On K3 Surfaces with Large Complex Structure

Adrian Clingher [*†] Charles F. Doran [‡]

Abstract

We discuss a notion of large complex structure for elliptic K3 surfaces with section inspired by the eight-dimensional F-theory/heterotic duality in string theory. This concept is naturally associated with the Type II Mumford partial compactification of the moduli space of periods for these structures. The paper provides an explicit Hodge-theoretic condition for the complex structure of an elliptic K3 surface with section to be large. We also establish certain geometric consequences of this large complex structure condition in terms of the Kodaira types of the singular fibers of the elliptic fibration.

1 Introduction

An elliptic K3 surface with section is a triple $(\mathrm{X}, \varphi, \mathrm{S})$ consisting of a K3 surface X, an elliptic fibration $\varphi\colon \mathrm{X} \to \mathbb{P}^1$ and a smooth rational curve S making a section of φ. The extra structure given by the elliptic fibration and section on X is equivalent to a pseudo-ample hyperbolic lattice polarization in the sense of Dolgachev [13] and one can use this property to construct a coarse moduli space for elliptic K3 surfaces with section. This space, which we shall denote here by $\mathcal{M}_{\mathrm{K3}}$, is a quasi-projective analytic variety of complex dimension eighteen. The properties of $\mathcal{M}_{\mathrm{K3}}$ have been extensively studied by means of the **period map**:

$$\mathrm{per}\colon \mathcal{M}_{\mathrm{K3}} \ \to \ \Gamma\backslash\Omega. \tag{1}$$

which associates to a given triple $(\mathrm{X}, \varphi, \mathrm{S})$ the equivalence class of its polarized Hodge structure. Here Ω represents the classical domain of periods:

$$\{ \ [\omega] \in \mathbb{P}\left(\mathrm{L} \otimes \mathbb{C}\right) \ | \ \omega \cdot \omega = 0, \ \omega \cdot \bar{\omega} > 0 \ \}$$

where L is the unimodular even lattice of signature $(2, 18)$ and Γ is the group of integral isometries of L acting on Ω in a natural way. A special case of the Global Torelli theorem for lattice polarized $K3$ surfaces (see [13]) asserts that (1) is an isomorphism of analytic spaces.

The target space of the period map (1) is connected but not compact. However, due to the nice arithmetic features of the period domain, there exists quite an array of methods at one's disposal for (partially) compactifying $\Gamma\backslash\Omega$. The simplest and most standard procedure is the Baily-Borel method [4] which exploits the natural holomorphic identification between Ω and the hermitian symmetric space:

$$\mathrm{O}(2, 18)/\mathrm{SO}(2) \times \mathrm{O}(18),$$

in order to fully compactify $\Gamma\backslash\Omega$ by adding a number of curves and points. However, the Baily-Borel construction does not capture the full geometric information encoded in the periods, a fact reflected in the high codimension of the Baily-Borel boundary components. We shall be concerned here with a different compactification method, which realizes, to a certain extent, a blow-up of Baily-Borel's construction. This

*Department of Mathematics, Stanford University, Stanford, CA 94305.
†Current address: Department of Mathematics, University of Missouri, St. Louis MO 63121. **e-mail:** *clingher@arch.umsl.edu*
‡Department of Mathematics, University of Washington, Seattle, WA 98195. **e-mail:** *doran@math.washington.edu*

method is a special case of Mumford's toroidal compactification construction [1]. The procedure, which was first applied in the K3 surface context by Friedman [14] [15], constructs a smooth partial compactification

$$\Gamma\backslash\Omega \;\subset\; \overline{\Gamma\backslash\Omega}, \tag{2}$$

by, essentially, adding to the quotient space $\Gamma\backslash\Omega$ two **Mumford boundary divisors** $\mathcal{D}_1$ and $\mathcal{D}_2$ associated to the two distinct classes of Type II maximal rational parabolic subgroups of $\mathrm{O}(2,18)$.

The Type II partial compactification (2), although purely arithmetic in nature, has an interesting geometric interpretation on the moduli space side. It corresponds to an enlargement

$$\mathcal{M}_{\mathrm{K3}} \;\subset\; \overline{\mathcal{M}}_{\mathrm{K3}}$$

obtained by allowing certain normal crossing degenerations of elliptic $K3$ surfaces with section, the Type II stable elliptic $K3$ surfaces with section (see section 3 of [9]). The period map (1) extends to an identification:

$$\overline{\mathrm{per}}\colon \overline{\mathcal{M}}_{\mathrm{K3}} \;\to\; \overline{\Gamma\backslash\Omega} \tag{3}$$

and allows one to regard the boundary points of $\mathcal{D}_1$ and $\mathcal{D}_2$ as periods for the singular surfaces.

One of the essential ingredients of this compactification method is the existence of a **large complex structure domain** associated to each of the two boundary divisors. Let us postpone to section 3 a more detailed definition of these domains and give just a brief description here. Consider a Type II maximal parabolic subgroup of $\mathrm{O}(2,18)$ defined over the rationals and, denote by P its intersection with Γ. One has then a holomorphic non-normal covering projection with infinitely many sheets:

$$\pi\colon \mathrm{P}\backslash\Omega \;\to\; \Gamma\backslash\Omega.$$

Moreover, the total space of this map fibers holomorphically, as in the diagram below, over the appropriate Mumford boundary component $\mathcal{D}$, with all fibers being isomorphic to complex open punctured discs.

$$\begin{tikzcd} \Omega \arrow[r] & \mathrm{P}\backslash\Omega \arrow[r, "\pi"] \arrow[d, "\alpha"'] & \Gamma\backslash\Omega \\ & \mathcal{D} & \end{tikzcd} \tag{4}$$

One can select then in Ω a subset with special properties.

Theorem 1.1. *There exists an open subset $\mathcal{V} \subset \Omega$ such that: $\mathcal{V}$ is left invariant by the action of P, the image of $\mathcal{V}$ under the natural projection to $\mathrm{P}\backslash\Omega$ intersects each fiber of α over an open neighborhood of the puncture and, the Γ-equivalence reduces to P-equivalence on $\mathcal{V}$.*

In the context of the above theorem, the restriction:

$$\pi_{|\mathrm{P}\backslash\mathcal{V}}\colon \mathrm{P}\backslash\mathcal{V} \;\to\; \Gamma\backslash\Omega \tag{5}$$

is an isomorphism onto its range

$$\mathcal{U} := \pi\left(\mathrm{P}\backslash\mathcal{V}\right). \tag{6}$$

The inverse map of (5) provides then the essential gluing map which allows one to smoothly fit the boundary component $\mathcal{D}$ together with $\Gamma\backslash\Omega$. Moreover, one obtains a natural holomorphic fibration of $\mathcal{U}$ over the boundary divisor $\mathcal{D}$, whose fibers are copies of $\mathbb{C}^*$. The open subset $\mathcal{U}$ is the **large complex structure domain** associated to $\mathcal{D}$. The inverse image of $\mathcal{U}$ under the period map determines an open region of $\mathcal{M}_{\mathrm{K3}}$ which is said to correspond to elliptic K3 surfaces with **large complex structure**.

The statement in Theorem 1.1 represents a special case of a general reduction theorem of Ash-Mumford-Rapoport-Tai proved in Chapter 5 of [1]. However, the method of proof in [1] does not yield an explicit description of the open subset $\mathcal{U}$. It is therefore not an easy task to decide whether a given period line $[\omega]$ corresponds to an elliptic K3 surface with section of large complex structure or not. The goal of this paper is to introduce a simple, easy-to-test condition on the period lines in Ω leading to an open subset $\mathcal{V} \subset \Omega$ which satisfies all requirements of Theorem 1.1. This, in addition to giving an alternative proof for Theorem 1.1, provides an effective Hodge-theoretic method of testing whether a given elliptic K3 surface with section has large complex structure.

Let us close this introductory section by also mentioning the string theory motivation underlying this work. This shall also serve as an explanation for the "large complex structure" terminology used for the subset $\mathcal{U}$ of (6).

Following the works of Vafa [34] and Sen [31] in 1996, it was noted that the geometry underlying elliptic $K3$ surfaces with section is related to the geometry of elliptic curves endowed with certain flat principal G-bundles. This non-trivial connection appears in string theory as the eight-dimensional manifestation of the phenomenon called F-theory/heterotic string duality. Over the past ten years the correspondence has been analyzed extensively ([9] [7] [11] [12] [17] [24] [25]) from a purely mathematical point of view. As it turns out, it leads to a beautiful geometric picture which links together moduli spaces for these two seemingly distinct types of geometrical objects: elliptic $K3$ surfaces with section and flat bundles over elliptic curves.

In a brief description, what happens is the following. Let G be one of the following two Lie groups:

$$(\mathrm{E}_8 \times \mathrm{E}_8) \rtimes \mathbb{Z}_2 \qquad \mathrm{Spin}(32)/\mathbb{Z}_2. \tag{7}$$

As argued in [16], one can define a moduli space $\mathcal{M}_{\mathrm{E,G}}$ of equivalence classes of flat G-bundles over elliptic curves as a quasi-projective analytic space of complex dimension seventeen. There exists then a holomorphic isomorphism between $\mathcal{M}_{\mathrm{E,G}}$ and one of the boundary divisors $\mathcal{D}_i$ introduced in the previous Type II partial compactification of $\Gamma\backslash\Omega$. Moreover, under this correspondence, the total space $\mathrm{P}\backslash\Omega$ of the fibration α in diagram (4) can be holomorphically identified with the moduli space $\mathcal{M}_{\mathrm{het}}$ of classical vacua in heterotic string theory[1].

The prediction made by the string duality is then that, although the two spaces $\mathcal{M}_{\mathrm{K3}}$ and $\mathcal{M}_{\mathrm{het}}$ are not globally identical, there should exist open regions in each of them, neighboring boundary divisors at infinity (regions that correspond, in physics language, to large levels of energy), that are analytically isomorphic. On the heterotic side, high energy levels appear when the two-torus has large volume and therefore, such a region has to be a tubular neighborhood of the punctures in the fibration α of diagram (4). On the F-theory side, the appropriate region is not a priori obvious and, by convention, is said to correspond to elliptic K3 surfaces with section of large complex structure.

Comparing the above paragraph with the arguments leading to the construction of the open subset $\mathcal{U}$ in (6), we note that the statement of Theorem 1.1 precisely captures the feature predicted by the duality. Hence, one is naturally led to characterize the complex structures associated to elliptic K3 surfaces with section in the open region

$$\mathcal{U} \subset \Gamma\backslash\Omega$$

as large[2].

The paper is organized as follows. In sections 2 and 3, we collect the basic facts needed to construct the period map and the Type II partial compactification of $\Gamma\backslash\Omega$. We introduce our Hodge-theoretic large complex structure condition in section 4. In section 5, we show that the domain $\mathcal{V}$ defined by this condition satisfies the properties required by Theorem 1.1. Finally, in section 6, we discuss some geometric consequences of the large complex structure condition, in terms of what types of singular fibers can appear in the elliptic fibration associated to a large structure K3 surface.

The authors would like to thank John Morgan for helpful discussions throughout the evolution of this project.

[1] $\mathcal{M}_{\mathrm{het}}$ represents the moduli space of equivalence classes of pairs of flat G-bundles and complexified Kähler classes over elliptic curves. We refer the interested reader to [9] and [8] for more details regarding this space.

[2] This notion of large complex structure differs from the similarly-named condition arising in the context of Type IIA/IIB string duality (mirror symmetry) for K3 surfaces [18] [23] [33].

2 Preliminaries

A K3 surface X is a non-singular, simply-connected complex surface with trivial canonical bundle. It is a well-known fact (see, for example, [3]) that any two surfaces with these properties are diffeomorphic. The cohomology group $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ is torsion free of rank 22 and, when endowed with the pairing $\langle \cdot, \cdot \rangle$ induced by the intersection form, it becomes an even unimodular lattice of signature $(3, 19)$. There exists a unique lattice with these features. This lattice can be constructed independent of geometry by taking the orthogonal direct sum of the following irreducible factors:

$$\mathrm{H} \oplus \mathrm{H} \oplus \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8 \tag{8}$$

where H represents the rank-two hyperbolic lattice and E_8 is the unique negative-definite even and unimodular lattice of rank eight.

The cohomology classes dual to algebraic cycles of X span a special sublattice $\mathrm{NS}(\mathrm{X})$ of $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$, called the **Neron-Severi lattice**. As a group, $\mathrm{NS}(\mathrm{X})$ is isomorphic to the Picard group of X, that is the group of algebraic equivalence classes of holomorphic line bundles over X. The rank the Neron-Severi lattice, denoted by $\mathrm{p_X}$, varies between 0 and 20. By the Hodge index theorem, the signature of $\mathrm{NS}(X)$ is $(1, p_X - 1)$. A generic K3 surface has rank $p_X = 0$ and hence is not projective.

Our objects of interest are **elliptically fibered K3 surfaces with section**. These are triples $(\mathrm{X}, \varphi, \mathrm{S})$ consisting of a K3 surface X, a proper analytic map $\varphi\colon \mathrm{X} \to \mathbb{P}^1$ whose general fibers are smooth elliptic curves, and a smooth rational curve S on X which makes a section in the elliptic fibration. Two elliptically fibered K3 surfaces with section $(\mathrm{X}, \varphi, \mathrm{S})$ and $(\mathrm{X}', \varphi', \mathrm{S}')$ are said to be **equivalent** if there exists an analytic isomorphism $\alpha\colon \mathrm{X} \to \mathrm{X}'$ with $\alpha(\mathrm{S}) = \mathrm{S}'$, inducing commutativity in the following diagram.

$$\begin{tikzcd} \mathrm{X} \arrow[rr, "\alpha"] \arrow[dr, "\varphi"'] & & \mathrm{X}' \arrow[dl, "\varphi'"] \\ & \mathbb{P}^1 & \end{tikzcd} \tag{9}$$

Given a triple $(\mathrm{X}, \varphi, \mathrm{S})$ as above, one has two special classes $f, s \in \mathrm{NS}(\mathrm{X})$ associated to the elliptic fiber and section. These classes are independent and span a sublattice of rank two:

$$\mathcal{H}_{(\varphi,\mathrm{S})} \subset \mathrm{NS}(\mathrm{X}). \tag{10}$$

In particular, $\mathrm{p_X} \geq 2$. The intersection form on $\mathcal{H}_{(\varphi,\mathrm{S})}$ with respect to the basis $\{f, s\}$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \tag{11}$$

and therefore $\mathcal{H}_{(\varphi,\mathrm{S})}$ is isometric to the standard rank-two hyperbolic lattice H. One has then a splitting of the Neron-Severi lattice of X as an orthogonal direct sum:

$$\mathrm{NS}(\mathrm{X}) \;=\; \mathcal{H}_{(\varphi,\mathrm{S})} \;\oplus\; \mathcal{W}_{\mathrm{X}} \tag{12}$$

where $\mathcal{W}_{\mathrm{X}}$ is negative-definite and of rank $\mathrm{p_X} - 2$.

Proposition 2.1. *The sublattice $\mathcal{H}_{(\varphi,\mathrm{S})} \subset \mathrm{NS}(\mathrm{X})$ completely determines the elliptic fibration with section (φ, S) on* X.

Proof. Let us assume that there exists a second elliptic structure with section (φ', S') on X such that:

$$\mathcal{H}_{(\varphi,\mathrm{S})} \;=\; \mathcal{H}_{(\varphi',\mathrm{S}')}. \tag{13}$$

Since the lattice (13) is isometric to H, it contains only two classes of self-intersection -2. These two classes are s and $-s$. By the Riemann-Roch theorem, only s is effective. It follows that the two sections S and S' represent the same -2 class and, since they are both irreducible curves, $\mathrm{S} = \mathrm{S}'$. Next, we note that there exist only two isotropic elements in (13) which have intersection 1 with s. These classes are f and $-f - s$. The second one cannot be effective and, therefore, the two elliptic pencils φ and φ' must represent the same isotropic class f. Since the generic element in each of them is a smooth irreducible curve, it follows that $\varphi = \varphi'$. □

It is not true that all embeddings of H in $\mathrm{NS}(\mathrm{X})$ are induced by an elliptic structure with section (φ, S) on X. However, this statement becomes true if one requires that the image of H contains a pseudo-ample class.

Definition 2.2. *A class $d \in \mathrm{NS}(\mathrm{X})$ is called* **pseudo-ample** *if it represents an effective divisor D which is nef (has non-negative intersection with any effective class) and has positive self-intersection.*

The above terminology is closely related to the classical notion of ampleness. Given a pseudo-ample class on X and D an effective divisor representing d, a result of Mayer [21] asserts that the linear system $|n\mathrm{D}|$ is base point free for $n \geq 3$ and the associated map:

$$\psi_{|n\mathrm{D}|} \colon \mathrm{X} \to \mathbb{P}^{\mathrm{N}}$$

is a birational morphism. Moreover, the image of $\psi_{|n\mathrm{D}|}$ is the normal model of X obtained by contracting all curves not met by D which are rational double point configurations.

Theorem 2.3. *Let $\mathcal{H} \subset \mathrm{NS}(\mathrm{X})$ be a sublattice isometric to H. There exists an elliptic structure with section (φ, S) on X such that:*

$$\mathcal{H}_{(\varphi,\mathrm{S})} \;=\; \mathcal{H}$$

if and only if $\mathcal{H}$ contains a pseudo-ample class.

Proof. We first check that the above condition is necessary. Indeed, given an elliptic structure with section (φ, S), one has the special classes $f, s \in \mathcal{H}_{(\varphi,\mathrm{S})}$. The set of effective classes on X is then the semi-group generated by f, the irreducible components of the singular fibers, and the possible (multi)-sections. It is then a simple verification to check that the class $d = 2f + s$ is pseudo-ample. In fact:

$$nf + ms \text{ is pseudo} - \text{ample for any } n \geq 2m > 0. \tag{14}$$

In order to prove the opposite implication, let us assume that $\mathcal{H}$ is a sublattice of $\mathrm{NS}(\mathrm{X})$ which is isometric to H and contains a pseudo-ample class d. Theorem 2.3 follows in two steps via the following pair of lemmas.

Lemma 2.4. *Denote by Γ_{X} the group of isometries of $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ whose $\mathbb{C}$-linear extensions preserve the Hodge filtration of X. There exists an elliptic fibration with section (φ, C_0) on X, and an isometry $\beta \in \Gamma_X$ such that:*

$$\mathcal{H} \;=\; \beta\left(\mathcal{H}_{(\varphi,\mathrm{C}_0)}\right). \tag{15}$$

Lemma 2.5. *There exists an analytic automorphism $\alpha \colon \mathrm{X} \to \mathrm{X}$ such that*

$$\alpha^{*}\left(\mathcal{H}_{(\varphi,\mathrm{C}_0)}\right) \;=\; \mathcal{H}. \tag{16}$$

Moreover α^* is either β or $\beta \circ \mathrm{R}_{[\mathrm{C}_0]}$ where β is the lattice isometry of Lemma 2.4 and $\mathrm{R}_{[\mathrm{C}_0]} \colon \mathrm{H}^2(\mathrm{X}, \mathbb{Z}) \to \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ is the reflection associated to the class $[\mathrm{C}_0]$.

Proof of Lemma 2.4: Let us note that there exist two distinct classes in $\mathcal{H}$ of self-intersection -2. The two classes differ by a change in sign. By the Riemann-Roch theorem, one and only one of these two classes is effective. We denote this class by s.

There exist two and only two primitive isotropic classes within $\mathcal{H}$ which intersect s with intersection number 1. If one of these classes is f, the other is given by $-s-f$. At least one of these two classes has positive intersection with d. We can assume therefore that $d \cdot f > 0$. It follows that:

$$d = nf + ms \;\text{ with }\; n \geq 2m > 0. \tag{17}$$

By a classical result of Pjateckiĭ-Šapiro and Šafarevič (see, for instance, Chapter 3 of [30]), there exists an isometry β_1 of $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ such that $\beta_1(f)$ is the class of a smooth elliptic curve F inducing an elliptic pencil

$$\varphi \colon \mathrm{X} \to \mathbb{P}^1.$$

Moreover, the isometry β_1 is a composition of reflections associated to effective -2 classes in $\mathrm{NS}(\mathrm{X})$, and therefore $\beta_1 \in \Gamma_{\mathrm{X}}$.

We construct a section for the elliptic pencil φ. The Riemann-Roch theorem combined with the fact that

$$[\mathrm{F}] \cdot \beta_1(s) = 1,$$

implies that $\beta_1(s)$ is an effective class. Therefore $\beta_1(s)$ can be associated to a formal sum of irreducible curves:

$$\mathrm{C} \;=\; \sum_i \, n_i \mathrm{C}_i, \;\; n_i \geq 1.$$

But $\mathrm{C}_i \cdot \mathrm{F} \geq 0$ and $\mathrm{C} \cdot \mathrm{F} = 1$. Hence, among the irreducible curves C_i there exists a unique one, say denoted C_0, such that its intersection pairing with F is not zero. In this context, we see that $\mathrm{F} \cdot \mathrm{C}_0 = 1$, $n_0 = 1$, and $\mathrm{F} \cdot \mathrm{C}_i = 0$ for $i \neq 0$. The restriction of the elliptic pencil φ to C_0 then defines a degree one map $\mathrm{C}_0 \to \mathbb{P}^1$ and from this we conclude that C_0 is a smooth rational curve. We have obtained therefore an elliptic fibration with section (φ, C_0) on X.

Let us now define $\mathcal{H}' \;=\; \beta_1(\mathcal{H})$. The lattice $\mathcal{H}'$ is spanned by [F] and [C] and induces an orthogonal direct sum decomposition:

$$\mathrm{NS}(X) \;=\; \mathcal{H}' \;\oplus\; \mathcal{Q}.$$

In this decomposition, one can write:

$$[\mathrm{C}_0] \;=\; -\frac{(q_o, q_o)}{2}[\mathrm{F}] + [\mathrm{C}] + q_o$$

for some fixed $q_o \in \mathcal{Q}$. We then define $\beta_2 \colon \mathrm{NS}(\mathrm{X}) \to \mathrm{NS}(\mathrm{X})$ by:

$$\beta_2\left(a[\mathrm{F}] + b[\mathrm{C}] + q\right) \;=\; \left(a - b\frac{(q_o, q_o)}{2} - (q, q_o)\right)[\mathrm{F}] + b[\mathrm{C}] + bq_o + q.$$

One verifies that β_2 is an isometry of $\mathrm{NS}(\mathrm{X})$ satisfying $\beta_2([\mathrm{F}]) = [\mathrm{F}]$ and $\beta_2([\mathrm{C}]) = [\mathrm{C}_0]$. Moreover, β_2 is clearly the restriction of an isometry in Γ_X which (by a slight abuse of notation) we shall also denote β_2.

Let us then set:

$$\beta = \left(\beta_2 \circ \beta_1\right)^{-1}.$$

Since $\beta([\mathrm{C}_0]) = s$ and $\beta([\mathrm{F}]) = f$, we find that (15) holds. This finishes the proof of Lemma 2.4.

Proof of Lemma 2.5: Recall the following classical result of Pjateckiĭ-Šapiro and Šafarevič (see, for instance, Theorem 1 in Chapter 6 of [30]):

Theorem 2.6. *(Pjateckiĭ-Šapiro, Šafarevič) One has a decomposition:*

$$\Gamma_{\mathrm{X}} \;=\; \Gamma_{\mathrm{X}}^{\mathrm{eff}} \cdot \mathrm{W}(\mathrm{X}) \cdot \{\pm \mathrm{id}\}$$

where $\Gamma_{\mathrm{X}}^{\mathrm{eff}}$ *is the subgroup of effective isometries*[3] *of* Γ_{X} *and* $\mathrm{W}(\mathrm{X})$ *is the subgroup generated by reflections with respect to effective* -2 *classes in* $\mathrm{NS}(\mathrm{X})$.

One can then express:

$$\beta = \gamma_1 \circ \gamma_2 \circ \gamma_3$$

with $\gamma_1 \in \Gamma_{\mathrm{X}}^{\mathrm{eff}}$, $\gamma_2 \in \mathrm{W}(\mathrm{X})$, and $\gamma_3 = \pm\mathrm{id}$.

Let:

$$d' = n[\mathrm{F}] + m[\mathrm{C}_0]$$

with n, m chosen as in (17). Since $[\mathrm{F}]$ and $[\mathrm{C}_0]$ are classes representing the elliptic fiber and section of an elliptic fibration, by an argument similar to (14) it follows that $n[\mathrm{F}] + m[\mathrm{C}_0]$ is a pseudo-ample class. Moreover, since $\beta(d') = d$, we deduce that β preserves the positive cone of X. By definition, so do γ_1 and γ_2. It follows therefore that $\gamma_3 = \mathrm{id}$.

Claim 2.7. *The isometry* γ_2 *preserves the hyperbolic sublattice* $\mathcal{H}_{(\varphi, \mathrm{C}_0)}$.

In order to justify the above claim, let us assume that $\gamma_2 \neq \mathrm{id}$. Denote then by $\mathcal{C}_{\mathrm{X}}^{+}$ the Kähler cone of X and let $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$ be its closure inside the positive cone. It follows that both d' and d belong to $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$. Therefore, $\beta = \gamma_1 \circ \gamma_2$ sends an element of $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$ to another element of $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$. But, as is well-known (see, for instance, Proposition 3.10 in Chapter VIII of [3]), the isometries of $\Gamma_{\mathrm{X}}^{\mathrm{eff}}$ preserve the Kähler cone and therefore they preserve $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$. This leads to:

$$\gamma_2\,(d') \;=\; \gamma_1^{-1}\,(d) \tag{18}$$

with both d' and $\gamma_1^{-1}\,(d)$ being elements of $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$. However, another classical result here (see Proposition 3.9 in Chapter VIII of [3]) asserts that $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$ is a fundamental domain for the action of $\mathrm{W}(\mathrm{X})$, in the sense that any $\mathrm{W}(\mathrm{X})$-orbit meets $\overline{\mathcal{C}}_{\mathrm{X}}^{+}$ in exactly one point. Our assumption that $\gamma_2 \neq \mathrm{id}$, together with (18), implies:

$$d' = \gamma_1^{-1}\,(d) \;\text{ and }\; \gamma_2(d') = d'. \tag{19}$$

It follows then that d' belongs to the boundary of the fundamental domain mentioned above, that is there exists an effective (-2)-class $e \in \mathrm{NS}(\mathrm{X})$ such that $(e, d') = 0$. Next, we show that a class that satisfies this condition has to be either $[\mathrm{C}_0]$ or orthogonal to both $[\mathrm{C}_0]$ and $[\mathrm{F}]$. This implies the claim.

Every effective (-2)-class as above can be realized as a formal sum (with multiplicities) of smooth rational curves (see, for instance, Section 2.3 of [5]). We can assume therefore that:

$$e \;=\; \delta\,\mathrm{C}_0 + \sum_i \lambda_i\,\mathrm{F}_i \;+\; \sum_j \eta_j\,\mathrm{S}_j \tag{20}$$

where C_0 is the rational curve from above, F_i are smooth rational curves that lie within the singular fibers of φ, S_j are rational curves (distinct from C_0) making multi-sections of φ, and δ, λ_i, η_j are non-negative integers.

In this setting, one can write:

$$(e,\,\mathrm{F}) \;=\; \delta + \sum_j \eta_j\,(\,\mathrm{S}_j,\,\mathrm{F}\,) \;\geq\; \delta \tag{21}$$

$$(e,\,\mathrm{C}_0) \;=\; -2\delta + \sum_i \lambda_i\,(\,\mathrm{F}_i,\,\mathrm{C}_0\,) \;+\; \sum_j \eta_j\,(\mathrm{S}_j,\,\mathrm{C}_0\,) \;\geq\; -2\delta. \tag{22}$$

[3] By definition, an isometry $\gamma \in \Gamma_{\mathrm{X}}$ is effective if it preserves the set of effective classes of X.

But $d' = n[\mathrm{F}] + m[\mathrm{C}_0]$ with $n \geq 2m > 0$, and therefore we obtain:

$$0 \;=\; (e, d') \;=\; n(e,\ \mathrm{F}) + m(e,\ \mathrm{C}_0) \;\geq n\delta - 2m\delta \;\geq\; 0. \tag{23}$$

For the sake of clarity, let us here divide our discussion into two cases, depending on whether $\delta > 0$ or $\delta = 0$.

In the first case, (23) implies that $n = 2m$ and, in addition, one has equalities in (21) and (22). In turn, this leads to:

$$\eta_j \;=\; 0 \;\text{ and }\; (\mathrm{F}_i, \mathrm{C}_0) = 0. \tag{24}$$

This means that e is given by the sum of $\delta\ \mathrm{C}_0$ with an effective divisor whose cohomology class lies in

$$\left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right)^{\perp} \,\cap\, \mathrm{NS}(\mathrm{X}).$$

However, the above lattice is negative definite. Therefore, in order for e to satisfy $(e, e) = -2$, the only viable option is $e = \mathrm{C}_0$.

Finally, in the second case ($\delta = 0$) relation (23) still implies equalities in (21) and (22). This leads to the conclusion that:

$$e \;=\; \sum_i \lambda_i\ \mathrm{F}_i,\;\; \text{with } (\mathrm{F}_i, \mathrm{C}_0) = 0.$$

Therefore $e \in \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right)^{\perp} \,\cap\, \mathrm{NS}(\mathrm{X})$ and hence, the reflection with respect to e restricts to identity over the hyperbolic sublattice $\mathcal{H}_{(\varphi, \mathrm{C}_0)}$. This finishes the proof of Claim 2.7

Based on Claim 2.7, we can finish the proof of Lemma 2.5. Since $\gamma_1 \in \Gamma_X^{\mathrm{eff}}$, by the Strong Torelli Theorem for K3 surfaces (see Theorem 11.1 in Chapter VIII of [3] or the similar results in [30] and [20]), there exists an analytic automorphism $\alpha \in \mathrm{Aut}(X)$ such that $\gamma_1 = \alpha^*$. One has then:

$$\alpha^* \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right) \;=\; \gamma_1 \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right) \;=\; \beta \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right),$$

because, according to Claim 2.7, $\gamma_2 \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right) = \mathcal{H}_{(\varphi, \mathrm{C}_0)}$. But $\beta \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right) = \mathcal{H}$ and therefore

$$\alpha^* \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right) = \mathcal{H}.$$

In order to conclude the proof of Theorem 2.3, note that, in the above context, $(\varphi \circ \alpha, \alpha^{-1}(\mathrm{C}_0))$ is an elliptic fibration with section on X and:

$$\mathcal{H}_{(\varphi \circ \alpha, \alpha^{-1}(\mathrm{C}_0))} \;=\; \alpha^* \left(\mathcal{H}_{(\varphi, \mathrm{C}_0)}\right) \;=\; \mathcal{H}. \tag{25}$$

□

In [13, 28], Dolgachev and Nikulin have considered the notion of **pseudo-ample lattice polarization** of a K3 surface X. Given an even lattice M of signature $(1, t)$, $t \leq p_{\mathrm{X}} - 1$, which can be embedded in the K3 lattice, a pseudo-ample **M-polarization** of X is a lattice embedding

$$i \colon \mathrm{M} \hookrightarrow \mathrm{NS}(\mathrm{X})$$

whose image contains a pseudo-ample class. In this context, if H is the standard rank-two hyperbolic lattice, Theorem 2.3 shows that there exists a bijective correspondence

$$\left\{ \begin{array}{c} \text{elliptic fibrations} \\ \text{with section on X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{pseudo-ample} \\ \text{H-polarizations of X} \end{array} \right\}. \tag{26}$$

Moreover, under the above correspondence the equivalence relation on elliptic fibrations with section defined in (9) translates precisely to Dolgachev's notion of equivalence for lattice polarizations. This leads to a canonical bijective correspondence between the equivalence classes of the two structures. One can therefore obtain a moduli space for triples $(\mathrm{X}, \varphi, \mathrm{S})$ by constructing a moduli space for pairs $(\mathrm{X}, \mathcal{H})$ consisting of a K3 surface X and a pseudo-ample H-polarization $\mathcal{H}$.

The construction of such a moduli space of lattice polarizations has been done in [13]. We shall present just the main features.

Given a triple $(\mathrm{X}, \varphi, \mathrm{S})$, the orthogonal complement:

$$\left(\mathcal{H}_{(\varphi,\mathrm{S})}\right)^{\perp} \subset \mathrm{H}^2(\mathrm{X}, \mathbb{Z}) \tag{27}$$

is an even, unimodular lattice of signature $(2, 18)$, and therefore, by standard lattice theory, it is isometric to:

$$\mathrm{L} = \mathrm{H} \oplus \mathrm{H} \oplus \mathrm{E}_8 \oplus \mathrm{E}_8. \tag{28}$$

An isometry between (27) and (28) is called a marking. If two elliptic K3 surfaces with section $(\mathrm{X}, \varphi, \mathrm{S})$ and $(\mathrm{X}', \varphi', \mathrm{S}')$ are given markings q and q', an analytic isomorphism $\alpha\colon \mathrm{X} \to \mathrm{X}'$ preserving the elliptic fibrations and sections is said to be compatible with the markings if $q' = q \circ (\alpha^*)^{\perp}$, where $(\alpha^*)^{\perp}$ is the isometry induced by the restriction:

$$\alpha^*\colon \left(\mathcal{H}_{(\varphi',\mathrm{S}')}\right)^{\perp} \to \left(\mathcal{H}_{(\varphi',\mathrm{S})}\right)^{\perp}.$$

One defines then the **H-polarized period domain**:

$$\Omega = \{ [\omega] \in \mathbb{P}^1 (\mathrm{L} \otimes \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \ \langle \omega, \bar{\omega} \rangle > 0 \}. \tag{29}$$

Every equivalence class of a marked elliptic K3 surface with section $(\mathrm{X}, \varphi, \mathrm{S}, q)$ determines uniquely a **period line**

$$[\omega] = q\left(H^{0,2}(\mathrm{X})\right) \in \Omega. \tag{30}$$

This period correspondence can be naturally seen as an analytic morphism. Indeed, there exists a fine moduli space $\mathcal{M}_{K3}^{\mathrm{marked}}$ for marked elliptic K3 surfaces with section. $\mathcal{M}_{K3}^{\mathrm{marked}}$ is an analytic space of complex dimension 18 and it is constructed, along the lines of [2] and [30], by gluing together local moduli spaces of marked elliptic K3 surfaces with section. A lattice polarized version of the Global Torelli Theorem ([6, 30, 32]) then asserts that the period correspondence:

$$\mathcal{M}_{\mathrm{K3}}^{\mathrm{marked}} \to \Omega \tag{31}$$

defined by (30) is a surjective morphism of analytic spaces.

This picture can be further refined by removing the markings. The discrete group Γ of integral isometries of L acts on Ω as well as on $\mathcal{M}_{\mathrm{K3}}^{\mathrm{marked}}$ and the morphism (31) is equivariant with respect to the two actions. The quotient space:

$$\mathcal{M}_{\mathrm{K3}} := \Gamma \backslash \mathcal{M}_{\mathrm{K3}}^{\mathrm{marked}}$$

is a coarse moduli space for K3 surfaces with pseudo-ample H-polarizations (see remark 3.4 of [13]). Then, by taking into account Theorem 2.3, one can show that $\mathcal{M}_{\mathrm{K3}}$ can be seen as a coarse moduli space for elliptic K3 surfaces with section. The induced **period map**:

$$\mathrm{per}\colon \mathcal{M}_{\mathrm{K3}} \to \Gamma \backslash \Omega$$

is an isomorphism of analytic spaces.

The quotient $\Gamma\backslash\Omega$ is a connected space. To see this, note that Ω is an open subset of an 18-dimensional complex quadric and consists of two connected components interchanged by complex conjugation. But, the discrete group Γ acts transitively on the set of these two connected components and hence the quotient space is connected.

The space $\Gamma\backslash\Omega$ is, however, not compact. There are nonetheless methods available to compactify this space (and hence the moduli space $\mathcal{M}_{\mathrm{K3}}$), due to the fact that $\Gamma\backslash\Omega$ can be identified with an Hermitian symmetric space factored by the action of an arithmetic group. Indeed, the real Lie group $\mathrm{O}(2,18)$ of real isometries of $\mathrm{L} \otimes \mathbb{R}$ acts transitively on Ω. The action leads to an identification:

$$\Omega \;=\; \mathrm{O}(2,18)/\mathrm{SO}(2) \times \mathrm{O}(18). \tag{32}$$

The discrete group Γ has an obvious action on the right term above, and the correspondence (32) becomes Γ-invariant. Therefore the above identification can be pushed to the level of quotients, where one obtains:

$$\Gamma \backslash \Omega \;=\; \Gamma \backslash \mathrm{O}(2,18)/\mathrm{SO}(2) \times \mathrm{O}(18). \tag{33}$$

In this context, a special case of Mumford's toroidal compactification [1] allows one to enlarge $\Gamma \backslash \Omega$ by adding boundary components associated to maximal rational parabolic subgroups of Type II in $\mathrm{O}(2,18)$. These groups are, essentially, subgroups of isometries stabilizing a given rank-two primitive and isotropic sublattice $\mathrm{V} \subset \mathrm{L}$.

3 Review of the Type II Partial Compactification

One performs the Type II partial compactification of $\Gamma \backslash \Omega$ by adding two specific boundary divisors. Let us give here a brief account of the procedure. For a detailed presentation we refer the reader to section 3 of [9] as well as to [14] (for a closely related case).

As mentioned at the end of the previous section, the procedure we are about to describe is centered around rational Type II parabolic subgroups of $\mathrm{O}(2,18)$ which, in turn, are defined as stabilizer groups for rank-two primitive and isotropic sublattices V of L. By classical lattice theory (see [29]), there are two distinct types of sublattices with these features. Let $\mathrm{I}_2(\mathrm{L})$ be the set of all possible such sublattices. The group Γ acts on $\mathrm{I}_2(\mathrm{L})$ and the action generates two distinct orbits. One can differentiate between these two orbits by analyzing the isomorphism class of the induced quotient lattice $\mathrm{V}^{\perp}/\mathrm{V}$. This quotient lattice is always negative-definite, even, unimodular and of rank 16 and, as it is well-known, there exist only two non-equivalent lattices with these features. One is $\mathrm{E}_8 \oplus \mathrm{E}_8$, the orthogonal direct sum of two copies of the unique negative definite, unimodular and even lattice of rank 8. The other is usually denoted by D_{16}^{+} (see [10]) and represents the unimodular index-two over-lattice of the usual Dynkin lattice D_{16}.

Let us then outline the compactification construction. As a first step, pick one of the connected components of Ω and denote it by Ω^{+}. One has then a natural identification:

$$\Gamma \backslash \Omega \;=\; \Gamma^{+} \backslash \Omega^{+} \tag{34}$$

where Γ^{+} is the index-two subgroup of Γ corresponding to isometries preserving Ω^{+}. Select then a sublattice V with the properties described above. As the next step, the technical ingredient required by Mumford's method is a primitive integral and nilpotent element N in the Lie algebra of $\mathrm{O}(2,18)$ such that $\mathrm{Im}(\mathrm{N}) = \mathrm{V}$. At this point in the general construction of [1], one has to make a choice of such N. However, in the present situation, the choice is canonical. An endomorphism N with above characteristics is unique, up to a sign change. Moreover, given such an N, for any $[\omega] \in \Omega$, the quantity $i\langle \mathrm{N}\omega, \bar{\omega} \rangle$ is real and non-zero. This quantity is positive on one connected component of Ω and negative on the other. We can then canonically select N such that $i\langle \mathrm{N}\omega, \bar{\omega} \rangle > 0$ for $[\omega]$ in Ω^{+}.

We next introduce the following group:

$$\mathrm{U}(\mathrm{N})_{\mathbb{C}} \;=\; \{ \, \exp(z\mathrm{N}) \mid z \in \mathbb{C} \, \}.$$

This group is by definition isomorphic to $(\mathbb{C}, +)$ and acts naturally on the compact quadric:

$$\Omega^{\vee} \;=\; \{ \, [\omega] \in \mathbb{P}^1 \left(\mathrm{L} \otimes \mathbb{C} \right) \mid \langle \omega, \omega \rangle = 0 \, \}$$

in which Ω embeds as an open subset. Denote by $\mathrm{U}(\mathrm{N})_{\mathbb{Z}}$ its subgroup corresponding to $z \in \mathbb{Z}$ and set:

$$\Omega^{+}(V) = \mathrm{U}(\mathrm{N})_{\mathbb{C}} \cdot \Omega^{+}.$$

This allows one to construct the following projection:

$$\mathrm{U(N)}_{\mathbb{Z}} \backslash \Omega^+(V) \;\to\; \mathrm{U(N)}_{\mathbb{C}} \backslash \Omega^+(V). \tag{35}$$

It can be easily verified that (35) is a holomorphic principal bundle with structure group $\mathbb{T} = \mathrm{U(N)}_{\mathbb{Z}} \backslash \mathrm{U(N)}_{\mathbb{C}}$ and that $\mathrm{U(N)}_{\mathbb{Z}} \backslash \Omega^+$ embeds as an open subset of the total space $\mathrm{U(N)}_{\mathbb{Z}} \backslash \Omega^+(V)$.

Let then $\mathrm{P} \subset \Gamma$ be the parabolic subgroup associated to V, that is, the subgroup of isometries stabilizing V. Denote $\mathrm{P}^+ = \mathrm{P} \cap \Gamma^+$. One checks that $\mathrm{U(N)}_{\mathbb{Z}} \lhd \mathrm{P}^+$. Moreover, the discrete group P^+ acts on both the base space and total space of the principal bundle (35) and the action is compatible with both the projection, and the action of the structure group $\mathbb{T}$ on the total space $\mathrm{U(N)}_{\mathbb{Z}} \backslash \Omega^+(V)$. Therefore, (35) descends to a holomorphic principal bundle[4] of structure group $\mathbb{T}$:

$$\alpha_V : \mathrm{P}^+ \backslash \Omega^+(\mathrm{V}) \;\to\; \mathrm{P}^+ \backslash \left(\mathrm{U(N)}_{\mathbb{C}} \backslash \Omega^+(\mathrm{V}) \right). \tag{36}$$

For simplicity, we shall denote the base space of (36) by $\mathcal{D}(V)$. This is the **Mumford boundary component associated to** V that one makes use of in order to partially compactify $\Gamma^+ \backslash \Omega^+$.

Mumford's toroidal compactification idea applies here in the following form. Since the group $\mathbb{T}$ is naturally identified with $\mathbb{C}^*$ by the exponential map, one can make this group act on a copy of the complex plane $\mathbb{C}$ in the standard way. This action is then used to construct the associated fibration with lines:

$$\overline{\mathrm{P}^+ \backslash \Omega^+(V)} \;:=\; \mathrm{P}^+ \backslash \Omega^+(V) \times_{\mathbb{T}} \mathbb{C}\,, \tag{37}$$

a procedure that has the effect of adding to the total space of (36) a divisor corresponding to the compactification of $\mathbb{C}^*$ to $\mathbb{C}$ in each fiber. With this construction in place, one defines:

$$\overline{\mathrm{P}^+ \backslash \Omega^+} \;:=\; \text{interior of the closure of } \mathrm{P}^+ \backslash \Omega^+ \text{ in } \overline{\mathrm{P}^+ \backslash \Omega^+(V)}. \tag{38}$$

Set-theoretically,

$$\overline{\mathrm{P}^+ \backslash \Omega^+} \;=\; \mathrm{P}^+ \backslash \Omega^+ \;\bigsqcup\; \mathcal{D}(\mathrm{V})$$

and therefore we have just performed a holomorphic gluing of the boundary component $\mathcal{D}(\mathrm{V})$ to $\mathrm{P}^+ \backslash \Omega^+$.

The last step in the procedure is the attaching of $\mathcal{D}(\mathrm{V})$ to $\Gamma^+ \backslash \Omega^+$. The key ingredient here, as mentioned in the introduction, is the existence of a large complex structure period domain. That is, there exists an open subset:

$$\mathcal{V}^+ \;\subset\; \Omega^+ \tag{39}$$

that satisfies the following properties:

(a) $\mathcal{V}^+$ is invariant under the action of P^+,

(b) the restriction of $\mathrm{U(N)}_{\mathbb{Z}} \backslash \mathcal{V}^+$ to any given fiber of (35) is an open neighborhood of the cusp,

(c) on $\mathcal{V}^+$, the equivalence under the action of Γ^+ reduces to P^+-equivalence.

If these three conditions are satisfied then the non-normal covering projection $\pi\colon \mathrm{P}^+ \backslash \Omega^+ \to \Gamma^+ \backslash \Omega^+$, when restricted to $\mathrm{P}^+ \backslash \mathcal{V}^+$ induces a holomorphic isomorphism between $\mathrm{P}^+ \backslash \mathcal{V}^+$ and its image. The following commutative diagram:

$$\begin{tikzcd}
\mathrm{P}^+ \backslash \mathcal{V}^+ \arrow[r, hook] \arrow[d, "\simeq"'] & \mathrm{P}^+ \backslash \Omega^+ \arrow[r] \arrow[d, "\pi"'] & \overline{\mathrm{P}^+ \backslash \Omega^+} \\
\pi\left(\mathrm{P}^+ \backslash \mathcal{V}^+\right) \arrow[r, hook] & \Gamma^+ \backslash \Omega^+ &
\end{tikzcd} \tag{40}$$

allows one to use (the inverse of) this isomorphism as gluing map, thus attaching $\mathcal{D}(V)$ holomorphically onto $\Gamma^+ \backslash \Omega^+$.

[4]Strictly speaking, the fibration (36) is not a principal bundle, as it is not locally trivial and the group $\mathbb{T}$ does not act freely on certain fibers. A more rigorous description here would be that (36) is a holomorphic Seifert $\mathbb{T}$-fibration.

The open set

$$\mathcal{U} = \pi\left(\mathrm{P}^+ \backslash \mathcal{V}^+\right),$$

is the **large complex structure domain** associated to $\mathcal{D}(\mathrm{V})$.

Remark 3.1. *Note that the subset $\mathcal{U}$, as well as the gluing map, depends only on the Γ-orbit of V in $\mathrm{I}_2(\mathrm{L})$. If one repeats the above procedure starting with a different sublattice $V' = \gamma(V)$ for $\gamma \in \Gamma$ the construction produces the same partial compactification. However, isotropic sublattices V and V' inequivalent under Γ lead to distinct large complex structure regions and different compactifications.*

Finally, the Type II partial compactification $\overline{\Gamma\backslash\Omega}$ is the space obtained after gluing onto $\Gamma\backslash\Omega$ the two boundary components $\mathcal{D}_1$ and $\mathcal{D}_2$ associated to the two Γ-orbits of $\mathrm{I}_2(\mathrm{V})$. It follows that $\overline{\Gamma\backslash\Omega}$ is a quasi-projective analytic space of complex dimension eighteen. It contains $\Gamma\backslash\Omega$ as a Zariski open subset. The complement

$$\overline{\Gamma\backslash\Omega} \ \backslash \ \Gamma\backslash\Omega$$

is the disjoint union of two irreducible divisors. Each of the two divisors $\mathcal{D}_1$, $\mathcal{D}_2$ is a quotient of a smooth space by a finite group action. We refer the reader to [1] for proofs of these statements.

4 A Large Complex Structure Condition

The main goal of this paper is to introduce an effective Hodge-theoretic condition for an elliptic $K3$ surface with section to have large complex structure. In other words, we wish to construct explicitly a large complex structure period region $\mathcal{V}^+$ as in (39) leading to a large complex structure domain $\mathcal{U} = \pi\left(\mathrm{P}^+ \backslash \mathcal{V}\right)$.

Let (X, φ, S) be an elliptically fibered $K3$ surface with section. Assume that V is a rank-two primitive and isotropic sublattice of

$$V \subset \left(\mathcal{H}_{(X,\varphi,S)}\right)^{\perp} \subset H^2(X, \mathbb{Z}).$$

We shall attach to such a quadruple, (X, φ, S, V), two Hodge-theoretic quantities denoted (by abuse of notation) $\tau(X, V)$ and $\widetilde{u}_2(X, V)$. The first quantity is a complex number with positive imaginary part. The second one is a positive real number.

Consider $\omega \in H^2(X, \mathbb{C})$ to be a class representing a holomorphic two-form on X. The class ω is unique up to a scaling by a non-zero complex number. Moreover, since the lattice $\mathcal{H}_{(X,\varphi,S)}$ is spanned by classes representing algebraic cycles, ω belongs to

$$\left(\mathcal{H}_{(X,\varphi,S)}\right)^{\perp} \otimes \mathbb{C}. \tag{41}$$

The intersection form on the real version of (41) has signature $(2, 18)$, and hence (see also Lemma 5.1), $\langle \omega, y \rangle \neq 0$ for any $y \in V$. We select then an oriented basis $\{y_1, y_2\}$ in V such that the $\mathbb{R}$-linear map:

$$\langle \omega, \cdot \rangle \colon V \otimes \mathbb{R} \to \mathbb{C} \tag{42}$$

is orientation preserving. We then define:

$$\tau(X, V) \ := \ \frac{\langle \omega, y_1 \rangle}{\langle \omega, y_2 \rangle}. \tag{43}$$

The quantity $\tau(X, V)$ is a well-defined complex number which does not change when one scales the class ω. Due to the orientation preserving assumption in (42), its imaginary part $\tau_2(X, V)$ is positive. Moreover, if one modifies the oriented basis $\{y_1, y_2\}$ on V, $\tau(X, V)$ varies under the usual $\mathrm{SL}(2, \mathbb{Z})$-action on the upper half-plane $\mathbb{H}$.

The second parameter mentioned earlier is introduced in the following form:

$$\widetilde{u}_2(X, V) \ := \ \frac{\langle \omega, \overline{\omega} \rangle}{4\, |\langle \omega, y_2 \rangle|^2 \cdot \ \mathrm{Im}\left[\tau(X, V)\right]}. \tag{44}$$

Lemma 4.1. *The term $\widetilde{u}_2(X,V)$ is a positive real number and is independent of the choice of ω. Moreover $\widetilde{u}_2(X,V)$ remains unchanged when one modifies the oriented basis $\{y_1, y_2\}$.*

Proof. The first assertion is straightforward. To check the second assertion, note that if one changes the oriented basis in V by way of an $\mathrm{SL}(2,\mathbb{Z})$ matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then this change induces the following variations:

$$\mathrm{Im}\,[\tau(X,V)] \;\mapsto\; \frac{\mathrm{Im}\,[\tau(X,V)]}{|c\tau(X,V)+d|^2}$$

$$|\langle \omega, y_2\rangle|^2 \;\mapsto\; |c\tau(X,V)+d|^2 \cdot |\langle \omega, y_2\rangle|^2.$$

The denominator of the right side of (44) remains invariant. □

Finally, we introduce the following auxiliary function:

$$\rho\colon \mathbb{H} \to (0,\infty)$$

$$\rho(\tau) \;=\; \sup_{m\in \mathrm{SL}(2,\mathbb{Z})} \mathrm{Im}\,(m\cdot\tau)\,.$$

The function $\rho(\tau)$ is well-defined and clearly invariant under the $\mathrm{SL}(2,\mathbb{Z})$ action. An alternative way of defining $\rho(\tau)$ is as the imaginary part of the unique representative of the $\mathrm{SL}(2,\mathbb{Z})$ orbit of τ in the standard fundamental domain of $\mathbb{H}$ with respect to the $\mathrm{SL}(2,\mathbb{Z})$ action. This formulation shows, in particular, that $\rho(\tau) \geq \sqrt{3}/2$.

Definition 4.2. *An elliptic $K3$ surface with section (X,φ,S) is said to have* **large complex structure with respect to V** *if:*

$$\widetilde{u}_2(X,V) \;>\; \max\left(\rho(\tau(X,V)),\; \frac{2}{\sqrt{3}}\right). \tag{45}$$

In order to connect this definition with the discussion in section 3, note that $\tau(X,V)$ and $\widetilde{u}_2(X,V)$ can be seen as C^∞ functions on Ω (τ is in fact analytic). One defines then the open subset $\mathcal{V} \subset \Omega$ of period lines satisfying condition (45). This set decomposes as a disjoint union:

$$\mathcal{V} \;=\; \mathcal{V}^+ \,\sqcup\, \mathcal{V}^- \tag{46}$$

where $\mathcal{V}^+ = \mathcal{V} \cap \Omega^+$ and $\mathcal{V}^-$ is the complex conjugate of $\mathcal{V}^+$. We claim then that $\mathcal{V}^+$ satisfies the features described in (39). Namely:

Theorem 4.3. *The open subset $\mathcal{V}^+$ satisfies the following:*

(a) $\mathcal{V}^+$ is left invariant by the action of P^+,

(b) $\mathrm{U(N)}_{\mathbb{Z}}\backslash\mathcal{V}^+$ is an open subset of the total space of the holomorphic principal $\mathbb{T}$-bundle:

$$\mathrm{U(N)}_{\mathbb{Z}}\backslash\Omega^+(V) \;\to\; \mathrm{U(N)}_{\mathbb{C}}\backslash\Omega^+(V)$$

defined in (35) and its intersection with every fiber of the bundle is an open neighborhood of the cusp,

(c) for any two $[\omega_1], [\omega_2] \in \mathcal{V}^+$ and $\gamma \in \Gamma^+$ such that $\gamma([\omega_1]) = [\omega_2]$, one has that $\gamma \in \mathrm{P}^+$.

The proof of Theorem 4.3 is presented in the next section.

We finish this section by formulating an extension of the large complex structure definition which is independent of V. Recall from section 3 that the set of primitive and isotropic rank-two sublattices in $\left(\mathcal{H}_{(X,\varphi,S)}\right)^{\perp}$ is acted upon by the discrete group Γ of isometries and that this action has two distinct orbits, essentially related to the two possible rank 16 negative definite even and unimodular lattices $E_8 \oplus E_8$ and D_{16}^{+}. As a consequence of Theorem 4.3, one has that, given an elliptic $K3$ surface with section (X, φ, S) and two distinct isotropic lattices V and V' which belong to the same Γ-orbit, (X, φ, S) can be of large complex structure with respect to at most one of V and V'. This allows us to formulate:

Definition 4.4. *An elliptic $K3$ surface with section X is said to have* **large complex structure in the $\mathbf{E_8} \oplus \mathbf{E_8}$ (or $\mathbf{D_{16}^{+}}$) sense** *if there exists a primitive isotropic rank-two lattice V in the appropriate Γ-orbit such that X has large complex structure with respect to V.*

5 Proof of Theorem 4.3

As a first step, in order to gain a better understanding of the partial compactification outlined in section 3, we introduce special coordinates on Ω^+. We are going to use two distinct, but closely related, sets of parameterizations. The first parametrization is holomorphic and represents the standard **tube domain realization** of Ω^+ (see, for example, [13]). The second coordinate system is a (non-holomorphic) perturbation of the tube domain coordinates and presents Ω^+ as a Siegel domain of the third kind. We shall refer to the latter coordinates as the **Narain parametrization**, as the description defined by them is related to a string theory construction of Narain [27].

5.1 Narain Coordinates

As before, we start with a fixed choice of a rank-two sublattice V of L which is primitive and isotropic. For any ω representing a class in Ω, the homomorphism:

$$\langle \omega, \cdot \rangle \, : \, \mathrm{V} \otimes \mathbb{R} \; \rightarrow \; \mathbb{C} \tag{47}$$

is an isomorphism of real vector spaces. This fact follows immediately from the following lemma.

Lemma 5.1. *Let $v \in \mathrm{L}$ be a non-zero element satisfying $\langle v, v \rangle = 0$. Then $\langle \omega, v \rangle \neq 0$ for any $[\omega] \in \Omega$.*

Proof. Assume that $\langle \omega, v \rangle = 0$ for some $[\omega] \in \Omega$. Denote by Q the plane in $\mathrm{L} \otimes \mathbb{R}$ spanned by the real and imaginary parts of ω. Then Q is positive definite with respect to $\langle \cdot, \cdot \rangle$ and since v is not zero, $v \notin Q$. Then v and Q span a three-dimensional subspace of $\mathrm{L} \otimes \mathbb{R}$ on which the pairing $\langle \cdot, \cdot \rangle$ is non-negative. This contradicts the fact that the signature of L is $(2, 18)$. $\square$

The above observation provides another effective method of differentiating between the two connected components of Ω. If an orientation is chosen on $\mathrm{V} \otimes \mathbb{R}$, then the map (47) is either orientation preserving or orientation reversing, depending on the component in which $[\omega]$ lies.

For the purpose of streamlining future computations we shall fix at this point an orientation on $\mathrm{V} \otimes \mathbb{R}$ such that (47) is orientation reversing for any ω representing a class in Ω^+. We make then the first step toward parameterizing Ω^+ by selecting a set $\{x_1, x_2, y_1, y_2\}$ of four linearly independent isotropic elements of L such that:

- $\{y_1, y_2\}$ forms an oriented basis in V
- $\langle x_1, x_2 \rangle \; = \; \langle x_1, y_2 \rangle \; = \; \langle x_2, y_1 \rangle \; = \; 0$
- $\langle x_1, y_1 \rangle \; = \; \langle x_2, y_2 \rangle \; = \; 1.$

It is clear that collections with the above features do exist. Such a collection is nothing but an embedding of an orthogonal direct sum $H \oplus H$ into L in a way such the image contains V.

Denote then by Λ the orthogonal complement in L of the sublattice generated by x_1, x_2, y_1 and y_2. It follows that Λ is of rank 16 and is unimodular and negative definite. Moreover, this construction induces a decomposition of L as a direct sum:

$$\mathrm{L} \ = \ (\mathbb{Z}x_1 \oplus \mathbb{Z}x_2) \ \oplus \ (\mathbb{Z}y_1 \oplus \mathbb{Z}y_2) \ \oplus \ \Lambda. \tag{48}$$

This allows us to identify an element of L as:

$$(a_1, a_2)(b_1, b_2)(c) \tag{49}$$

where a_1, a_2, b_1, b_2 are integers, $c \in \Lambda$ and the parameters a_1, a_2, b_1, b_2, c are uniquely determined. In this setting, the pairing $\langle \cdot, \cdot \rangle$ is recovered as:

$$\langle \ (a_1, a_2)(b_1, b_2)(c), \ (a_1', a_2')(b_1', b_2')(c') \ \rangle \ = \ a_1 b_1' + a_2 b_2' + b_1 a_1' + b_2 a_2' + (c, c') \tag{50}$$

where $(\cdot, \cdot)$ is the negative-definite pairing on Λ.

Remark 5.2. *Note that the projection on the third term in* (48) *induces a natural isometry:*

$$\mathrm{V}^{\perp}/\mathrm{V} \ \simeq \ \Lambda.$$

Let then $[\omega]$ be an element of Ω^+. Using Lemma 5.1, one can always pick a normalized representative ω such that $\langle \omega, y_2 \rangle = 1$. Then, under the identification (48), one can write:

$$\omega \ = \ (\tau, 1)(u, w)(z) \tag{51}$$

where $\tau, u, v \in \mathbb{C}^*$, $z \in \Lambda_{\mathbb{C}}$. The first Hodge-Riemann condition $\langle \omega, \omega \rangle = 0$ becomes:

$$2\,(\tau u + w) + (z, z) \ = \ 0$$

and therefore:

$$w \ = \ -\tau u - \frac{(z, z)}{2}. \tag{52}$$

After substituting w in (51), the second Hodge -Riemann condition $\langle \omega, \bar{\omega} \rangle > 0$ can be written as:

$$4\tau_2 u_2 + 2(z_2, z_2) > 0 \tag{53}$$

where the subscript 2 indicates that one takes the imaginary part. But $[\omega] \in \Omega^+$ and, because of the convention assuming that (47) reverses orientations, we obtain that $\tau_2 > 0$. Inequality (53) implies then:

$$u_2 \ > \ -\frac{(z_2, z_2)}{2\tau_2} \ > \ 0. \tag{54}$$

It follows that:

Proposition 5.3. *(Tube Domain Realization) One has an analytic isomorphism:*

$$\Omega^+ \ \xrightarrow{\simeq} \ \{ \ (\tau, u, z) \in \mathbb{H} \times \mathbb{H} \times \Lambda_{\mathbb{C}} \mid 2\tau_2 u_2 + (z_2, z_2) > 0 \ \} \tag{55}$$

where $\mathbb{H}$ denotes the complex upper half-plane and $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$. The inverse of (55) *maps a coordinate triple (τ, u, z) to a period line $[\omega]$ with:*

$$\omega \ = \ (\tau, 1) \left(u, -\tau u - \frac{(z, z)}{2} \right) (z). \tag{56}$$

The **Narain parametrization** perturbs slightly the second tube domain coordinate above. Indeed, under the same assumptions as before, let:

$$\widetilde{u} \; : = \; u + \frac{(z, z_2)}{2\tau_2}. \tag{57}$$

The inequality (54) assures us that $\widetilde{u} \in \mathbb{H}$ and it follows that:

Proposition 5.4. *(Narain Coordinates) One has a C^∞ isomorphism:*

$$\Omega^+ \; \xrightarrow{\simeq} \; \mathbb{H} \times \mathbb{H} \times \Lambda_{\mathbb{C}} \tag{58}$$

which associates to a period line $[\omega]$ the triple $(\tau, \widetilde{u}, z)$. The inverse of (58) maps a coordinate triple $(\tau, \widetilde{u}, z)$ to a period line $[\omega]$ with:

$$\omega \; = \; (\tau, \; 1) \left(\widetilde{u} - \frac{(z, z_2)}{2\tau_2}, \; -\tau\widetilde{u} + \frac{\tau(z, z_2)}{2\tau_2} - \frac{(z, z)}{2} \right) (z).$$

Note at this point that, for both parameterizations described above, the projection on the first and third coordinates:

$$\Omega^+ \; \rightarrow \; \mathbb{H} \times \Lambda_{\mathbb{C}}, \;\; [\omega] \mapsto (\tau, z) \tag{59}$$

is holomorphic. This map is nothing but the restriction to Ω^+ of the projection:

$$\Omega^+(\mathrm{V}) \; \rightarrow \; \mathrm{U(N)}_{\mathbb{C}} \backslash \Omega^+(\mathrm{V}) \tag{60}$$

from section 3. Indeed, the integral nilpotent endomorphism N used in the construction of section 3 can be written here explicitly as:

$$\mathrm{N} \colon \mathrm{L} \rightarrow \mathrm{L}, \quad \mathrm{N}(\gamma) = \langle \gamma, y_2 \rangle y_1 - \langle \gamma, y_1 \rangle y_2 \tag{61}$$

and, using the coordinates of (49), it can be described as:

$$\mathrm{N}\left((a_1, a_2)(b_1, b_2)(c) \right) \; = \; (0, 0)(a_2, -a_1)(0).$$

The group $U(\mathrm{N})_{\mathbb{C}}$ acts then on a period line $[\omega]$ associated to (56) as follows:

$$\exp(w\mathrm{N}) \cdot [\omega] \; = \; \left[(\tau, 1) \left(u + w, -\tau(u + w) - \frac{(z, z)}{2} \right) (z) \right]. \tag{62}$$

Hence, in the framework of tube domain parametrization (as well as in Narain parametrization) the action of $U(\mathrm{N})_{\mathbb{C}}$ has the effect of a translation in the second coordinate.

We also note that the two quantities $\tau(X, V)$ and $\widetilde{u}_2(X, V)$ used in the definition of the large complex structure condition of section 4 are the first Narain coordinate and the imaginary part of the second Narain coordinate, respectively, of the corresponding period line in Ω^+. The Narain coordinate description of the large complex structure domain is then:

$$\mathcal{V}^+ \; = \; \left\{ \; (\tau, \widetilde{u}, z) \in \Omega^+ \mid \widetilde{u}_2 > \max\left(\rho(\tau), \frac{2}{\sqrt{3}} \right) \right\}.$$

5.2 The Action of the Parabolic Group P^+

Next, we use the framework constructed by the Narain coordinates to explicitly describe the action of the parabolic group P^+ on the period domain Ω^+. Note that, under decomposition (48), one can view an isometry $\gamma \in \Gamma$ as a matrix:

$$\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix} \tag{63}$$

with A, B, D, E in $\mathrm{End}(\mathbb{Z}^2)$, C, F in $\mathrm{Hom}(\Lambda, \mathbb{Z}^2)$, G, H in $\mathrm{Hom}(\mathbb{Z}^2, \Lambda)$ and $K \in \mathrm{O}(\Lambda)$. The conditions required for the entries of (63) to determine an actual isometry of L can be written explicitly and, they lead one to the following conclusion:

Proposition 5.5. *A parabolic isometry $\gamma \in \mathrm{P}$ is given by a matrix:*

$$\begin{pmatrix} m & 0 & 0 \\ R & \widetilde{m} & -Qf \\ Q^t m & 0 & f \end{pmatrix} \tag{64}$$

with entries as follows.

1. $m \in \mathrm{GL}(2,\mathbb{Z})$ *and* $\widetilde{m}$ *represents* $(m^t)^{-1}$.

2. $Q \in \mathrm{Hom}(\Lambda, \mathbb{Z}^2)$, $R \in \mathrm{End}(\mathbb{Z}^2)$ *and they satisfy:*

$$R^t m + m^t R + m^t Q Q^t m = 0. \tag{65}$$

3. f *is an isometry of* Λ.

A matrix as above corresponds to an isometry in P^+ *if and only if* m *is an element of* $\mathrm{SL}(2,\mathbb{Z})$.

The upper-script "t" refers to the adjoint of the homomorphism in question with respect to the pairings existing on its domain and target space.

For simplicity, we shall refer to the isometry given by the matrix (64) as $\gamma(m, Q, R, f)$. The composition law on P can then be read in this context as:

$$\gamma(m_1, Q_1, R_1, f_1) \circ \gamma(m_2, Q_2, R_2, f_2) \;=\; \gamma(m_1 m_2,\ Q_1 + \widetilde{m}_1 Q_2 f_1^{-1},\ R_1 m_2 + \widetilde{m}_1 R_2 - Q_1 f_1 Q_2^t m_2,\ f_1 f_2).$$

In particular, we have:

$$\gamma(m, Q, R, f) \;= \gamma(I, Q, Rm^{-1}, I) \;\circ\; \gamma(m, 0, 0, I) \;\circ\; \gamma(I, 0, 0, f). \tag{66}$$

This shows that P^+ is generated by three special subgroups with familiar-looking structure.

(a) $\mathcal{S} = \{R = 0, Q = 0, f = \mathrm{id}_\Lambda\}$. This group is naturally isomorphic to a copy of $\mathrm{SL}(2,\mathbb{Z})$.

(b) $\mathcal{W} = \{m = \mathrm{I}_2, R = 0, Q = 0\}$. This group is just the orthogonal group of the lattice Λ.

(c) $\mathcal{T} = \{m = \mathrm{I}_2, f = \mathrm{id}_\Lambda\}$. This is, essentially, the Heisenberg group of the lattice Λ. Note that, imposing $Q = 0$ in $\mathcal{T}$, we obtain a normal subgroup which is no more but the abelian group $\mathrm{U(N)}_\mathbb{Z}$. The quotient $\mathcal{T}/\mathrm{U(N)}_\mathbb{Z}$ is naturally isomorphic to $\Lambda \oplus \Lambda$ and this fact leads to a presentation of $\mathbb{T}$ as a semi-direct product:

$$(\Lambda \oplus \Lambda) \ltimes \mathrm{U(N)}_\mathbb{Z}.$$

By similar considerations, one verifies the following:

Remark 5.6. *The subgroups* $\mathcal{S}$, $\mathcal{W}$ *and* $\mathcal{T}$ *generate the entire* P^+. *The following features also hold:*

(1) $\mathrm{U(N)}_\mathbb{Z} \subset \mathrm{Z}\left(\mathrm{P}^+\right)$

(2) $\mathcal{T} \lhd \mathrm{P}^+$

(3) $[\mathcal{S},\ \mathcal{W}] \;=\; \{I\}$

(4) $\mathrm{P}^+ \;=\; \mathcal{T} \rtimes (\mathcal{S} \times \mathcal{W})$.

Let us then describe the action of P^+ on Ω^+ by writing the action for each of the three types of generators. We use Narain coordinates to make this description.

Theorem 5.7. *Let* $[\omega]$ *be a period line in* Ω^+ *of Narain coordinates* $(\tau, \widetilde{u}, z)$. *Consider an integral isometry* $\gamma \in \mathrm{P}^+$ *and denote by* $(\tau', \widetilde{u}', z')$ *the Narain coordinates of* $\gamma\left([\omega]\right)$. *Then:*

(a) If $\gamma = \gamma(m, 0, 0, I) \in \mathcal{S}$ with m given by the $\mathrm{SL}(2, \mathbb{Z})$ matrix:

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$$

then:

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad \widetilde{u}' = \widetilde{u}, \quad z' = \frac{z}{c\tau + d}. \tag{67}$$

(b) If $\gamma = \gamma(I, 0, 0, f) \in \mathcal{W}$ with $f \in \mathrm{O}(\Lambda)$ then:

$$\tau' = \tau, \quad \widetilde{u}' = \widetilde{u}, \quad z' = f(z).$$

(c) Assume that $\gamma = \gamma(I, R, Q, I) \in \mathcal{T}$. The homomorphism $Q \in \mathrm{Hom}(\Lambda, \mathbb{Z})$ induces two uniquely defined $c_1, c_2 \in \Lambda$ such that $Q(c) = ((c, c_1), (c, c_2))$ for any $c \in \Lambda$. Let

$$R = \left(\begin{array}{cc} r_{11} & r_{12} \\ r_{21} & r_{22} \end{array} \right).$$

Then:

$$\tau' = \tau, \quad \widetilde{u}' = \widetilde{u} + r_{12} + \frac{(c_1, c_2)}{2} - \frac{(z, c_1)}{2} + \frac{(\tau c_1 + c_2, z_2)}{2\tau_2}, \quad z' = \tau c_1 + c_2 + z. \tag{68}$$

Proof. Let us analyze case (a). Note that:

$$\widetilde{m} = \left(\begin{array}{cc} d & -c \\ -b & a \end{array} \right).$$

Therefore, the parabolic transformation $\gamma(m, 0, 0, I)$ sends the period line:

$$[\omega] = \left[(\tau, 1) \left(u, -\tau u - \frac{(z, z)}{2} \right) (z) \right]$$

to:

$$\left[(a\tau + b, c\tau + d) \left(du + c\tau u + \frac{c(z, z)}{2}, \ -bu - a\tau u - \frac{a(z, z)}{2} \right) (z) \right]. \tag{69}$$

After the proper normalization, the holomorphic (tube domain) coordinates of (69) are:

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad u' = u + \frac{c(z, z)}{2(c\tau + d)}, \quad z' = \frac{z}{c\tau + d}.$$

We have then two of the conditions required by (67). The only thing missing is the second Narain coordinate of (69). But that value can be derived as:

$$\widetilde{u}' = u' + \frac{(z', z_2')}{2\tau_2'} = u + \frac{c(z, z)}{2(c\tau + d)} + \frac{(z', z_2')}{2\tau_2'} = \widetilde{u} - \frac{(z, z_2)}{2\tau_2} + \frac{c(z, z)}{2(c\tau + d)} + \frac{(z', z_2')}{2\tau_2'}. \tag{70}$$

The imaginary parts of the primed terms above are:

$$\tau_2' = \frac{\tau_2}{|c\tau + d|^2},$$

$$z_2' = \frac{c\tau_1 + d}{|c\tau + d|^2} \, z_2 - \frac{c\tau_2}{|c\tau + d|^2} \, z_1$$

and, after introducing these expressions in (70) and carefully removing the canceling terms, we obtain that $\widetilde{u}' = \widetilde{u}$. Here the subscript 1 refers to the real part of the corresponding term.

Case (b) is straightforward. For case (c), we follow a strategy similar with the one used to check case (a). The parabolic isometry $\gamma(I,R,Q,I)$ sends the parabolic line:

$$[\omega] \;=\; \left[(\tau,1)\left(u,-\tau u-\frac{(z,z)}{2}\right)(z)\right]$$

to

$$\left[(\tau,1)\left(r_{11}\tau+r_{12}+u-(z,c_1),\; r_{21}\tau+r_{22}-\tau u-\frac{(z,z)}{2}-(z,c_2)\right)(z+\tau c_1+c_2)\right]. \tag{71}$$

The tube domain coordinates of the above period line are then:

$$\tau'=\tau,\quad u'=r_{11}\tau+r_{12}+u-(z,c_1),\quad z'=z+\tau c_1+c_2.$$

As with the case (a), what remains to be computed is the second Narain coordinate of (71). In order to evaluate this term, we write:

$$\begin{aligned}\widetilde{u}' \;&=\; u'+\frac{(z',z_2')}{2\tau_2'} \;=\; r_{11}\tau+r_{12}+u-(z,c_1)+\frac{(z+\tau c_1+c_2,z_2+\tau_2c_1)}{2\tau_2} \;=\\ &=\; r_{11}\tau+r_{12}+\widetilde{u}-\frac{(z,z_2)}{2\tau_2}-(z,c_1)+\frac{(z+\tau c_1+c_2,z_2+\tau_2c_1)}{2\tau_2}\end{aligned} \tag{72}$$

Condition (65) imposes though that:

$$r_{11} \;=\; -\frac{(c_1,c_1)}{2}.$$

After replacing r_{11} in relation (72) and making the appropriate cancellation, one obtains:

$$\widetilde{u}' \;=\; \widetilde{u}+r_{12}+\frac{(c_1,c_2)}{2}-\frac{(z,c_1)}{2}+\frac{(\tau c_1+c_2,z_2)}{2\tau_2}$$

which is exactly the middle equality of (68). □

Let us point out the following feature of the above result. Under parabolic transformations as in cases (a) and (b), the Narain coordinate $\widetilde{u}$ does not change. In case (c), $\widetilde{u}$ is not invariant anymore. However, its imaginary part $\widetilde{u}_2$ is. Indeed, it follows from the middle equality of (68) that:

$$\widetilde{u}_2' \;=\; \widetilde{u}_2-\frac{(z_2,c_1)}{2}+\frac{(\tau_2c_1,z_2)}{2\tau_2} \;=\; \widetilde{u}_2.$$

Since the three types of parabolic isometries of cases (a),(b), and (c) generate the entire group P^+, we obtain that:

Corollary 5.8. *The imaginary part $\widetilde{u}_2$ of the second Narain coordinate is left invariant by the action of* P^+.

We note that the above corollary could also be derived from Lemma 4.1.

5.3 Proof of (a) and (b)

From Theorem 5.7, we see that a parabolic transformation $\gamma\in\mathrm{P}^+$, acting on Γ^+, induces an $\mathrm{SL}(2,\mathbb{Z})$ change in the first Narain coordinate τ while preserving the imaginary part of the second Narain coordinate $\widetilde{u}$. It follows that the quantities $\rho(\tau)$ and $\widetilde{u}_2$ do not change under P^+. The open subset

$$\mathcal{V}^+ \;=\; \{\,(\tau,\widetilde{u},z)\in\Omega^+ \mid \widetilde{u}_2>\max\left(\rho(\tau),\frac{2}{\sqrt{3}}\right)\}$$

is therefore invariant under the action of P^+. This establishes part(a) of Theorem 4.3.

In order to verify (b), recall that the restriction to Ω^+ of the holomorphic fibration:

$$\Omega^+(V) \xrightarrow{p} \mathrm{U(N)}_{\mathbb{Z}} \backslash \Omega^+(V)$$

is the projection in the first and last Narain coordinates:

$$\Omega^+ \;\to\; \mathbb{H} \times \Lambda_{\mathbb{C}}, \quad (\tau, \widetilde{u}, z) \mapsto (\tau, z).$$

Since the group $\mathrm{U(N)}_{\mathbb{Z}}$ acts on $(\tau, \widetilde{u}, z)$ by translation of $\widetilde{u}$ by integers, we see that, under the exponential identification, $\mathcal{V}^+ \cap p^{-1}(\tau, z)$ represents an open punctured disc:

$$\Delta^*_{(\tau,z)} \;=\; \{\, t \in \mathbb{C}^* \mid |t| < e^{-2\pi \cdot \max(\rho(\tau), 2/\sqrt{3})} \,\}.$$

The center of the disc is the compactifying cusp.

5.4 Proof of (c)

Let $[\omega]$ and $[\omega']$ be period lines in $\mathcal{V}^+$. Assume that $\gamma \in \Gamma^+$ has the the property that $\gamma([\omega]) = [\omega']$. We shall prove here that, under these conditions, the isometry γ must belong to P^+.

Most of the considerations required to check this fact will rely on the following technical argument:

Lemma 5.9. *Let r be an element of L which does not belong to $\mathrm{V}^\perp$. Denote by a_1 and a_2 the integers representing the intersection numbers $\langle r, y_1\rangle$ and $\langle r, y_2\rangle$, respectively. Then, for any $[\omega] \in \Omega^+$ with $\langle \omega, y_2\rangle = 1$, one has:*

$$|\langle \omega, r\rangle| \;-\; \frac{\langle r, r\rangle \tau_2}{2|a_2\tau - a_1|} \;\geq\; |a_2\tau - a_1| \cdot \widetilde{u}_2$$

where $\widetilde{u}_2$ is the imaginary part of the second Narain coordinate of $[\omega]$.

Proof. Let $(a_1, a_2)(b_1, b_2)(c)$ be the representation of r in the decomposition (49). Then, a_1, a_2 are not simultaneously zero and $2a_1b_1 + 2a_2b_2 + (c, c) = \langle r, r\rangle$.

The period line $[\omega]$ belongs to Ω^+ and, hence, it is given by a set of holomorphic (tube domain) coordinates (τ, u, z). For the sake of simplicity in the following computations, we shall denote $\langle \omega, r\rangle$ by A. We have:

$$A \;=\; b_1\tau + b_2 + a_1u - a_2\tau u - \frac{a_2}{2}(z, z) + (c, z)$$

and therefore,

$$u \;=\; \frac{b_1\tau + b_2 - \frac{a_2}{2}(z, z) + (c, z) - A}{a_2\tau - a_1}. \tag{73}$$

The Narain coordinate $\tilde{u}$ is given by:

$$\tilde{u} = u + \frac{(z, z_2)}{2\tau_2}$$

so its imaginary part can be obtained as:

$$\tilde{u}_2 = u_2 + \frac{(z_2, z_2)}{2\tau_2} = \frac{2\tau_2 u_2 + (z_2, z_2)}{2\tau_2}. \tag{74}$$

But, from (73), one computes:

$$u_2 \;=\; \frac{\mathrm{Im}\left\{\left[b_1\tau + b_2 - \frac{a_2}{2}(z, z) + (c, z) - A\right]\,\left[\overline{a_2\tau - a_1}\right]\right\}}{|a_2\tau - a_1|^2}. \tag{75}$$

The numerator of the right-hand side of (75) has the following form:

$$\begin{aligned}
\mathrm{Num} \;=\; & \Big[b_1\tau_1 + b_2 - \frac{a_2}{2}(z_1,z_1) + \frac{a_2}{2}(z_2,z_2) + (c,z_1) - A_1\Big]\,[-a_2\tau_2] \;+ \\
& +\; [b_1\tau_2 - a_2(z_1,z_2) + (c,z_2) - A_2]\,[a_2\tau_1 - a_1] \;= \\
& = -[b_1a_1 + b_2a_2]\tau_2 + \frac{a_2^2\tau_2}{2}(z_1,z_1) - \frac{a_2^2\tau_2}{2}(z_2,z_2) - \\
& - a_2[a_2\tau_1 - a_1](z_1,z_2) - a_2\tau_2(c,z_1) + [a_2\tau_1 - a_1](c,z_2) + \\
& + A_1a_2\tau_2 - A_2a_2\tau_1 + A_2a_1.
\end{aligned}$$

Then:

$$\tilde{u}_2 \;=\; \frac{2\tau_2(\mathrm{Num}) + |a_2\tau - a_1|^2(z_2,z_2)}{2\tau_2|a_2\tau - a_1|^2} \tag{76}$$

and one obtains the numerator of the right-hand side of (76) as:

$$\begin{aligned}
& -2[b_1a_1 + b_2a_2]\tau_2^2 + a_2^2\tau_2^2(z_1,z_1) - a_2^2\tau_2^2(z_2,z_2) + \\
& - 2a_2[a_2\tau_1 - a_1]\tau_2(z_1,z_2) - 2a_2\tau_2^2(c,z_1) + 2[a_2\tau_1 - a_1]\tau_2(c,z_2) + \\
& + [a_2\tau_1 - a_1]^2(z_2,z_2) + a_2^2\tau_2^2(z_2,z_2) + \\
& + 2A_1a_2\tau_2^2 - 2A_2a_2\tau_1\tau_2 + 2A_2a_1\tau_2 \;= \\
& = \; -2[b_1a_1 + b_2a_2]\tau_2^2 + a_2^2\tau_2^2(z_1,z_1) + [a_2\tau_1 - a_1]^2(z_2,z_2) + \\
& - 2a_2[a_2\tau_1 - a_1]\tau_2(z_1,z_2) - 2a_2\tau_2^2(c,z_1) + 2[a_2\tau_1 - a_1]\tau_2(c,z_2) + \\
& + 2A_1a_2\tau_2^2 - 2A_2a_2\tau_1\tau_2 + 2A_2a_1\tau_2 \;= \\
& -[2b_1a_1 + 2b_2a_2 + (c,c)]\tau_2^2 + \\
& + \; ([-a_2\tau_2]z_1 + [a_2\tau_1 - a_1]z_2 + \tau_2 c,\; -[a_2\tau_2]z_1 + [a_2\tau_1 - a_1]z_2 + \tau_2 c) \;+ \\
& + 2A_1a_2\tau_2^2 - 2A_2a_2\tau_1\tau_2 + 2A_2a_1\tau_2.
\end{aligned} \tag{77}$$

Denote

$$w := [-a_2\tau_2]z_1 + [a_2\tau_1 - a_1]z_2 + \tau_2 c.$$

We have then:

$$\tilde{u}_2 \;=\; \frac{-\langle r,r\rangle\tau_2^2 + (w,w) + 2A_1a_2\tau_2^2 - 2A_2a_2\tau_1\tau_2 + 2A_2a_1\tau_2}{2\tau_2|a_2\tau - a_1|^2}\;. \tag{78}$$

One obtains therefore:

$$\begin{aligned}
\tilde{u}_2 - \frac{(w,w)}{2\tau_2|a_2\tau - a_1|^2} \;&=\; \frac{-\langle r,r\rangle\tau_2}{2|a_2\tau - a_1|^2} \;+\; \frac{2A_1a_2\tau_2^2 - 2A_2a_2\tau_1\tau_2 + 2A_2a_1\tau_2}{2\tau_2|a_2\tau - a_1|^2} \;= \\
&= \; \frac{-\langle r,r\rangle\tau_2}{2|a_2\tau - a_1|^2} \;+\; \frac{A_1a_2\tau_2 - A_2a_2\tau_1 + A_2a_1}{|a_2\tau - a_1|^2} \;= \\
&= \; \frac{-\langle r,r\rangle\tau_2}{2|a_2\tau - a_1|^2} \;-\; \frac{\mathrm{Im}\,[A(\overline{a_2\tau - a_1})]}{|a_2\tau - a_1|^2}\;.
\end{aligned}$$

Since the pairing $(\cdot,\cdot)$ is negative definite, it follows that:

$$\tilde{u}_2 \;\le\; \frac{-\langle r,r\rangle\tau_2}{2|a_2\tau - a_1|^2} \;-\; \frac{\mathrm{Im}\,[A(\overline{a_2\tau - a_1})]}{|a_2\tau - a_1|^2} \;\le\; \frac{-\langle r,r\rangle\tau_2}{2|a_2\tau - a_1|^2} \;+\; \frac{|A|}{|a_2\tau - a_1|}. \tag{79}$$

Multiplying the above line by $|a_2\tau - a_1|$ produces the inequality stated in Lemma 5.9. $\square$

Remark 5.10. *During the course of the above proof, we actually show a slightly stronger result than the one stated in Lemma* 5.9*. Namely, from the line just before* (79)*, it follows that:*

$$\left| \mathrm{Im}\left(\frac{\langle \omega, r\rangle}{a_2\tau - a_1} \right) \right| \; - \; \frac{\langle r, r\rangle \tau_2}{2|a_2\tau - a_1|^2} \; \geq \; \widetilde{u}_2.$$

Let us return to the proof of 4.3 (c). Our strategy is as follows. In holomorphic (tube domain) coordinates:

$$\omega \; = \; (\tau, 1)\left(u, -\tau u - \frac{1}{2}(z, z) \right)(z)$$

$$\omega' \; = \; (\tau', 1)\left(u', -\tau' u' - \frac{1}{2}(z', z') \right)(z').$$

It suffices to prove 4.3 (c) for the case when

$$\rho(\tau) = \tau_2 \;\text{ and }\; \rho(\tau') = \tau_2', \tag{80}$$

as, from the description in Theorem 5.7, one can always find parabolic isometries in $\mathcal{S}$ that transform ω and ω' to period lines satisfying the above conditions.

We shall therefore assume that (80) holds. We prove then that, under these conditions, if $\gamma([\omega]) = [\omega']$ then $\gamma^{-1}(y_1)$ and $\gamma^{-1}(y_2)$ must belong to V. This claim implies $\gamma^{-1} \in \mathrm{P}^+$, which in turn gives $\gamma \in \mathrm{P}^+$.

Claim 5.11. $\gamma^{-1}(y_2) \in V$.

Proof. Since $\gamma([\omega]) = [\omega']$, one can write that, for some $\alpha \in \mathbb{C}^*$,

$$\alpha\left(\, (\tau', 1)(u', -\tau u' - \frac{1}{2}(z', z'))(z') \, \right) \; = \; \gamma\left(\, (\tau, 1)(u, -\tau u - \frac{1}{2}(z, z))(z) \, \right). \tag{81}$$

The scaling factor α can be determined as follows:

$$\alpha \; = \; \langle \gamma(\omega), y_2\rangle \; = \; \langle \omega, \gamma^{-1}(y_2)\rangle.$$

Since γ is an isometry of L, $\gamma^{-1}(y_2)$ is integral, primitive and isotropic. In the framework of (49), one can then write:

$$\gamma^{-1}(y_2) = (a_1, a_2)(b_1, b_2)(c)$$

with $2(a_1 b_1 + a_2 b_2) + (c, c) = 0$.

Recall then that $\langle \omega, \overline{\omega}\rangle \; = \; 4\tau_2 u_2 + 2(z_2, z_2) \; = \; 4\tau_2 \tilde{u}_2$. This fact, combined with (81), shows:

$$\tau_2 \tilde{u}_2 \; = \; \frac{\langle \omega, \overline{\omega}\rangle}{4} \; = \; \frac{|\alpha|^2 \langle \omega', \overline{\omega}'\rangle}{4} \; = \; |\alpha|^2 \tau_2' \tilde{u}_2'. \tag{82}$$

Hence:

$$\tau_2' \tilde{u}_2' \; = \; \frac{\tau_2 \tilde{u}_2}{|\alpha|^2}. \tag{83}$$

We will show that the above condition implies that a_1 and a_2 are simultaneously zero. Indeed, let us assume the contrary and argue to a contradiction. Lemma 5.9 applied with $r = \gamma^{-1}(y_2)$ implies that:

$$|\alpha| \; \geq \; |a_2\tau - a_1| \tilde{u}_2.$$

Therefore,

$$\tau_2' \tilde{u}_2' \; \leq \; \frac{\tau_2}{|a_2\tau - a_1|^2 \tilde{u}_2}. \tag{84}$$

But, since the assumption is that a_1 and a_2 are not both zero, one has that:

$$\frac{\tau_2}{|a_2\tau - a_1|^2} \ \leq \ \rho(\tau).$$

Then (84) and the fact that $[\omega] \in \mathcal{V}^+$ lead to:

$$\tau_2' \tilde{u}_2' \ \leq \ \frac{\rho(\tau)}{\tilde{u}_2} \ < \ 1.$$

One has then that:

$$\tilde{u}_2' \ < \ \frac{1}{\tau_2'} = \frac{1}{\rho(\tau')} \ \leq \ \frac{2}{\sqrt{3}}$$

which clearly contradicts $[\omega'] \in \mathcal{V}^+$.

Now, since we have shown $a_1 = a_2 = 0$ and, since $\gamma^{-1}(y_2)$ is isotropic, we have that $(c, c) = 0$. This, in turn, implies that $c = 0$. It follows that $\gamma^{-1}(y_2) \in V$. ☐

Next, we complete the final step in the proof of 4.3(c).

Claim 5.12. $\gamma^{-1}(y_1) \in V$.

Proof. Claim 5.11 assures us that:

$$\gamma^{-1}(y_2) = (0,0)(b_1, b_2)(0)$$

where b_1, b_2 are not simultaneously zero and are relatively prime. Let us then analyze the integral element:

$$\gamma^{-1}(y_1) \ = \ (a_1', a_2')(b_1', b_2')(c').$$

If $a_1' = a_2' = 0$, this fact together with the fact that $\gamma^{-1}(y_1)$ is isotropic implies that $c' = 0$. It follows then that $\gamma^{-1}(y_1) \in$ V and the proof of Claim 5.12 is done.

It suffices therefore to assume that at least one of a_1' and a_2' is non-zero and derive a contradictiction. Indeed, since:

$$\langle \gamma^{-1}(y_1), \gamma^{-1}(y_2) \rangle \ = \ \langle y_1, y_2 \rangle \ = \ 0$$

we have that:

$$a_1' b_1 + a_2' b_2 = 0. \tag{85}$$

Relation (85) combined with the fact that b_1, b_2 are relatively prime implies then that:

$$a_1' = q b_2, \ a_2' = -q b_1$$

for some $q \in \mathbb{Z}$, $q \neq 0$.

But then, recalling equation (81) of Claim 5.11, we deduce that:

$$\tau' \ = \ \frac{\langle \omega, \gamma^{-1}(y_1) \rangle}{\langle \omega, \gamma^{-1}(y_2) \rangle} \ = \ \frac{\langle \omega, \gamma^{-1}(y_1) \rangle}{b_1 \tau + b_2}.$$

This implies:

$$\tau' \ = \ - \, q \, \frac{\langle \omega, \gamma^{-1}(y_1) \rangle}{a_2' \tau - a_1'}. \tag{86}$$

At this point, Remark 5.10 applied with $r = \gamma^{-1}(y_1)$ provides the following estimate:

$$\left| \mathrm{Im} \left(\frac{\langle \omega, \gamma^{-1}(y_1) \rangle}{a_2' \tau - a_1'} \right) \right| \ \geq \ \widetilde{u}_2.$$

From (86), one then obtains:

$$\tau_2' \;=\; |q|\cdot\left|\mathrm{Im}\left(\frac{\langle\omega,\gamma^{-1}(y_1)\rangle}{a_2'\tau - a_1'}\right)\right| \;\geq\; |q|\cdot\widetilde{u}_2. \tag{87}$$

But, equation (83) tells us that:

$$\tau_2'\widetilde{u}_2' \;=\; \frac{\tau_2\widetilde{u}_2}{|b_1\tau+b_2|^2}. \tag{88}$$

This fact, together with inequality (87) implies:

$$\frac{\tau_2}{|b_1\tau+b_2|^2\widetilde{u}_2'} \;\geq\; |q|. \tag{89}$$

Let us analyze the possibilities that can appear. If one assumes:

$$\frac{\tau_2}{|b_1\tau+b_2|^2} \;\leq\; 1$$

then inequality (89) implies that

$$|q| \;\leq\; \frac{1}{\widetilde{u}_2'} \;<\; \frac{\sqrt{3}}{2}$$

which contradicts the fact that q is integral and non-zero. Therefore, it must be the case that:

$$\frac{\tau_2}{|b_1\tau+b_2|^2} \;>\; 1.$$

However, since $\tau_2 = \rho(\tau)$, τ_2 represents the maximum imaginary part over the $\mathrm{SL}(2,\mathbb{Z})$ orbit of τ. The above condition can then only hold if $b_1 = 0$ and $b_2 = 1$. In such a situation, taking into account (89) and (88), one finds:

$$1 \;\leq\; \frac{\tau_2'}{\widetilde{u}_2} \;=\; \frac{\tau_2}{\widetilde{u}_2'}.$$

Therefore:

$$\tau_2' \;\geq \widetilde{u}_2 \;\text{ and }\; \tau_2 \geq \widetilde{u}_2'.$$

But the above inequalities contradict the fact that $[\omega]$ and $[\omega']$ are in $\mathcal{V}^+$, which would imply:

$$\widetilde{u}_2 > \tau_2 \;\text{ and }\; \widetilde{u}_2' > \tau_2'.$$

We conclude therefore that $a_1' = a_2' = 0$, and, as explained earlier, this implies $\gamma^{-1}(y_1) \in \mathrm{V}$. $\square$

6 A Consequence of the Large Complex Structure Condition

In this section we analyze the types of singular fibers (or rather their so-called ADE types) that appear in the elliptic fibration of a triple $(\mathrm{X}, \varphi, \mathrm{S})$ satisfying the large complex structure condition. Recall that, if

$$\mathcal{U} \;\subset\; \Gamma\backslash\Omega$$

is one of the two large complex structure domains defined in Section 4, there exists a natural holomorphic fibration with fibers isomorphic to open complex punctured discs:

$$\mathcal{U} \;\rightarrow\; \mathcal{D} \tag{90}$$

whose base space is the appropriate Type II Mumford boundary divisor. We shall prove here that the ADE type does not change over the fibers of (4).

In order to place the above statement on a rigorous footing, let us review a few classical definitions and results. For details and explicit proofs we refer the reader to [19, 3, 22].

Let $(\mathrm{X}, \varphi, \mathrm{S})$ be an elliptic K3 surface with section. As mentioned earlier, in such a context one has a decomposition of the Neron-Severi lattice:

$$\mathrm{NS(X)} \;= \mathcal{H}_{\mathrm{X}} \oplus \mathcal{W}_{\mathrm{X}}$$

where $\mathcal{W}_{\mathrm{X}}$ is the negative-definite sublattice of NS(X) generated by classes associated to algebraic cycles orthogonal to both the elliptic fiber and the section.

Definition 6.1. *The sublattice $\mathcal{W}_{\mathrm{X}}^{\mathrm{root}}$ of $\mathcal{W}_{\mathrm{X}}$ spanned by:*

$$\{\; r \in \mathcal{W}_{\mathrm{X}} \mid \langle r, r \rangle = -2 \;\}$$

is called the **ADE type** *of the elliptic fibration with section* $(\mathrm{X}, \varphi, \mathrm{S})$.

The reason for the above terminology is that the lattice $\mathcal{W}_{\mathrm{X}}^{\mathrm{root}}$ has a special decomposition involving the classical root lattices A_n, D_n and E_n and, this decomposition encodes important information about the geometry of the singular fibers of the elliptic pencil φ. In order to explain this feature, denote by Σ the finite set of points $v \in \mathbb{P}^1$ for which the corresponding fiber F_v, in the elliptic fibration $\varphi \colon \mathrm{X} \to \mathbb{P}^1$, is singular. For each $v \in \Sigma$ one has a formal decomposition into irreducible components:

$$\mathrm{F}_v \;=\; \Theta_{v,0} \;+\; \sum_{j=1}^{t_v - 1} \mu_{v,j} \Theta_{v,j}. \tag{91}$$

Here $t_v \geq 1$ represents the number of irreducible components of F_v and $\Theta_{v,0}$ is the unique irreducible component of F_v meeting the section S. One denotes then by T_v the sublattice in $\mathcal{W}_{\mathrm{X}}$ spanned by the classes:

$$c_1\left(\Theta_{v,j}\right), \;\; 1 \leq j \leq t_v - 1.$$

The following classical result due to Kodaira [19] relates the lattices T_v with the geometry of the singular fibers.

Theorem 6.2. *If $t_v \geq 2$ then $\Theta_{v,j}$ is a smooth rational curve for $0 \leq j \leq t_v - 1$. Moreover, one can deduce the isomorphism class of the lattice T_v from the Kodaira type of the singular fiber F_v as follows:*

Type of F_v	T_v
I_1, II	{0}
I_2, III	A_1
I_3, IV	A_2
I_n ($n \geq 4$)	A_{n-1}
I_n^*	D_{n+4}
IV^*	E_6
III^*	E_7
II^*	E_8

Note furthermore that $\mathrm{T}_v \subset \mathcal{W}_{\mathrm{X}}^{\mathrm{root}}$. Also, for $v_1 \neq v_2$, the two lattices T_{v_1} and T_{v_2} are orthogonal. This allows one to define the direct sum:

$$\mathrm{T} \;=\; \bigoplus_{v \in \Sigma} \mathrm{T}_v \;\subset\; \mathcal{W}_{\mathrm{X}}^{\mathrm{root}}. \tag{92}$$

Proposition 6.3. $\mathrm{T} = \mathcal{W}_{\mathrm{X}}^{\mathrm{root}}$.

Proof. It suffices to check that any root of $\mathcal{W}_{\mathrm{X}}$ also belongs to T. Let r be such a root. Since $\langle r, r\rangle = -2$, by the Riemann-Roch theorem either r or $-r$ represents an effective divisor D on X. Let:

$$\mathrm{D} = \sum n_i \mathrm{D}_i$$

be the formal decomposition of D into irreducible components and denote by F the divisor class of the elliptic fiber. Since $\mathrm{F} \cdot \mathrm{D} = 0$ and F is nef, we deduce that F has vanishing intersection with each of the irreducible components D_i. It follows that each D_i is either equivalent to F, or it is an irreducible component of a singular fiber F_v for some $v \in \Sigma$. But $\mathrm{S} \cdot \mathrm{D} = 0$ and a simple look at the decomposition (91) assures us that $\mathrm{D}_i \in \mathrm{T}_v$. $\square$

We have therefore a decomposition:

$$\mathcal{W}_{\mathrm{X}}^{\mathrm{root}} \;=\; \bigoplus_{v \in \Sigma} \mathrm{T}_v \tag{93}$$

in which every term is isomorphic to one of the classical root lattices A_n ($n \geq 1$), D_n ($n \geq 1$) or E_n ($n = 6, 7, 8$). Moreover, since the root lattices are known to be indecomposable, the decomposition (93) is unique. Therefore, by knowing the isomorphism class of $\mathcal{W}_{\mathrm{X}}^{\mathrm{root}}$, one is able to detect, in a lattice-theoretic manner, most of the geometric types of singular fiber appearing in the actual elliptic fibration. Of course, this method does not distinguish between the Kodaira types I_2 and III or between I_3 and IV and also cannot detect the appearance of the singular fiber types I_1 or II.

Let us formulate the result suggested at the beginning of the section. As in the construction of Section 3, we assume a choice of a rank-two primitive isotropic sublattice $\mathrm{V} \subset \mathrm{L}$. The associated Type II Mumford boundary component is:

$$\mathcal{D}(\mathrm{V}) \;=\; \mathrm{P}^{+} \backslash \left(\mathrm{U}(N)_{\mathbb{C}} \backslash \Omega^{+}(\mathrm{V}) \right).$$

The large complex structure domain associated to V

$$\mathcal{U} \;\subset\; \Gamma \backslash \Omega$$

fibers holomorphically:

$$\mathcal{U} \;\rightarrow\; \mathcal{D}(\mathrm{V}) \tag{94}$$

with all fibers being isomorphic to open punctured complex discs.

Theorem 6.4. *The ADE type lattice $\mathcal{W}_{\mathrm{X}}^{\mathrm{root}}$ remains constant over the fibers of* (94).

Proof. It suffices to prove the above statement for the projection:

$$\mathcal{V}^{+} \;\rightarrow\; \mathrm{U}(N)_{\mathbb{C}} \backslash \Omega^{+}(\mathrm{V}) \tag{95}$$

which covers (94). But, as we explained in the remarks at the end of Section 5.1, in the Narain coordinate framework (95) is just:

$$\left\{ \ (\tau, \widetilde{u}, z) \mid \widetilde{u}_2 > \max\left(\rho(\tau), \frac{2}{\sqrt{3}} \right) \ \right\} \;\rightarrow\; \mathbb{H} \times \Lambda_{\mathbb{C}} \tag{96}$$

$$(\tau, \widetilde{u}, z) \;\rightarrow\; (\tau, z).$$

Let then

$$\mathcal{W}_{(\tau, \widetilde{u}, z)}^{\mathrm{root}} \;\subset\; \mathrm{L}$$

be the ADE lattice associated to the period line $[\omega] \in \Omega^{+}$ of Narain coordinates $(\tau, \widetilde{u}, z)$. Theorem 6.4 is implied by the following two claims.

1. $\mathcal{W}_{(\tau, \widetilde{u}, z)}^{\mathrm{root}} \;\cap\; \mathrm{V}^{\perp}$ does not depend on $\widetilde{u}$.

2. If $\widetilde{u}_2 > \max\left(\rho(\tau), \frac{2}{\sqrt{3}}\right)$, then $\mathcal{W}^{\mathrm{root}}_{(\tau,\widetilde{u},z)} \subset \mathrm{V}^{\perp}$.

The first claim is almost straightforward. Recall the decomposition (49) and the Narain parametrization of Proposition 5.4. An element of L

$$r \ = \ (a_1, a_2)(b_1, b_2)(c)$$

is a root in $\mathcal{W}^{\mathrm{root}}_{(\tau,\widetilde{u},z)}$ if and only if the following two conditions hold.

$$\langle r, r\rangle \ = \ 2(a_1b_1 + a_2b_2) + (c, c) \ = \ -2 \tag{97}$$

$$\langle \omega, r\rangle \ = \ b_1\tau + b_2 + a_1\left(\widetilde{u} - \frac{(z, z_2)}{2\tau_2}\right) + a_2\left(-\tau\widetilde{u} + \frac{\tau(z, z_2)}{2\tau_2} - \frac{(z, z)}{2}\right) + (c, z) = 0. \tag{98}$$

But $\mathrm{V}^{\perp}$ corresponds to $a_1 = a_2 = 0$. Therefore:

$$\mathcal{W}^{\mathrm{root}}_{(\tau,\widetilde{u},z)} \ \cap \ \mathrm{V}^{\perp} \ = \ \{ \ (0,0)(b_1,b_2)(c) \mid (c,c) = -2, \ b_1\tau + b_2 + (c,z) = 0 \ \}$$

and clearly it does not depend on $\widetilde{u}$.

In order to justify the second claim, we show that, under the assumption $\widetilde{u}_2 > \max\left(\rho(\tau), 2/\sqrt{3}\right)$, conditions (97) and (98) imply that $a_1 = a_2 = 0$.

To check this assumption, let us assume that a_1 and a_2 are not simultaneously zero. Then, if conditions (97) and (98) are satisfied, Lemma 5.9 tells us that:

$$\widetilde{u}_2 \ \leq \ \frac{\tau_2}{|a_2\tau - a_1|^2}. \tag{99}$$

Let us discuss the possibilities that can occur.

(a) If $a_1 \neq 0$ and $a_2 \neq 0$, let $n = \gcd(|a_1|, |a_2|)$. Inequality (99) implies then

$$\widetilde{u}_2 \ \leq \ \frac{\rho(\tau)}{n^2} \ \leq \ \rho(\tau).$$

(b) If $a_1 = 0$ and $a_2 \neq 0$ then (99) gives:

$$\widetilde{u}_2 \ \leq \ \frac{\tau_2}{|a_2|^2|\tau|^2} \ \leq \ \frac{\rho(\tau)}{|a_2|^2} \ \leq \ \rho(\tau).$$

(c) If $a_1 \neq 0$ and $a_2 = 0$ then, from (99):

$$\widetilde{u}_2 \ \leq \ \frac{\tau_2}{|a_1|^2} \ \leq \ \frac{\rho(\tau)}{|a_1|^2} \ \leq \ \rho(\tau).$$

Hence, all three possible cases produce contradictions with the large complex structure assumption $\widetilde{u}_2 > \rho(\tau)$. The above assumption must therefore be false. It has to be that $a_1 = a_2 = 0$. The second claim is fully justified and this concludes the proof of Theorem 6.4. □

We close this section with a final comment about the above result. We have shown that, if one moves inside the moduli space $\Gamma\backslash\Omega$, from a Type II boundary point and following the fibers of (94), the ADE type of the elliptic pencils associated to the points encountered stays constant. The fibers of (94) are coming from nilpotent orbits on Ω^+ and, therefore, they can be continued indefinitely inside $\Gamma\backslash\Omega$. However, outside $\mathcal{U}$ they no longer form the fibers of a fibration over the boundary divisor. The images of nilpotent orbits corresponding to distinct boundary points may intersect inside the moduli space.

On each nilpotent orbit, one encounters points where the lattice $\mathcal{W}^{\mathrm{root}}_X$ contains roots no longer belonging to $\mathrm{V}^{\perp}$. At such points, some exceptional singular fibers appear in the corresponding elliptic pencils, as the ADE type lattice is no longer generic (within the given nilpotent orbit). These are the points physicists refer to as points of **enhanced gauge symmetry**. The reason for this terminology is that, under the eight-dimensional F-theory/heterotic string duality, these points correspond on the heterotic side to flat G-bundles with reduced structure group.

In light of this interpretation, Theorem 6.4 states that all enhanced gauge symmetry points lie outside of our large complex structure region.

References

[1] A. Ash, D. Mumford, M. Rapoport and Y.S. Tai. Smooth Compactifications of Locally Symmetric Varieties. Brookline, Math. Sci. Press, 1975

[2] A. Beauville, Application aux Espaces de Modules. In "Géometrie des Surfaces K3: Modules and Périodes". *Astérisque*, 126, 1985

[3] W. Barth, C. Peters and A. van de Ven, Compact Complex Surfaces. Springer, 1984

[4] W.L. Baily. and A. Borel. Compactification of Arithmetic Quotients of Bounded Symmetric Domains. *Ann. of Math.*, 84(2), 1966

[5] F. A. Bogomolov and Y. Tschinkel, Density of Rational Points on Elliptic K3 Surfaces. *Asian J. Math.* 2(4), 2000

[6] D. Burns and M. Rapoport, On the Torelli Problem for Kählerian K3 Surfaces. *Ann. E.N.S.* 8, 1975

[7] G. Cardoso, G. Curio, D. Lust and T. Mohaupt. On the Duality Between the Heterotic String and F-Theory in 8 Dimensions. *Phys. Lett. B*, 389(3), 1996

[8] A. Clingher, Heterotic String Data and Theta Functions. *Adv. Theor. Math. Phys.* 9(2), 2005

[9] A. Clingher and J.W. Morgan, Mathematics Underlying the F-Theory/Heterotic Duality in Eight Dimensions. *Comm. Math. Phys.* 254(3), 2005

[10] J.H. Conway and N.J.A. Sloane. Sphere Packings, Lattices and Groups. *Grundlehren der Mathematischen Wissenschaften*, Springer-Verlag, New York, 1988

[11] G. Curio, R. Donagi. Moduli in N=1 Heterotic/F-Theory Duality. *Nuclear Phys. B*, 512, 1998

[12] R. Donagi. Heterotic F-Theory Duality. *XIIth International Congress of Mathematical Physics (ICMP '97, Brisbane)*, Internat. Press, 1999

[13] I.V. Dolgachev, Mirror Symmetry for Lattice Polarized K3 Surfaces. *J. Math. Sci.* 81(3), 1996

[14] R. Friedman, A New Proof of the Global Torelli Theorem for K3 Surfaces. *Annals of Math.* 120(2), 1984

[15] R. Friedman. Hodge Theory, Degenerations and the Global Torelli Problem. *Thesis, Harvard University*, 1981

[16] R. Friedman, J.W. Morgan and E. Witten. Principal G-Bundles over Elliptic Curves. *Math. Res. Lett.* 5(1-2), 1998

[17] R. Friedman, J.W. Morgan and E. Witten. Vector Bundles and F-Theory. *Comm. Math. Phys*, 187, 1997

[18] M. Gross and P.M.H. Wilson. Large Complex Structure Limits of K3 Surfaces. *J. Diff. Geom.*, 55(3), 2000

[19] K. Kodaira, On Compact Analytic Surfaces. (II) *Annals of Math.* 77, 1963

[20] E. Looijenga and C. Peters, Torelli Theorems for Kähler K3 Surfaces. *Compositio. Math.* 42(2), 1981

[21] A. Mayer, Families of K3 Surfaces. *Nagoya J. Math.* 48, 1972

[22] R. Miranda, The Basic Theory of Elliptic Surfaces. *ETS Editrice* Pisa, 1989

[23] D. Morrison, Compactifications of Moduli Spaces Inspired by Mirror Symmetry. In "Journées de Géometrie Algébrique d'Orsay". *Astérisque*, 218, 1993

[24] D. Morrison and C. Vafa. Compactifications of F-Theory on Calabi-Yau Threefolds (I). *Nuclear Phys. B* 473, 1996

[25] D. Morrison and C. Vafa. Compactifications of F-Theory on Calabi-Yau Threefolds (II). *Nuclear Phys. B* 476, 1996

[26] D. Mumford, A New Approach to Compactifying Locally Symmetric Varieties. Discrete Subgroups of Lie Group and Applications to Moduli. Bombay Colloquium. *Oxford University Press* 1975

[27] K.S. Narain. New Heterotic String Theories in Uncompactified Dimensions < 10. *Phys. Lett. B* 169(1), 1986

[28] V. Nikulin, Finite Groups of Automorphisms of K¨ahler K3 Surfaces. *Proc. Moscow Math. Society* 38, 1980

[29] V. Nikulin, Integral Symmetric Bilinear Forms and Some of Their Applications. *Math. USSR Izv.* 14, 1980

[30] I.I. Pjateckiĭ-Šapiro and I.R. Šafarevič A Torelli Theorem for Algebraic Surfaces of Type K3. *Math. USSR Izv.* 35, 1971

[31] A. Sen, F-Theory and Orientifolds. *Nuclear Phys. B*, 475(3), 1996

[32] A. Todorov, Applications of the Kähler-Einstein-Calabi-Yau Metric to Moduli of K3 Surfaces. *Invent. Math.* 61(3), 1980

[33] B.H. Lian, A. Todorov and S.-T. Yau, Maximal Unipotent Monodromy for Complete Intersection CY Manifolds. *Amer. J. Math.* 127(1), 2005

[34] C. Vafa, Evidence for F-theory. *Nuclear Phys. B*, 469(3), 1996