diagrams (19) and (21) get augmented (with two or one rational curves, respectively) to nineteen-curve diagrams as follows.



(23)



(24)

Note the similarity with diagrams (15) and (16)

Theorem 4.2. *The section S_2^{a}, interpreted as an element of the Mordell-Weil group $\mathrm{MW}(\varphi_{\mathrm{X}}^{\mathrm{a}}, S_1^{\mathrm{a}})$, has order two. Fiber-wise translations by S_2^{a} extend to a Van Geemen-Sarti involution $\Phi_{\mathrm{X}} \colon \mathrm{X} \to \mathrm{X}$.*

Proof. One needs to verify the criterion of Proposition 1.2. We shall do this check assuming a non-special polarization (X, i). Similar arguments hold for the special polarizations.

Assume that (X, i) is a non-special polarization and take the orthogonal decomposition

$$\mathrm{NS}(\mathrm{X}) = \langle \mathrm{F}^{\mathrm{a}}, a_2 \rangle \oplus \mathcal{W}.$$

This provides the negative-definite lattice $\mathcal{W}$ which has rank $\mathrm{p}_{\mathrm{X}} - 2$. The root sublattice $\mathcal{W}_{\mathrm{root}}$ contains as orthogonal factors:

$$\langle a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ S^{\mathrm{s}},\ b_8,\ b_7,\ b_6,\ b_5,\ b_4,\ b_3 \rangle \ \oplus\ \langle b_1,\ \cdots \rangle\ \oplus \langle d_1,\ d_2 \cdots \rangle. \tag{25}$$

The second factor above is spanned by the classes of the irreducible components of the singular fiber in $\varphi_{\mathrm{X}}^{\mathrm{a}}$ containing b_1 and not meeting S_1^{a}. The third factor $\langle d_1,\ d_2 \cdots \rangle$ is spanned by the irreducible components of the singular fiber containing a_1 and not meeting S_1^{a}. For a generic non-special (X, i), one has $\langle b_1,\ \cdots \rangle = \langle b_1 \rangle$ and $\langle d_1,\ d_2 \cdots \rangle = \langle d \rangle$ where d is the rational curve of diagram (23).

One needs to check that $2b_2 - 2a_2 - 4\mathrm{F}^{\mathrm{a}} \in \mathcal{W}_{\mathrm{root}}$. By taking into account (20) one obtains:

$$2b_2 - \mathrm{F}^{\mathrm{s}} \ = \ -\left(b_8 + 2b_7 + 3b_6 + 4b_4 + 2b_5 + 3b_3 + b_1\right) \ \in \mathcal{W}_{\mathrm{root}} \tag{26}$$

$$\mathrm{F}^{\mathrm{s}} - (a_1 + 2a_2 + 3a_3) \ = \ (4a_4 + 2a_5 + 3a_6 + 2a_7 + a_8) \ \in \mathcal{W}_{\mathrm{root}}. \tag{27}$$

Taking the sum of (26) and (26), we have:

$$(2b_2 - 2a_2) - (a_1 + 3a_3) \ \in \mathcal{W}_{\mathrm{root}}. \tag{28}$$

Note also that, by comparing with (22), we also have:

$$\mathrm{F}^{\mathrm{a}} - a_3 \ = \ a_5 + 2\left(a_4 + a_6 + a_7 + a_8 + S^{\mathrm{s}} + b_8 + b_7 + b_6 + b_4\right) + b_3 + b_5 \ \in \mathcal{W}_{\mathrm{root}} \tag{29}$$

$$\mathrm{F}^{\mathrm{a}} - a_1 \ \in \langle d_1,\ d_2 \cdots \rangle \ \subset \ \mathcal{W}_{\mathrm{root}}. \tag{30}$$

One obtains therefore that:

$$4\mathrm{F}^{\mathrm{a}} - (a_1 + 3a_3) \ \in \mathcal{W}_{\mathrm{root}}. \tag{31}$$

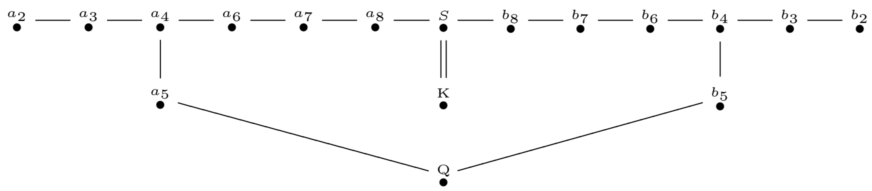
By subtracting (31) from (28), we obtain $2b_2 - 2a_2 - 4\mathrm{F}^{\mathrm{a}} \in \mathcal{W}_{\mathrm{root}}$. ☐

4.2 Properties of the Involution Φ_{X}

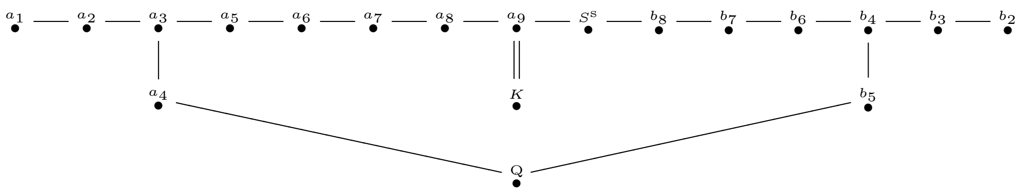
We assume that the polarized K3 surface (X, i) is such that, in both cases (non-special or special), the alternate elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{X}}$ has singular fiber types $\mathrm{I}_8^* + 2 \times \mathrm{I}_2 + 6 \times \mathrm{I}_1$ or $\mathrm{I}_{10}^* + \mathrm{I}_2 + 6 \times \mathrm{I}_1$, respectively. We shall therefore make use of the diagrams of rational curves (23) and (24)

The fixed locus $\{n_1, n_2, n_3, \cdots n_8\}$ of the Van Geemen-Sarti involution Φ_{X} appears as follows. The first six points $n_1, n_2, n_3, \cdots n_6$ are the singularities of the six I_1 fibers of the alternate fibration. The remaining n_7, n_8 are distinct points lying on the rational curve S^{s}, if the polarization (X, i) is non-special, and on the curve a_9 if (X, i) is special.

Two additional effective reduced divisors on X play a role in the construction. The first divisor, denoted Q is obtained from compactifying the set of order-two points in the smooth fibers of $\varphi^{\mathrm{a}}_{\mathrm{X}}$ that do not lie on $\mathrm{S}^{\mathrm{a}}_1$ or $\mathrm{S}^{\mathrm{a}}_2$. The divisor Q is a bi-section of the alternate fibration. It contains $n_1, n_2, n_3, n_4, n_5, n_6$ but not n_1, n_2. Generally, Q is a smooth genus-two curve and the restriction of the alternate fibration provides a double cover $\mathrm{Q} \to \mathbb{P}^1$ ramified at the six points $n_1, n_2, n_3, n_4, n_5, n_6$. The second divisor, denoted K is obtained from compactifying the points x in the smooth fibers of $\varphi^{\mathrm{a}}_{\mathrm{X}}$ that, with respect to the elliptic group law with neutral element at $\mathrm{S}^{\mathrm{a}}_1$, satisfy $2x = \mathrm{S}^{\mathrm{a}}_2$. The divisor K is a four-section of the alternate fibration. It contains all eight points of the fixed locus of Φ_{X}. Generally, K is a smooth curve of genus three in the non-special case and of genus two in the special case, respectively. The restriction of the alternate fibration gives a four-sheeted cover $\mathrm{K} \to \mathbb{P}^1$ branched at the base-points corresponding to singular fibers in the alternate fibration (nine points in the non-special case and eight points in the special case, respectively). Both divisors Q and K are mapped to themselves by the involution Φ_{X}. Their intersections with the curves of the big singular fiber of the alternate fibration are presented in the diagrams below. The first diagram corresponds to the case of a non-special (X, i). The second is associated with the special case.



(32)



(33)

The intersections of Q and K with the I_2 fiber curves are as follows. In the non-special case, one has.

$$\mathrm{Q} \cdot a_1 = \mathrm{Q} \cdot d = \mathrm{Q} \cdot c = \mathrm{Q} \cdot a_1 = 1, \quad \mathrm{K} \cdot a_1 = \mathrm{K} \cdot d = \mathrm{K} \cdot c = \mathrm{K} \cdot a_1 = 2.$$

In the special case:

$$\mathrm{Q} \cdot c = \mathrm{Q} \cdot b_1 = 1, \quad \mathrm{K} \cdot c = \mathrm{K} \cdot b_1 = 2.$$

Definition 4.3. *A K3 surface (X, i) polarized by the lattice $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$ is called* **generic** *if the following two conditions are satisfied:*

(a) The alternate fibration $\varphi^{\mathrm{a}}_{\mathrm{X}} \colon \mathrm{X} \to \mathbb{P}^1$ has singular fiber types $\mathrm{I}_8^* + 2 \times \mathrm{I}_2 + 6 \times \mathrm{I}_1$ or $\mathrm{I}_{10}^* + \mathrm{I}_2 + 6 \times \mathrm{I}_1$ depending on whether (X, i) is non-special, or special, respectively.

(b) The effective divisors Q and K introduced above are both irreducible.

Note that for K3 surfaces Z associated to generic six-line configurations $\mathcal{L}$ (in the sense of Definition 3.4), the $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$-polarized K3 surfaces W, given by the Nikulin construction of Section 3, are all generic in the sense of Definition 4.3.

We are in position to prove the inverse result:

Theorem 4.4. *Let (X, i) be a generic K3 surface polarized by the lattice $\mathrm{H} \oplus \mathrm{E}_7 \oplus \mathrm{E}_7$. Denote by Y the K3 surface obtained by the Nikulin construction associated to the Van Geemen-Sarti involution Φ_{X}. Then, the surface Y is isomorphic to the minimal resolution of a double cover of the projective plane $\mathbb{P}^2$ branched at a six-line configuration $\mathcal{L}$. No three of the six lines are concurrent and the configuration $\mathcal{L}$ is generic in the sense of Definition 3.4. If the polarization (X, i) is non-special then the six-line configuration $\mathcal{L}$ is non-Kummer. For special polarizations (X, i), the configuration $\mathcal{L}$ is Kummer.*

Proof. We present a detailed proof for the case when (X, i) is a non-special generic polarization. The same set of ideas together with a slight modification of the arguments provide the proof in the generic special case.

This proof uses the notation of diagrams (23) and (32). Note that the Van Geemen-Sarti involution Φ_{X} maps the three curves S, Q and K to themselves and interchanges the following nine pairs of rational curves:

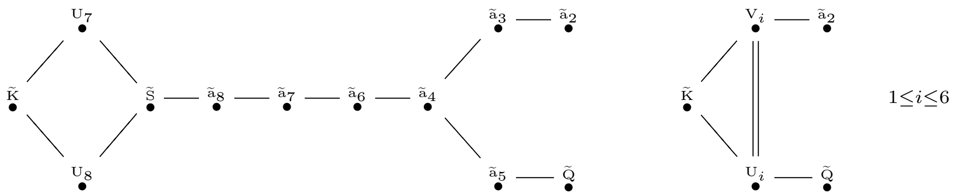
$$(\mathrm{a}_8, \mathrm{b}_8),\ (\mathrm{a}_7, \mathrm{b}_7),\ (\mathrm{a}_6, \mathrm{b}_6),\ (\mathrm{a}_5, \mathrm{b}_5),\ (\mathrm{a}_4, \mathrm{b}_4),\ (\mathrm{a}_3, \mathrm{b}_3),\ (\mathrm{a}_2, \mathrm{b}_2),\ (\mathrm{a}_1, \mathrm{d}),\ (\mathrm{c}, \mathrm{b}_1). \tag{34}$$

Under the push-forward by the rational degree-two map $\mathrm{X} \dashrightarrow \mathrm{Y}$ of the Nikulin construction, the three curves S, Q and K, as well as the curves in the first seven pairs of (34) determine ten smooth rational curves. We shall denote these curves by:

$$\widetilde{\mathrm{S}},\ \widetilde{\mathrm{Q}},\ \widetilde{\mathrm{K}},\ \widetilde{\mathrm{a}}_8,\ \widetilde{\mathrm{a}}_7,\ \widetilde{\mathrm{a}}_6,\ \widetilde{\mathrm{a}}_5,\ \widetilde{\mathrm{a}}_4,\ \widetilde{\mathrm{a}}_3,\ \widetilde{\mathrm{a}}_2. \tag{35}$$

The last two pairs in (34) determine rational curves with a single ordinary node. These form two I_1 fibers in the elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{Y}} \colon \mathrm{Y} \to \mathbb{P}^1$ induced from the alternate fibration on X.

Denote by U_i, $1 \leq i \leq 8$, the rational curves on Y appearing as exceptional curves associated to the fixed points p_i. Let us also consider V_i, $1 \leq i \leq 6$, as the resolutions of the I_1 fibers (of the alternate fibration) with singularities at p_i. One obtains twenty-four smooth rational curves on Y whose intersection pattern is summarized by the following dual diagram.



(36)

The elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{Y}} \colon \mathrm{Y} \to \mathbb{P}^1$ has the singular fiber type

$$\mathrm{I}_4^* + 6 \times \mathrm{I}_2 + 2 \times \mathrm{I}_1\ .$$

The two curves $\widetilde{\mathrm{a}}_2, \widetilde{\mathrm{Q}}$ form sections in $\varphi^{\mathrm{a}}_{\mathrm{Y}}$ while $\widetilde{\mathrm{K}}$ is a bi-section. As explained in Section 1, fiber-wise translations by the section $\widetilde{\mathrm{Q}}$ determine the dual Van Geemen-Sarti involution Φ_{Y}.

Let $\sigma \colon \mathrm{Y} \to \mathrm{Y}$ be the non-symplectic involution associated to $x \mapsto -x$ in the group law with origin at $\widetilde{\mathrm{a}}_2$ on the smooth fibers of $\varphi^{\mathrm{a}}_{\mathrm{Y}}$. By classical theory on elliptic curve deformations, the fixed locus of the involution σ is given by the six disjoint rational curves:

$$\widetilde{\mathrm{K}},\ \widetilde{\mathrm{S}},\ \widetilde{\mathrm{a}}_7,\ \widetilde{\mathrm{a}}_4,\ \widetilde{\mathrm{a}}_2,\ \widetilde{\mathrm{Q}}. \tag{37}$$

In addition, the rational curves

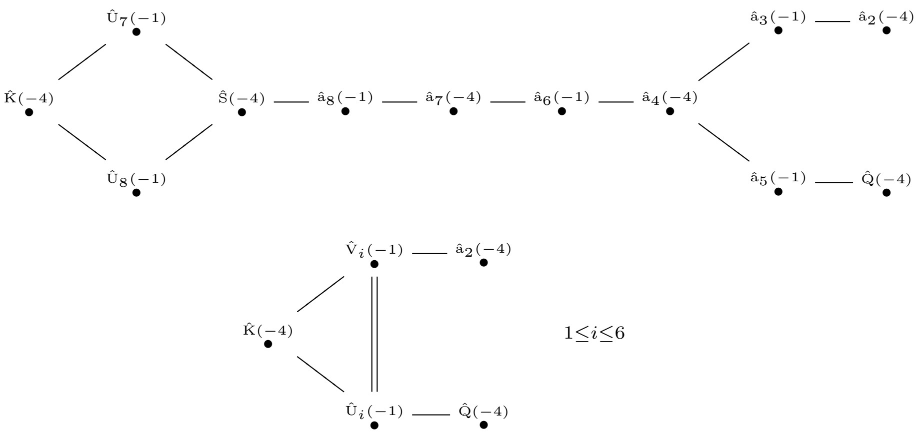
$$\widetilde{\mathrm{a}}_8,\ \widetilde{\mathrm{a}}_6,\ \widetilde{\mathrm{a}}_5,\ \widetilde{\mathrm{a}}_3,\ \mathrm{U}_i \text{ with } 1 \leq i \leq 8,\ \mathrm{V}_i \text{ with } 1 \leq i \leq 6 \tag{38}$$

are mapped onto themselves under σ.

The quotient of the K3 surface Y by the involution σ is a rational ruled surface R with a ruling

$$\varphi_{\mathrm{R}} \colon \mathrm{R} \to \mathbb{P}^1 \tag{39}$$

induced from the elliptic fibration $\varphi^{\mathrm{a}}_{\mathrm{Y}}$. We shall use the superscript $\hat{}$ to denote the rational curves on R obtained as push-forward under the quotient map of the curves in (37) and (38). A dual diagram similar to (36) appears.



(40)

The self-intersection numbers are included.

The ruling (39), as well as the rational curves of (40), will be used to prove that the rational surface R is isomorphic to the blow-up of $\mathbb{P}^2$ at a configuration of fifteen distinct points corresponding to the intersection of six distinct lines. The considerations of Section 3 bring some insight into the construction, allowing us to write explicitly the cohomology classes associated to the fifteen would-be exceptional curves.

$$
\begin{aligned}
\mathrm{E}_{12} &= \hat{\mathrm{a}}_2 \\
\mathrm{E}_{13} &= 12\mathrm{F}^{\mathrm{a}} + 3\hat{\mathrm{a}}_2 - 3\hat{\mathrm{a}}_4 - 3\hat{\mathrm{a}}_5 - 8\hat{\mathrm{a}}_6 - 5\hat{\mathrm{a}}_7 - 12\hat{\mathrm{a}}_8 - 7\hat{\mathrm{S}} - \hat{\mathrm{U}}_2 - \hat{\mathrm{U}}_3 - 3\hat{\mathrm{U}}_4 - 2\hat{\mathrm{U}}_5 - 2\hat{\mathrm{U}}_6 - 7\hat{\mathrm{U}}_7 - 8\hat{\mathrm{U}}_8 \\
\mathrm{E}_{14} &= 8\mathrm{F}^{\mathrm{a}} + 2\hat{\mathrm{a}}_2 - 2\hat{\mathrm{a}}_4 - 2\hat{\mathrm{a}}_5 - 5\hat{\mathrm{a}}_6 - 3\hat{\mathrm{a}}_7 - 7\hat{\mathrm{a}}_8 - 4\hat{\mathrm{S}} - \hat{\mathrm{U}}_3 - 2\hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_5 - 2\hat{\mathrm{U}}_6 - 4\hat{\mathrm{U}}_7 - 5\hat{\mathrm{U}}_8 \\
\mathrm{E}_{15} &= \mathrm{F}^{\mathrm{a}} - \hat{\mathrm{a}}_4 - \hat{\mathrm{a}}_5 - 2\hat{\mathrm{a}}_6 - \hat{\mathrm{a}}_7 - 2\hat{\mathrm{a}}_8 - \hat{\mathrm{S}} - \hat{\mathrm{U}}_7 - \hat{\mathrm{U}}_8 \\
\mathrm{E}_{16} &= \hat{\mathrm{a}}_5 \\
\mathrm{E}_{23} &= \hat{\mathrm{a}}_8 \\
\mathrm{E}_{24} &= 4\mathrm{F}^{\mathrm{a}} + \hat{\mathrm{a}}_2 - \hat{\mathrm{a}}_4 - \hat{\mathrm{a}}_5 - 3\hat{\mathrm{a}}_6 - 2\hat{\mathrm{a}}_7 - 4\hat{\mathrm{a}}_8 - 2\hat{\mathrm{S}} - \hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_5 - \hat{\mathrm{U}}_6 - 2\hat{\mathrm{U}}_7 - 2\hat{\mathrm{U}}_8 \\
\mathrm{E}_{25} &= 9\mathrm{F}^{\mathrm{a}} + 2\hat{\mathrm{a}}_2 - 2\hat{\mathrm{a}}_4 - 2\hat{\mathrm{a}}_5 - 6\hat{\mathrm{a}}_6 - 4\hat{\mathrm{a}}_7 - 9\hat{\mathrm{a}}_8 - 5\hat{\mathrm{S}} - \hat{\mathrm{U}}_2 - \hat{\mathrm{U}}_3 - 2\hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_5 - 2\hat{\mathrm{U}}_6 - 5\hat{\mathrm{U}}_7 - 6\hat{\mathrm{U}}_8 \\
\mathrm{E}_{26} &= 8\mathrm{F}^{\mathrm{a}} + 2\hat{\mathrm{a}}_2 - 2\hat{\mathrm{a}}_4 - 2\hat{\mathrm{a}}_5 - 6\hat{\mathrm{a}}_6 - 4\hat{\mathrm{a}}_7 - 9\hat{\mathrm{a}}_8 - 5\hat{\mathrm{S}} - \hat{\mathrm{U}}_3 - 2\hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_5 - \hat{\mathrm{U}}_6 - 5\hat{\mathrm{U}}_7 - 6\hat{\mathrm{U}}_8 \\
\mathrm{E}_{34} &= \hat{\mathrm{U}}_7 \\
\mathrm{E}_{35} &= 5\mathrm{F}^{\mathrm{a}} + \hat{\mathrm{a}}_2 - \hat{\mathrm{a}}_4 - \hat{\mathrm{a}}_5 - 3\hat{\mathrm{a}}_6 - 2\hat{\mathrm{a}}_7 - 5\hat{\mathrm{a}}_8 - 3\hat{\mathrm{S}} - \hat{\mathrm{U}}_3 - \hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_5 - \hat{\mathrm{U}}_6 - 3\hat{\mathrm{U}}_7 - 3\hat{\mathrm{U}}_8 \\
\mathrm{E}_{36} &= 4\mathrm{F}^{\mathrm{a}} + \hat{\mathrm{a}}_2 - \hat{\mathrm{a}}_4 - \hat{\mathrm{a}}_5 - 3\hat{\mathrm{a}}_6 - 2\hat{\mathrm{a}}_7 - 5\hat{\mathrm{a}}_8 - 3\hat{\mathrm{S}} - \hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_6 - 3\hat{\mathrm{U}}_7 - 3\hat{\mathrm{U}}_8 \\
\mathrm{E}_{45} &= \mathrm{F}^{\mathrm{a}} - \hat{\mathrm{U}}_4 = \hat{\mathrm{V}}_4 \\
\mathrm{E}_{46} &= \hat{\mathrm{U}}_1 \\
\mathrm{E}_{56} &= 5\mathrm{F}^{\mathrm{a}} + \hat{\mathrm{a}}_2 - \hat{\mathrm{a}}_4 - \hat{\mathrm{a}}_5 - 3\hat{\mathrm{a}}_6 - 2\hat{\mathrm{a}}_7 - 5\hat{\mathrm{a}}_8 - 3\hat{\mathrm{S}} - \hat{\mathrm{U}}_4 - \hat{\mathrm{U}}_5 - \hat{\mathrm{U}}_6 - 3\hat{\mathrm{U}}_7 - 4\hat{\mathrm{U}}_8
\end{aligned}
$$

We shall show that the classes

$$\mathrm{E}_{13},\ \mathrm{E}_{14},\ \mathrm{E}_{15},\ \mathrm{E}_{24},\ \mathrm{E}_{25},\ \mathrm{E}_{26},\ \mathrm{E}_{35},\ \mathrm{E}_{36},\ \mathrm{E}_{56} \tag{41}$$

are effective and their associated linear system consists of a single smooth rational curve. In order to accomplish this goal, we start a blow-down process

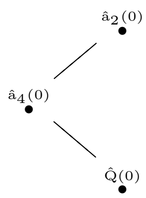
$$\mathrm{R} = \mathrm{R}_{15} \ \rightarrow\ \mathrm{R}_{14} \ \rightarrow\ \mathrm{R}_{13} \ \rightarrow\ \cdots \ \rightarrow \mathrm{R}_3 \ \rightarrow\ \mathrm{R}_2 \ \rightarrow\ \widetilde{\mathrm{R}}_1 \tag{42}$$

by collapsing a sequence of exceptional rational curves in the fibers of the ruling φ_{R}. The sequence of fourteen exceptional curves is as follows:

$$\hat{\mathrm{E}}_{15} = \hat{\mathrm{U}}_1,\ \hat{\mathrm{E}}_{14} = \hat{\mathrm{U}}_2,\ \hat{\mathrm{E}}_{13} = \hat{\mathrm{U}}_3,\ \hat{\mathrm{E}}_{12} = \hat{\mathrm{V}}_4,\ \hat{\mathrm{E}}_{11} = \hat{\mathrm{V}}_5,\ \hat{\mathrm{E}}_{10} = \hat{\mathrm{V}}_6,\ \hat{\mathrm{E}}_9 = \hat{\mathrm{U}}_7,\ \hat{\mathrm{E}}_8 = \hat{\mathrm{U}}_8,$$

$$\hat{\mathrm{E}}_7 = \hat{\mathrm{a}}_8,\ \hat{\mathrm{E}}_6 = \hat{\mathrm{a}}_6,\ \hat{\mathrm{E}}_5 = \hat{\mathrm{S}},\ \hat{\mathrm{E}}_4 = \hat{\mathrm{a}}_7,\ \hat{\mathrm{E}}_3 = \hat{\mathrm{a}}_5,\ \hat{\hat{\mathrm{E}}}_2 = \hat{\mathrm{a}}_3.$$

By a slight abuse, we keep the notation for the various rational curves involved, as they get pushed-forward under blow-downs. The resulting surface $\widetilde{\mathrm{R}}_1$ is smooth, rational and minimally ruled. Hence, by standard results on ruled surfaces (see, for instance, Chapter III of [1]), the surface $\widetilde{\mathrm{R}}_1$ is isomorphic to one of the Hirzebruch surfaces $\mathbb{F}_n$, $n \geq 0$. The cohomology group $\mathrm{H}^2(\widetilde{\mathrm{R}}_1, \mathbb{Z})$ has rank two and is spanned by the classes of two rulings, one induced from φ_{R} and having $\hat{\mathrm{a}}_2$ and $\hat{\mathrm{Q}}$ as fibers, and a second ruling with $\hat{\mathrm{a}}_4$ as fiber.



It follows then that $\widetilde{\mathrm{R}}_1$ is isomorphic to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, if we remove the last blow-down in (42) and instead we collapse the exceptional curves

$$\hat{\mathrm{E}}_2 = \hat{\mathrm{a}}_2 \text{ and } \hat{\mathrm{E}}_1 = \hat{\mathrm{a}}_4$$

$$\mathrm{R}_2 \;\rightarrow\; \mathrm{R}_1 \;\rightarrow\; \mathrm{R}_0,$$

the resulting surface R_0 is a copy of the projective plane $\mathbb{P}^2$.

The blow-down construction determines the following configuration in R_0. First, there are x_1, x_2 - the two distinct points of R_0 where the last two exceptional curves $\hat{\mathrm{a}}_4$ and $\hat{\mathrm{a}}_2$ collapse. The push-forward of $\hat{\mathrm{a}}_2$ is the line l joining x_1 and x_2. One has seven other distinct lines

$$l',\; l_1,\; l_2,\; l_3,\; l_4,\; l_5,\; l_6$$

obtained as push-forward of

$$\widetilde{\mathrm{Q}},\; \hat{\mathrm{V}}_1,\; \hat{\mathrm{V}}_2,\; \hat{\mathrm{V}}_3,\; \hat{\mathrm{U}}_4,\; \hat{\mathrm{U}}_5,\; \hat{\mathrm{U}}_6.$$

The line l' passes through x_1 but not x_2. The six lines $l_1, l_2, \cdots l_6$ meet at x_2 but do not pass through x_1. Denote by y_i with $1 \leq i \leq 6$, the six points of intersection between the lines l' and l_i, respectively. Going backwards through the blow-down process (42), one recovers the rational surface R as the blow-up of the projective plane R_0 at a sequence of fifteen points:

$$p_1,\; p_2,\; p_3,\; \cdots\; p_{15}$$

where $p_1 = x_1$, $p_2 = x_2$, the points $p_3, p_4, \cdots p_9$ are infinitely near x_1 with p_3 representing the tangent direction of l', the points p_{10}, p_{11}, p_{12} are infinitely near x_2 and represent the tangent directions of l_6, l_5, l_4, and $p_{13} = y_3$, $p_{14} = y_2$, $p_{15} = y_1$.

Let $\hat{\mathrm{H}}$ be the class of a hyperplane section in R_0 and denote by $\hat{\mathrm{E}}_1,\ \hat{\mathrm{E}}_2,\ \cdots\ \hat{\mathrm{E}}_{15}$ the strict transforms of the fifteen exceptional curves associated to the blow-up $\mathrm{R} \rightarrow \mathrm{R}_0$. The sixteen classes $\hat{\mathrm{H}}, \hat{\mathrm{E}}_1,\ \hat{\mathrm{E}}_2,\ \cdots\ \hat{\mathrm{E}}_{15}$ form a basis over the integers for $\mathrm{H}^2(\mathrm{R}, \mathbb{Z})$ and, with respect to this basis, the classes of (41) are as follows:

$$\begin{aligned}
\mathrm{E}_{13} &= 5\hat{\mathrm{H}} - 3\hat{\mathrm{E}}_1 - 2\left(\hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4 + \hat{\mathrm{E}}_5\right) - \left(\hat{\mathrm{E}}_8 + \hat{\mathrm{E}}_{10} + \hat{\mathrm{E}}_{11} + \hat{\mathrm{E}}_{13} + \hat{\mathrm{E}}_{14}\right)\\
\mathrm{E}_{14} &= 3\hat{\mathrm{H}} - 2\hat{\mathrm{E}}_1 - \left(\hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4 + \hat{\mathrm{E}}_5 + \hat{\mathrm{E}}_8 + \hat{\mathrm{E}}_{11} + \hat{\mathrm{E}}_{13}\right)\\
\mathrm{E}_{15} &= \hat{\mathrm{H}} - \hat{\mathrm{E}}_1 - \hat{\mathrm{E}}_2\\
\mathrm{E}_{24} &= \hat{\mathrm{H}} - \hat{\mathrm{E}}_1 - \hat{\mathrm{E}}_4\\
\mathrm{E}_{25} &= 4\hat{\mathrm{H}} - 2\left(\hat{\mathrm{E}}_1 + \hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4\right) - \left(\hat{\mathrm{E}}_5 + \hat{\mathrm{E}}_8 + \hat{\mathrm{E}}_{11} + \hat{\mathrm{E}}_{13} + \hat{\mathrm{E}}_{14}\right)\\
\mathrm{E}_{26} &= 4\hat{\mathrm{H}} - 2\left(\hat{\mathrm{E}}_1 + \hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4\right) - \left(\hat{\mathrm{E}}_5 + \hat{\mathrm{E}}_8 + \hat{\mathrm{E}}_{10} + \hat{\mathrm{E}}_{11} + \hat{\mathrm{E}}_{13}\right)\\
\mathrm{E}_{35} &= 2\hat{\mathrm{H}} - \left(\hat{\mathrm{E}}_1 + \hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4 + \hat{\mathrm{E}}_5 + \hat{\mathrm{E}}_{13}\right)\\
\mathrm{E}_{36} &= 2\hat{\mathrm{H}} - \left(\hat{\mathrm{E}}_1 + \hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4 + \hat{\mathrm{E}}_5 + \hat{\mathrm{E}}_{11}\right)\\
\mathrm{E}_{56} &= 2\hat{\mathrm{H}} - \left(\hat{\mathrm{E}}_1 + \hat{\mathrm{E}}_2 + \hat{\mathrm{E}}_4 + \hat{\mathrm{E}}_5 + \hat{\mathrm{E}}_8\right).
\end{aligned}$$

Moreover, the class of the fiber of the ruling φ_{R} is

$$\mathrm{F}^{\mathrm{a}} \;=\; \hat{\mathrm{H}} - \hat{\mathrm{E}}_2.$$

One verifies then that the nine points $p_1, p_2, p_4, p_5, p_8, p_{10}, p_{11}, p_{13}, p_{14}$ are in a general enough position that all above classes are effective and each is represented by a unique smooth rational curve. Abusing the notation, we denote these rational curves of R by same symbol as their cohomology class.

We have obtained fifteen disjoint rational curves on R, denoted E_{ij} with $1 \leq i < j \leq 6$. All curves E_{ij} have self-intersection -1. By blowing down E_{ij}, one obtains another copy of the projective plane $\mathbb{P}^2$. Denote by q_{ij} the fifteen distinct points obtained by collapsing the exceptional curves. The push-forwards of the six curves:

$$\hat{\mathrm{a}}_4,\; \hat{\mathrm{a}}_7,\; \hat{\mathrm{S}},\; \hat{\mathrm{K}},\; \hat{\mathrm{a}}_2,\; \hat{\mathrm{Q}}$$

form a configuration $\mathcal{L} = \{\mathrm{L}_1, \mathrm{L}_2, \cdots \mathrm{L}_6\}$ of six lines in this projective plane, meeting at the fifteen points q_{ij}. The push-forward of $\hat{\mathrm{U}}_7$ is a conic passing through the five points $q_{13}, q_{14}, q_{25}, q_{26}, q_{56}$ but this conic does not contain q_{34}. Therefore, the six-line configuration $\mathcal{L}$ is non-Kummer.

A slight modification of the above arguments gives a proof for the case of a generic special polarized pair (X, i). One obtains the fifteen disjoint rational curves E_{ij} in the same manner as above. Then one checks that the conic through $q_{13}, q_{14}, q_{25}, q_{26}, q_{56}$ also contains q_{34}. This fact, in turn, implies the existence of a rational curve $\mathrm{E}_{\varnothing}$ tangent to all the six lines of the configuration $\mathcal{L}$. $\square$

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